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# Global in Time Results for a Parabolic Equation Solution in Non-rectangular Domains

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**Abstract.** This article deals with the parabolic equation

$$\partial_t w - c(t)\partial_x^2 w = f \text{ in } D,\ D = \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi_i : [0,+\infty[\to \mathbb{R}, i = 1,\ 2$ and $c : [0,+\infty[\to \mathbb{R}$ satisfying some conditions and the problem is supplemented with boundary conditions of Dirichlet-Robin type. We study the global regularity problem in a suitable parabolic Sobolev space. We prove in particular that for $f \in L^2(D)$ there exists a unique solution $w$ such that $w,\ \partial_t w,\ \partial^j w \in L^2(D)$, $j = 1, 2$. Notice that the case of bounded non-rectangular domains is studied in [9]. The proof is based on energy estimates after transforming the problem in a strip region combined with some interpolation inequality. This work complements the results obtained in [19] in the case of Cauchy-Dirichlet boundary conditions.

**Keywords:** parabolic equations, heat equation, non-rectangular domains, unbounded domains, anisotropic Sobolev spaces.

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## 1. Introduction and statement of the main result

Let $D$ be an open set of $\mathbb{R}^2$ defined by

$$D := \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

where $\varphi_i \in C([0,+\infty[) \cap C^1(0,+\infty)$, $i = 1,\ 2$,

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0 \quad \forall t > 0, \text{ and } \varphi(0) = 0.$$

The lateral boundaries of $D$ are defined by

$$\Gamma_i = \left\{(t, \varphi_i(t)) \in \mathbb{R}^2 : t > 0\right\},\ i = 1,\ 2.$$

Let us introduce the following functional space:

$$\mathcal{H}^{1,2}(D) := \left\{w \in L^2(D) : \partial_t w, \partial_x w, \partial_x^2 w \in L^2(D)\right\}$$

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where $L^2(D)$ stands for the usual Lebesgue space of square-integrable functions on $D$. The space $\mathcal{H}^{1,2}(D)$ is equipped with the natural norm, that is

$$\|w\|^2_{\mathcal{H}^{1,2}(D)} = \|w\|^2_{L^2(D)} + \|\partial_t w\|^2_{L^2(D)} + \sum_{j=1}^{2} \left\|\partial_x^j w\right\|^2_{L^2(D)}.$$

We consider the problem: to find a function $u \in \mathcal{H}^{1,2}(D)$ that satisfies the equation

$$\partial_t u - c(t)\partial_x^2 u = f \;\text{ a.e. on } D \tag{1.1}$$

and the boundary conditions

$$u|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \tag{1.2}$$

where $f \in L^2(D)$ and the coefficient $c$ is a continuous real-valued function defined on $[0, +\infty[$, differentiable on $]0, +\infty[$ and such that

$$0 < \alpha \leqslant c(t) \leqslant \beta$$

for every $t \in [0, +\infty[$, where $\alpha$ and $\beta$ are positive constants. Here, the coefficient $\beta_2$, in boundary conditions is a real number such that

$$\beta_2 > 0.$$

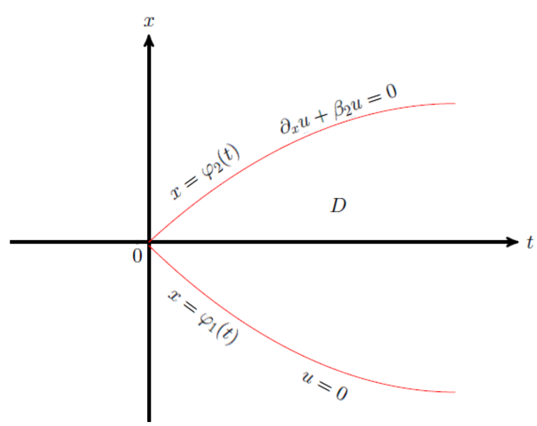


Fig. 1. The unbounded non-rectangular domain D

Problem (1.1)–(1.2) modelizes, for instance, the lateral diffusion of a pollutant in a flow of a river with variable width. Note that the Robin type condition

$$\partial_x u + \beta_2 u|_{\Gamma_2} = 0,$$

means for instance, that the flux of diffusion of the pollutant is proportional to its propagation along the wide of the river. The most interesting points of the parabolic problem studied here is the unboundedness of $D$ with respect to the time variable $t$ and the fact that $D$ shrinks at $t = 0$ ($\varphi(0) = 0$) which prevent one using the methods in [13] and [14]. It is well known that there are two main approaches for the study of boundary value problems in such non-regular domains. The analysis can be done in weighted spaces with the weight controlling the behavior of the solutions near the singularity of the boundary of the domain (see, for instance, [10, 11] and [12]). Our approach is different. Indeed, the space $\mathcal{H}^{1,2}$ used here has low smoothness but one must add assumptions on the type of the domain $D$, as well as conditions on the coefficients $c$ and $\beta_2$, near the singular point 0 and in the neighborhood of $+\infty$. So, our main result is the following:

**Theorem 1.1.** *Let us assume that*

$$|\varphi_i'(t)|\,\varphi(t) \to 0 \quad as \quad t \to 0^+, \quad i = 1,\, 2, \tag{1.3}$$

$$2c(t)\beta_2 - \varphi_2'(t) \geqslant 0 \quad a.e.\ t \in \left]0, +\infty\right[, \tag{1.4}$$

$$\varphi \text{ and } \varphi' \text{ are uniformly bounded in a neighborhood of} + \infty, \tag{1.5}$$

$$c \text{ is a decreasing function in} \left]0, +\infty\right[, \tag{1.6}$$

*and one of the following conditions is satisfied*

**(a)** *$\varphi$ is increasing in a neighborhood of $+\infty$,*

**(b)** $\exists M > 0 : |\varphi'|\,\varphi \leqslant Mc(t)$.

*Then Problem (1.1), (1.2) admits a unique solution $u \in \mathcal{H}^{1,2}(D)$.*

The case where $D$ is bounded (with $c(t) = 1$) is studied in [9]. The case where $\beta_2 = \infty$ corresponding to Cauchy-Dirichlet boundary conditions is studied in [19]. Whereas second-order parabolic equations in bounded non-cylindrical domains are well studied (see for instance [2, 5, 7, 15–18] and the references therein), the literature concerning unbounded non-cylindrical domains does not seem to be very rich. The regularity of the heat equation solution in a non-smooth and unbounded domain (in the $x$ direction) is obtained in [3, 6, 8] and [4].

In the next sections, we prove Theorem 1.1 in four steps:
(1) case of a bounded domain which can be transformed into a rectangle;
(2) case of an unbounded domain which can be transformed into a half strip;
(3) case of a small in time bounded triangular domain;
(4) finally, we use the previous steps and a trace result to complete the proof of Theorem 1.1.

## 2. The case of a bounded domain which can be transformed into a rectangle

Let $T$ be an arbitrary positive number. Denote by

$$D_1 := \left\{(t, x) \in \mathbb{R}^2 : 0 < t < T;\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi(t) > 0$ for all $t \in [0, T]$ and consider the following problem:

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \ \text{ a.e. on } D_1, \\ u|_{\Gamma_1} = u|_{\Gamma_0} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{2.1}$$

where $f_1 \in L^2(D_1)$ and $\Gamma_0$ is the part of $\partial D_1$ where $t = 0$.

Let us denote the inner product in $L^2(D_1)$ by $\langle ., . \rangle$. Then, the uniqueness of the solutions may be obtained by developing the inner product

$$\left\langle \partial_t u - c(t)\partial_x^2 u, u \right\rangle .$$

Indeed, Let us consider $u \in \mathcal{H}^{1,2}(D_1)$ a solution of Problem (2.1) with a null right-hand side term. So,

$$\partial_t u - c(t)\partial_x^2 u = 0 \ \text{ in } \ D_1.$$

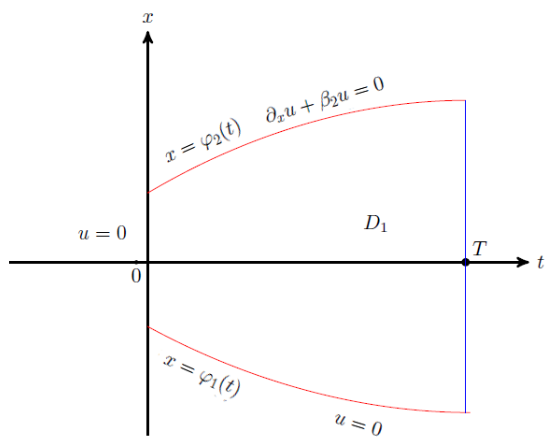


Fig. 2. The bounded domain $D_1$

In addition $u$ fulfils the boundary conditions

$$u|_{\Gamma_0} = u|_{\Gamma_1} = \partial_x u + \beta_2 u|_{\Gamma_2} = 0.$$

Using Green formula, we have

$$\int_{D_1} \left(\partial_t u - c(t)\partial_x^2 u\right) u \; dt \; dx = \int_{\partial D_1} \left(\frac{1}{2} |u|^2 \nu_t - c(t)\partial_x u.u\nu_x\right) d\sigma + \int_{D_1} c(t) \left(|\partial_x u|^2\right) dt \; dx,$$

where $\nu_t$, $\nu_x$ are the components of the unit outward normal vector at $\partial D_1$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of $D_1$ where $t = 0$, we have $u = 0$. Accordingly the corresponding boundary integral vanishes. On the part of the boundary of $D_1$ where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\frac{1}{2} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 (T, x) dx,$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + \left(\varphi_i'\right)^2 (t)}}, \quad \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + \left(\varphi_i'\right)^2 (t)}}$$

and

$$u\left(t, \varphi_1(t)\right) = \partial_x u\left(t, \varphi_2(t)\right) + \beta_2 u\left(t, \varphi_2(t)\right) = 0.$$

Consequently, the corresponding integral is

$$\int_0^T \frac{1}{2} \left(2c(t)\beta_2 - \varphi_2'(t)\right) u^2\left(t, \varphi_2(t)\right) dt.$$

Then, we obtain

$$\begin{aligned}\int_{D_1} \left(\partial_t u - c(t)\partial_x^2 u\right) u \; dt \; dx &= \frac{1}{2} \int_0^T \left(2c(t)\beta_2 - \varphi_2'(t)\right) u^2\left(t, \varphi_2(t)\right) dt \; + \\ &\quad + \frac{1}{2} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 (T, x) dx + \int_{D_1} c(t) \left(|\partial_x u|^2\right) dt \; dx.\end{aligned}$$

Consequently using the fact that $u$ is the solution yields

$$\int_{D_1} c(t)\left(|\partial_x u|^2\right) dt\ dx = 0,$$

because thanks to the condition (1.4) and to the fact that $c(t) > 0$ for every $t \in [0, +\infty[$, we have

$$\frac{1}{2}\int_0^T \left(2c(t)\beta_2 - \varphi_2'(t)\right) u^2(t, \varphi_2(t))\, dt + \frac{1}{2}\int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2(T, x) dx + \int_{D_1} c(t)\left(|\partial_x u|^2\right) dt\ dx \geqslant 0.$$

This implies that $|\partial_x u|^2 = 0$ and consequently $\partial_x^2 u = 0$. Then, the hypothesis $\partial_t u - c(t)\partial_x^2 u = 0$ gives $\partial_t u = 0$. Thus, $u$ is a constant. The boundary conditions and the fact that $\beta_2 \neq 0$, imply that $u = 0$ in $D_1$. This proves the uniqueness of the solution of Problem (2.1).
Now, let us look at the existence of solutions for Problem (2.1). The change of variables $(t, x)$ to $\left(t, \dfrac{x - \varphi_1(t)}{\varphi(t)}\right)$ transforms $D_1$ into the rectangle $Q = ]0, T[\times]0, 1[$ and Problem (2.1) becomes the following:

$$\begin{cases} \partial_t u + a(t, x)\partial_x u - \dfrac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \text{ a.e. on } Q, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2\varphi(t)u|_{x=1} = 0, \end{cases}$$

where $f_1 \in L^2(Q)$ and $a(t, x) = -\dfrac{x\varphi'(t) + \varphi_1'(t)}{\varphi(t)}$. Observe that the coefficient $a$ is bounded. So, the operator

$$a(t, x)\partial_x : \mathcal{H}^{1,2}(Q) \longrightarrow L^2(Q)$$

is compact. Hence, it is sufficient to study the following problem:

$$\begin{cases} \partial_t u - \dfrac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \text{ a.e. on } Q, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2\varphi(t)u|_{x=1} = 0, \end{cases} \tag{2.2}$$

where $f_1 \in L^2(Q)$. It is clear that Problem (2.2) admits a (unique) solution $u \in \mathcal{H}^{1,2}(Q)$ because the coefficient $\dfrac{c(t)}{\varphi^2(t)}$ satisfies the "uniform parabolicity" condition (see, for example [1]). On other hand, it is easy to verify that the aforementioned change of variable conserves the spaces $L^2$ and $\mathcal{H}^{1,2}$. Consequently, we have proved the following theorem:

**Theorem 2.1.** *Problem (2.1) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_1)$.*

## 3. The case of an unbounded domain which can be transformed into a half strip

In this case, we set

$$D_2 := \left\{(t, x) \in \mathbb{R}^2 : t > 0;\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

with $\varphi(0) > 0$ and consider the following problem:

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \ \text{ a.e. on } D_2, \\ u|_{\Gamma_1} = u|_{\Gamma_0} = 0, \\ \partial_x u + \beta_2 u|_{\Gamma_2} = 0, \end{cases} \tag{3.1}$$

where $f_1 \in L^2(D_2)$, and $\Gamma_0$ is the part of $\partial D_2$ where $t = 0$.

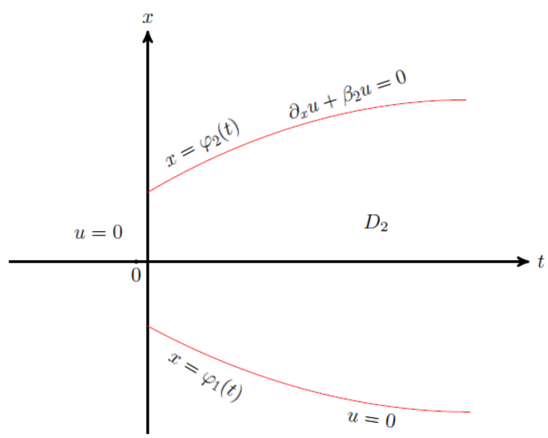


Fig. 3. The unbounded domain $D_2$

The change of variables indicated in the previous section transforms $D_2$ into the half strip $P = ]0, +\infty[ \times ]0, 1[$. So Problem (3.1) can be written as follows:

$$\begin{cases} \partial_t u + a(t,x)\,\partial_x u - \dfrac{c(t)}{\varphi^2(t)}\partial_x^2 u = f_1 \ \text{ a.e. on } P, \\ u|_{t=0} = u|_{x=0} = 0, \\ \partial_x u + \beta_2 \varphi(t) u|_{x=1} = 0, \end{cases} \tag{3.2}$$

where $f_1 \in L^2(P)$ and the coefficients $a$ is that defined in Section 2.

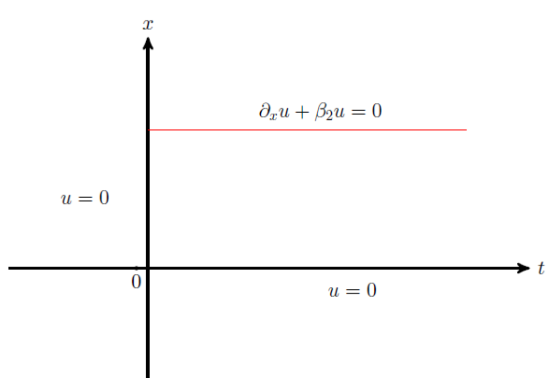


Fig. 4. The half strip P

Let $f_1^{(n)}$ be the restriction $f_1|_{]0,n[\times]0,1[}$, for all $n \in \mathbb{N}^*$. Then, Theorem 2.1 shows that for all

$n \in \mathbb{N}^*$, there exists a function $u_n \in \mathcal{H}^{1,2}(P_n)$ which solves the problem

$$\begin{cases} \partial_t u_n + a(t,x)\,\partial_x u_n - \dfrac{c(t)}{\varphi^2(t)}\partial_x^2 u_n = f_1^{(n)} \text{ a.e. on } P_n, \\ u_n|_{t=0} = u_n|_{x=0} = 0, \\ \partial_x u_n + \beta_2 \varphi(t) u_n|_{x=1} = 0, \end{cases} \tag{3.3}$$

where $f_1^{(n)} \in L^2(P_n)$, and $P_n = ]0,n[\times]0,1[$.

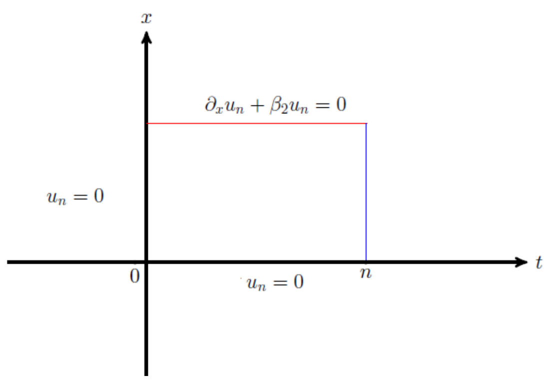


Fig. 5. The truncated half strip Pn

Now, let us prove an "energy" type estimate for the solutions $u_n$ which will allow us to solve Problem (3.2) and then equivalently Problem (3.1).

**Proposition 3.1.** *There exists a constant $K > 0$ independent of $n$ such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2}(P_n)} \leqslant K \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} \leqslant K \|f_1\|^2_{L^2(P)}.$$

In order to prove Proposition 3.1, we need the following result:

**Lemma 3.1.** *There exists a constant $K$ independent of $n$ such that*

$$\|u_n\|^2_{L^2(P_n)} \leqslant K \|\partial_x u_n\|^2_{L^2(P_n)} \leqslant K \|f_1\|^2_{L^2(P)}.$$

*Proof.* The Poincaré inequality gives $\|u_n\|_{L^2(P_n)} \leqslant K \|\partial_x u_n\|_{L^2(P_n)}$. Now, we estimate the inner product $\left\langle f_1^{(n)}, u_n \right\rangle$.

Estimation of $\left\langle f_1^{(n)}, u_n \right\rangle$:

$$\begin{aligned} \left\langle f_1^{(n)}, u_n \right\rangle &= \int_{P_n} u_n \partial_t u_n dtdx + \int_{P_n} a(t,x)\, u_n \partial_x u_n dtdx - \int_{P_n} \frac{c(t)}{\varphi^2(t)} u_n \partial_x^2 u_n dtdx = \\ &= \int_{\partial P_n} \left[ \frac{1}{2} |u_n|^2 \nu_t + a(t,x) \frac{1}{2} |u_n|^2 \nu_x - \frac{c(t)}{\varphi^2(t)} \partial_x u_n . u_n \nu_x \right] d\sigma + \\ &\quad + \int_{P_n} \frac{c(t)}{\varphi^2(t)} (\partial_x u_n)^2 dtdx - \frac{1}{2} \int_{P_n} \partial_x a(t,x) |u_n|^2 dtdx, \end{aligned}$$

where $\nu_t, \nu_x$ are the components of the unit outward normal vector at the boundary of $P_n$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of $P_n$ where $t = 0$, we have $u_n = 0$ and consequently the corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral is the following:

$$\int_0^1 \frac{1}{2}(u_n)^2(n,x)dx.$$

On the part of the boundary where $x = 0$, we have $\nu_x = -1$, $\nu_t = 0$ and $u_n(t,0) = 0$. Consequently, the corresponding integral vanishes. On the part of the boundary where $x = 1$, we have $\nu_x = 1$, $\nu_t = 0$ and

$$\partial_x u_n(t,1) + \beta_2 \varphi(t) u_n(t,1) = 0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{(2c(t)\beta_2 - \varphi_2'(t))}{2\varphi(t)}(u_n)^2(t,1)dt.$$

Finally,

$$\begin{aligned}\left\langle f_1^{(n)}, u_n\right\rangle &= \int_{\varphi_1(n)}^{\varphi_2(n)} \frac{1}{2}(u_n)^2(n,x)dx + \int_0^n \frac{(2c(t)\beta_2 - \varphi_2'(t))}{2\varphi(t)}(u_n)^2(t,1)dt + \\ &\quad + \int_{P_n} \frac{c(t)}{\varphi^2(t)}(\partial_x u_n)^2 dtdx + \frac{1}{2}\int_{P_n} \frac{\varphi'(t)}{\varphi(t)}|u_n|^2 dtdx.\end{aligned}$$

Thanks to the condition (1.4) and since the function $\varphi$ increases, we obtain

$$\left\langle f_1^{(n)}, u_n\right\rangle \geqslant \int_{P_n} \frac{c(t)}{\varphi^2(t)}(\partial_x u_n)^2 dtdx \geqslant C\|\partial_x u_n\|^2_{L^2(P_n)}.$$

Hence, for all $\epsilon > 0$,

$$\begin{aligned}\|\partial_x u_n\|^2_{L^2(P_n)} &\leqslant \frac{1}{C}\|u_n\|_{L^2(P_n)}\left\|f_1^{(n)}\right\|_{L^2(P_n)} \leqslant \\ &\leqslant \frac{1}{C\epsilon}\left\|f_1^{(n)}\right\|^2_{L^2(P_n)} + \frac{\epsilon}{C}\|u_n\|^2_{L^2(P_n)}.\end{aligned}$$

By using the Poincaré inequality, we obtain

$$\left(1 - \frac{\epsilon}{C}\right)\|\partial_x u_n\|^2_{L^2(P_n)} \leqslant \frac{1}{C\epsilon}\left\|f_1^{(n)}\right\|^2_{L^2(P_n)}.$$

Choosing $\epsilon$ small enough in the previous inequality, we prove the existence of a constant $K$ such that

$$\|\partial_x u_n\|^2_{L^2(P_n)} \leqslant K\left\|f_1^{(n)}\right\|^2_{L^2(P_n)}.$$

Since

$$\left\|f_1^{(n)}\right\|^2_{L^2(P_n)} \leqslant \|f_1\|^2_{L^2(P)},$$

we obtain

$$\|\partial_x u_n\|^2_{L^2(P_n)} \leqslant K\|f_1\|^2_{L^2(P)}.$$

□

**Remark 3.1.** *Similar computations show that the same result holds true when we substitute the condition that $\varphi$ increases in a neighborhood of $+\infty$ by the following:*

$$|\varphi'(t)|\,\varphi(t) \leqslant Mc(t).$$

*Proof of Proposition 3.1.*

Let us denote the inner product in $L^2(P_n)$ by $\langle .,. \rangle$, then we have

$$\left\| f_1^{(n)} \right\|^2_{L^2(P_n)} = \langle \partial_t u_n + a(t,x)\,\partial_x u_n - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u_n, \partial_t u_n + a(t,x)\,\partial_x u_n - \frac{c(t)}{\varphi^2(t)}\partial_x^2 u_n \rangle =$$

$$= \|\partial_t u_n\|^2_{L^2(P_n)} + \|a\partial_x u_n\|^2_{L^2(P_n)} + \left\| \frac{c(t)}{\varphi^2(t)}\partial_x^2 u_n \right\|^2_{L^2(P_n)} + 2\int_{P_n} a\partial_t u_n \partial_x u_n dt dx -$$

$$-2\int_{P_n} a\frac{c(t)}{\varphi^2(t)}\partial_x u_n \partial_x^2 u_n dt dx - 2\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dt dx.$$

Observe that the coefficients $a$ and $\dfrac{c(t)}{\varphi^2(t)}$ are bounded. So, thanks to Lemma 3.1, for all $\epsilon > 0$ we obtain

$$\|\partial_t u_n\|^2_{L^2(P_n)} + \left\| \frac{c(t)}{\varphi^2(t)}\partial_x^2 u_n \right\|^2_{L^2(P_n)} - 2\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dt dx \leqslant$$

$$\leqslant \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} + \|a\partial_x u_n\|^2_{L^2(P_n)} + 2\,\|\partial_t u_n\|_{L^2(P_n)}\,\|a\partial_x u_n\|_{L^2(P_n)} +$$

$$+ 2\left\|\partial_x^2 u_n\right\|_{L^2(P_n)} \left\| a\frac{c(t)}{\varphi^2(t)}\partial_x u_n \right\|_{L^2(P_n)} \leqslant$$

$$\leqslant \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} + K_1\left(1+\frac{2}{\epsilon}\right)\|\partial_x u_n\|^2_{L^2(P_n)} + \epsilon\,\|\partial_t u_n\|^2_{L^2(P_n)} + \epsilon\left\|\partial_x^2 u_n\right\|_{L^2(P_n)} \leqslant$$

$$\leqslant K_\epsilon \left\| f_1^{(n)} \right\|^2_{L^2(P_n)} + \epsilon\,\|\partial_t u_n\|^2_{L^2(P_n)} + \epsilon\left\|\partial_x^2 u_n\right\|_{L^2(P_n)}$$

where $K_\epsilon$ and $K_1$ are constants independent of $n$. Consequently

$$(1-\epsilon)\left(\|\partial_t u_n\|^2_{L^2(P_n)} + \left\|\partial_x^2 u_n\right\|^2_{L^2(P_n)}\right) \leqslant 2\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dt dx + K_\epsilon \left\| f_1^{(n)} \right\|^2_{L^2(P_n)}. \quad (3.4)$$

Estimation of $2\displaystyle\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dt dx$ :

We have

$$\partial_t u_n \partial_x^2 u_n = \partial_x(\partial_t u_n \partial_x u_n) - \frac{1}{2}\partial_t(\partial_x u_n)^2.$$

Then

$$2\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t u_n \partial_x^2 u_n dt dx = 2\int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_x(\partial_t u_n \partial_x u_n)\, dt dx - \int_{P_n} \frac{c(t)}{\varphi^2(t)}\partial_t(\partial_x u_n)^2\, dt dx =$$

$$= \int_{\partial P_n} \frac{c(t)}{\varphi^2(t)}\left[-(\partial_x u_n)^2\nu_t + 2\partial_t u_n \partial_x u_n \nu_x\right] d\sigma + \int_{P_n} \left(\frac{c(t)}{\varphi^2(t)}\right)'(\partial_x u_n)^2 dt dx$$

where $\nu_t, \nu_x$ are the components of the outward normal vector at the boundary of $P_n$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of $P_n$ where $t = 0$, we have $u_n = 0$ and consequently $\partial_x u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = n$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$-\int_0^1 \frac{c(n)}{\varphi^2(n)}(\partial_x u_n)^2(n,x)dx$$

is negative. On the part of the boundary where $x=0$, we have $\nu_x=-1,\ \nu_t=0$ and $u_n(t,0)=0$. Consequently, the corresponding integral vanishes. On the part of the boundary where $x=1$, we have $\nu_x=1,\ \nu_t=0$ and

$$\partial_x u_n(t,1)+\beta_2\varphi(t)\,u_n(t,1)=0.$$

Consequently, the corresponding integral is

$$\int_0^n \frac{-2\beta_2 c(t)}{\varphi(t)})\partial_t u_n(t,1)u_n(t,1)dt=\frac{-\beta_2 c(n)}{\varphi(n)}u_n^2(n,1)+\int_0^n \beta_2\Big(\frac{c(t)}{\varphi(t)}\Big)'u_n^2(t,1)dt,$$

which is negative thanks to the condition (1.6) and to the fact that $\beta_2>0$. Finally,

$$\begin{aligned}2\int_{P_n}\frac{c(t)}{\varphi^2(t)}\partial_t u_n\partial_x^2 u_n dtdx=&-\int_0^1\frac{c(n)}{\varphi^2(n)}(\partial_x u_n)^2(n,x)dx-\frac{\beta_2 c(n)}{\varphi(n)}u_n^2(n,1)+\\&+\int_0^n\beta_2\Big(\frac{c(t)}{\varphi(t)}\Big)'u_n^2(t,1)dt+\int_{P_n}\Big(\frac{c(t)}{\varphi^2(t)}\Big)'(\partial_x u_n)^2dtdx.\end{aligned}$$

Note that the functions $\dfrac{c(t)}{\varphi^2(t)}$ and $\Big(\dfrac{c(t)}{\varphi^2(t)}\Big)'$ are bounded. So, by using Lemma 3.1, we deduce

$$\begin{aligned}2\int_{P_n}\frac{c(t)}{\varphi^2(t)}\partial_t u_n\partial_x^2 u_n dtdx &\leqslant \int_{P_n}\Big(\frac{c(t)}{\varphi^2(t)}\Big)'(\partial_x u_n)^2dtdx\leqslant\\&\leqslant K_2\left\|\partial_x u_n\right\|^2_{L^2(P_n)}\leqslant\\&\leqslant K_3\left\|f_1\right\|^2_{L^2(P)},\end{aligned}$$

where $K_2$ and $K_3$ are constants independent of $n$. Consequently, Choosing $\epsilon=\dfrac{1}{2}$ in the relationship (3.4), we obtain

$$\left\|\partial_t u_n\right\|^2_{L^2(P_n)}+\left\|\partial_x^2 u_n\right\|^2_{L^2(P_n)}\leqslant K\left\|f_1\right\|^2_{L^2(P)}.$$

Consequently, making use of Lemma 3.1 and the previous estimate, then, there exists a constant $K>0$, independent of $n$ satisfying

$$\left\|u_n\right\|^2_{\mathcal{H}^{1,2}(P_n)}\leqslant K\left\|f_1\right\|^2_{L^2(P)}.$$

This ends the proof of Proposition 3.1.

**Remark 3.2.** *We obtain the solution $u$ of Problem (3.1) by letting $n$ go to infinity in the previous proposition. The uniqueness can be proved as in Theorem 2.1.*

Finally, we have proved the following Theorem:

**Theorem 3.1.** *Problem (3.1) admits a (unique) solution $u\in\mathcal{H}^{1,2}(D_2)$.*

## 4. The case of a small in time bounded triangular domain

Let $T$ be a small enough positive real number. We set

$$D_3:=\left\{(t,x)\in\mathbb{R}^2:0<t<T;\ \varphi_1(t)<x<\varphi_2(t)\right\}$$

with $\varphi(0)=0$ and consider the following problem:

$$\begin{cases}\partial_t u-c(t)\partial_x^2 u=f_1 \ \text{ a.e. on } D_3,\\ u|_{\Gamma_1}=0,\\ \partial_x u+\beta_2 u|_{\Gamma_2}=0,\end{cases}\tag{4.1}$$

where $f_1 \in L^2(D_3)$. Set

$$Q_n = \left\{(t,x) \in D_3 : \frac{1}{n} < t < T\right\},\ n \in \mathbb{N}^* \ \text{ and } \ \frac{1}{n} < T.$$

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we set $f_1^{(n)} = f_1|_{Q_n} \in L^2(Q_n)$ and denote by $u_n \in \mathcal{H}^{1,2}(Q_n)$ the solution of the following problem:

$$\begin{cases} \partial_t u_n - c(t)\partial_x^2 u_n = f_1 \ \text{ a.e. on } Q_n, \\ u_n|_{t=\frac{1}{n}} = u_n|_{x=\varphi_1(t)} = 0, \\ \partial_x u_n + \beta_2 u_n|_{x=\varphi_2(t)} = 0. \end{cases} \tag{4.2}$$

Such a solution exists by Theorem 2.1.

**Proposition 4.1.** *There exists a constant $K > 0$ independent of $n$ such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2}(Q_n)} \leqslant K \left\|f_1^{(n)}\right\|^2_{L^2(Q_n)} \leqslant K \|f_1\|^2_{L^2(D_3)}.$$

**Remark 4.1.** *Let $\epsilon > 0$ be a real which we will choose small enough. The hypothesis (1.3) implies the existence of a real number $T > 0$ small enough such that*

$$|\varphi_i'(t)\,\varphi(t)| \leqslant \epsilon, \ \textit{for all } t \in (0,T),\ i = 1,2. \tag{4.3}$$

In order to prove Proposition 4.1, we need some preliminary results.

**Lemma 4.1.** *There exists a constant $K$ independent of $n$ such that for all $t \in ]0,T[$:*
*1)* $\|u_n\|_{L^2(Q_n)} \leqslant K \|\varphi \partial_x u_n\|_{L^2(Q_n)}$;
*2)* $\int\limits_{\varphi_1(t)}^{\varphi_2(t)} u_n^2(t,x)dx \leqslant K\varphi^4 \int\limits_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t,x)dx$;
*3)* $\int\limits_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2(t,x)dx \leqslant K\varphi^2 \int\limits_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t,x)dx$;
*4)* $\|\partial_x u_n\|_{L^2(Q_n)} \leqslant K \|f_1\|_{L^2(D_3)}$.

*Proof.* Inequality (1) is a consequence of the Poincaré inequality.
The following operator is an isomorphism (see, [9])

$$H^2_\gamma(0,1) \longrightarrow L^2(0,1),\ u \mapsto u'',$$

where,

$$H^2_\gamma(0,1) = \left\{u \in H^2(0,1) : u(0) = 0, u'(1) + \beta_2 u(1) = 0\right\}.$$

So, there exists a constant $K > 0$ such that

$$\begin{aligned} \|u\|_{L^2(0,1)} &\leqslant \|u''\|_{L^2(0,1)}, \\ \|u'\|_{L^2(0,1)} &\leqslant \|u''\|_{L^2(0,1)}. \end{aligned}$$

The change of variables (for a fixed $t$)

$$[0,1] \to [\varphi_1(t), \varphi_2(t)];\ x \longmapsto y = (1-x)\varphi_1(t) + x\varphi_2(t),$$

leads to the estimates (2) and (3).
To prove (4), it is sufficient to expand the inner product $\left\langle f_1^{(n)}, u_n \right\rangle$ and use the inequality (1). Indeed, we deduce for all $\epsilon > 0$, (see the proof of uniqueness of solutions in Theorem 2.1)

$$\begin{aligned}\int_{Q_n} c(t)(\partial_x u_n)^2 dtdx &\leqslant \left|\left\langle f_1^{(n)}, u_n \right\rangle\right| \leqslant \\ &\leqslant \frac{1}{\epsilon}\left\|f_1^{(n)}\right\|^2_{L^2(Q_n)} + \epsilon \left\|u_n\right\|^2_{L^2(Q_n)} \leqslant \\ &\leqslant \frac{1}{\epsilon}\left\|f_1\right\|^2_{L^2(D_3)} + \epsilon K \left\|\varphi \partial_x u_n\right\|^2_{L^2(Q_n)} .\end{aligned}$$

However, $\varphi$ is bounded and $c > \alpha > 0$. Choosing $\epsilon$ small enough yields the desired result. □

*Proof of Proposition 4.1.* Let us denote the inner product in $L^2(Q_n)$ by $\langle ., . \rangle$ and set $\mathcal{L} := \partial_t - c(t)\partial_x^2$, then we have

$$\left\|f_1^{(n)}\right\|^2_{L^2(Q_n)} = \langle \mathcal{L}u_n, \mathcal{L}u_n\rangle = \|\partial_t u_n\|^2_{L^2(Q_n)} + \left\|c(t)\partial_x^2 u_n\right\|^2_{L^2(Q_n)} - 2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle.$$

**Estimation of** $-2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle$ :
We have

$$\partial_t u_n \partial_x^2 u_n = \partial_x(\partial_t u_n \partial_x u_n) - \tfrac{1}{2}\partial_t (\partial_x u_n)^2 .$$

Then,

$$\begin{aligned}-2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle &= -2\int_{Q_n} c(t)\partial_t u_n \partial_x^2 u_n dtdx = \\ &= -2\int_{Q_n} c(t)\partial_x (\partial_t u_n \partial_x u_n) dtdx + \int_{Q_n} c(t)\partial_t (\partial_x u_n)^2 dtdx = \\ &= \int_{\partial Q_n} c(t)\left[(\partial_x u_n)^2 \nu_t - 2\partial_t u_n \partial_x u_n \nu_x\right] d\sigma - \int_{Q_n} c'(t)(\partial_x u_n)^2 dtdx\end{aligned}$$

where $\nu_t, \nu_x$ are the components of the unit outward normal vector at the boundary of $Q_n$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of $Q_n$ where $t = \dfrac{1}{n}$, we have $u_n = 0$ and consequently $\partial_x u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} c(T) (\partial_x u_n)^2 dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2 (t)}},\ \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2 (t)}},\ u_n (t, \varphi_1 (t)) = \partial_x u_n (t, \varphi_2 (t)) + \beta_2 u_n (t, \varphi_2 (t)) = 0.$$

Consequently, the corresponding integral is

$$\begin{aligned}&-\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t) [\partial_x u_n (t, \varphi_1 (t))]^2 dt - 2\int_{\frac{1}{n}}^{T} c(t)\partial_t u_n (t, \varphi_2 (t)) .\partial_x u_n (t, \varphi_2 (t)) dt - \\ &-\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) [\partial_x u_n (t, \varphi_2 (t))]^2 dt.\end{aligned}$$

By putting $h(t) := u_n(t, \varphi_2(t)),\ t \in [\frac{1}{n}, T]$, we obtain

$$\partial_t u_n(t, \varphi_2(t))\partial_x u_n(t, \varphi_2(t)) = h'(t)\partial_x u_n(t, \varphi_2(t)) - \varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2.$$

So, by using the boundary conditions, we get

$$\begin{aligned}
&-2\int_{\frac{1}{n}}^{T} c(t)\partial_t u_n(t, \varphi_2(t))\partial_x u_n(t, \varphi_2(t))\, dt = \\
&= -2\int_{\frac{1}{n}}^{T} c(t)h'(t)\partial_x u_n(t, \varphi_2(t))\, dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2\, dt = \\
&= 2\beta_2\int_{\frac{1}{n}}^{T} c(t)h'(t)h(t)\, dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2\, dt = \\
&= \beta_2\int_{\frac{1}{n}}^{T} c(t)(h(t)^2)'\, dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2\, dt = \\
&= \beta_2 c(T)(h(T))^2 - \beta_2\int_{\frac{1}{n}}^{T} c'(t)u_n^2(t, \varphi_2(t)\, dt + 2\int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left(\partial_x u_n(t, \varphi_2(t))\right)^2\, dt.
\end{aligned}$$

Observe that, thanks to the condition (1.6) and the fact that $\beta_2 > 0,\ c(t) > 0$, we have

$$\beta_2 c(T)(h(T))^2 - \beta_2\int_{\frac{1}{n}}^{T} c'(t)u_n^2(t, \varphi_2(t)\, dt \quad \geqslant \quad 0.$$

So, by setting

$$\begin{aligned}
I_{n,1} &= -\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t) \left[\partial_x u_n(t, \varphi_1(t))\right]^2 dt, \\
I_{n,2} &= \int_{\frac{1}{n}}^{T} c(t)\varphi_2'(t) \left[\partial_x u_n(t, \varphi_2(t))\right]^2 dt,
\end{aligned}$$

we have

$$-2\langle \partial_t u_n, c(t)\partial_x^2 u_n\rangle \geqslant -\left|I_{n,1}\right| - \left|I_{n,2}\right|. \tag{4.4}$$

**Estimation of** $I_{n,k},\ k = 1,\ 2.$

**Lemma 4.2.** *There exists a constant* $K > 0$ *independent of* $n$ *such that*

$$\max(\left|I_{n,1}\right|, \left|I_{n,2}\right|) \quad \leqslant \quad K\epsilon \left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2.$$

*Proof.* We convert the boundary integral $I_{n,1}$ into a surface integral by setting

$$\left[\partial_x u_n(t, \varphi_1(t))\right]^2 = \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \left[\partial_x u_n(t, x)\right]^2 \Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)} = \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\partial}{\partial x}\left\{\frac{\varphi_2(t) - x}{\varphi(t)} \left[\partial_x u_n(t, x)\right]^2\right\} dx =$$

$$= 2\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2(t) - x}{\varphi(t)} \partial_x v_n(t, x)\, \partial_x^2 u_n(t, x)\, dx - \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi(t)} \left[\partial_x u_n(t, x)\right]^2 dx.$$

Then, we have

$$\begin{aligned} I_{n,1} &= -\int_{\frac{1}{n}}^{T} c(t)\varphi_1'(t)\left[\partial_x u_n(t,\varphi_1(t))\right]^2 dt = \\ &= -\int_{Q_n} \frac{c(t)\varphi_1'(t)}{\varphi(t)}\left(\partial_x u_n\right)^2 dtdx + 2\int_{Q_n} \frac{\varphi_2(t)-x}{\varphi(t)} c(t)\varphi_1'(t)\left(\partial_x u_n\right)\left(\partial_x^2 u_n\right) dtdx. \end{aligned}$$

Thanks to Lemma 4.1, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x u_n(t,x)\right]^2 dx \leqslant C\left[\varphi(t)\right]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 u_n(t,x)\right]^2 dx.$$

Therefore,

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x u_n(t,x)\right]^2 \frac{|\varphi_1'|}{\varphi} dx \leqslant C\left|\varphi_1'\right|\left[\varphi\right] \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_x^2 u_n(t,x)\right]^2 dx,$$

consequently,

$$|I_{n,1}| \leqslant C\int_{Q_n} c(t)\left|\varphi_1'\right|\left[\varphi\right]\left(\partial_x^2 u_n\right)^2 dtdx + 2\int_{Q_n} c(t)\left|\varphi_1'\right|\left|\partial_x u_n\right|\left|\partial_x^2 u_n\right| dtdx,$$

since $\left|\frac{\varphi_2(t)-x}{\varphi(t)}\right| \leqslant 1$. So, for all $\epsilon > 0$, we have

$$|I_{n,1}| \leqslant C\int_{Q_n} \left|c(t)\varphi_1'\right|\left[\varphi\right]\left(\partial_x^2 u_n\right)^2 dtdx + \epsilon\int_{Q_n} c(t)\left(\partial_x^2 u_n\right)^2 dtdx + \frac{1}{\epsilon}\int_{Q_n} c(t)\left(\varphi_1'\right)^2\left(\partial_x u_n\right)^2 dtdx.$$

Lemma 4.1 yields

$$\frac{1}{\epsilon}\int_{Q_n} c(t)\left(\varphi_1'\right)^2\left(\partial_x u_n\right)^2 dtdx \leqslant C\frac{1}{\epsilon}\int_{Q_n} c(t)\left(\varphi_1'\right)^2\left[\varphi\right]^2\left(\partial_x^2 u_n\right)^2 dtdx.$$

Thus, there exists a constant $M > 0$ independent of $n$ such that

$$\begin{aligned} |I_{n,1}| &\leqslant C\int_{Q_n} c(t)\left[\left|\varphi_1'\right|\left|\varphi\right| + \frac{1}{\epsilon}\left(\varphi_1'\right)^2\left|\varphi\right|^2\right]\left(\partial_x^2 u_n\right)^2 dtdx + \epsilon\int_{Q_n} c(t)\left(\partial_x^2 u_n\right)^2 dtdx \leqslant \\ &\leqslant M\epsilon\int_{Q_n}\left(\partial_x^2 u_n\right)^2 dtdx, \end{aligned}$$

because $\left|\varphi_1'\varphi\right| \leqslant \epsilon$. The inequality

$$|I_{n,2}| \leqslant K\epsilon\left\|\partial_x^2 u_n\right\|^2_{L^2(Q_n)},$$

can be proved by a similar argument. □

Now, we can complete the proof of Proposition 4.1. Summing up the estimates (4.4) and those of Lemma 4.2, we then obtain

$$\begin{aligned} \left\|f_1^{(n)}\right\|^2_{L^2(Q_n)} &\geqslant \left\|\partial_t u_n\right\|^2_{L^2(Q_n)} + \left\|c(t)\partial_x^2 u_n\right\|^2_{L^2(Q_n)} - K_1\epsilon\left\|\partial_x^2 u_n\right\|^2_{L^2(Q_n)} \geqslant \\ &\geqslant \left\|\partial_t u_n\right\|^2_{L^2(Q_n)} + \left(\alpha^2 - K_1\epsilon\right)\left\|\partial_x^2 u_n\right\|^2_{L^2(Q_n)}, \end{aligned}$$

where $K_1$ is a positive number. Then, it is sufficient to choose $\epsilon$ such that

$$\alpha^2 - K_1\epsilon > 0,$$

to get a constant $K_0 > 0$ independent of $n$ such that

$$\left\| f_1^{(n)} \right\|_{L^2(Q_n)}^2 \geqslant K_0\big( \|\partial_t u_n\|_{L^2(Q_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2 \big).$$

But

$$\left\| f_1^{(n)} \right\|_{L^2(Q_n)} \leqslant \|f_1\|_{L^2(D_3)},$$

then, there exists a constant $K > 0$, independent of $n$ satisfying

$$\|\partial_t u_n\|_{L^2(Q_n)}^2 + \left\|\partial_x^2 u_n\right\|_{L^2(Q_n)}^2 \leqslant K \|f_1\|_{L^2(D_3)}^2 .$$

Consequently, making use of Lemma 4.1 and the previous estimates, then, there exists a constant $K > 0$, independent of $n$ satisfying.

$$\|u_n\|_{\mathcal{H}^{1,2}(Q_n)}^2 \leqslant C \|f_1\|_{L^2(D_3)}^2 .$$

This ends the proof of Proposition 4.1. Finally, we have proved the following Theorem:

**Theorem 4.1.** *Problem (4.1) admits a (unique) solution $u \in \mathcal{H}^{1,2}(D_3)$.*

*Proof.* We obtain the solution $u$ of Problem (4.1) by letting $n$ go to infinity in the previous proposition. The uniqueness can be proved as in Theorem 2.1. □

## 5. Back to Problems (1.1)–(1.2) and proof of Theorem 1.1

The proof of Theorem 1.1 can be obtained by subdividing the domain

$$D := \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\}$$

into three open sub-domains $\Omega_1,\ \Omega_2$ and $\Omega_3$. So, we set $D = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Gamma_{T_1} \cup \Gamma_{T_2}$ where

$$\Omega_1 = \{(t,x) \in D : 0 < t < T_1\},\quad \Omega_2 = \{(t,x) \in D : T_1 < t < T_2\},\quad \Omega_3 = \{(t,x) \in D : t > T_2\},$$

$$\Gamma_{T_1} = \left\{(T_1,x) \in \mathbb{R}^2 : \varphi_1(T_1) < x < \varphi_2(T_1)\right\} \text{ and } \Gamma_{T_2} = \left\{(T_2,x) \in \mathbb{R}^2 : \varphi_1(T_2) < x < \varphi_2(T_2)\right\}$$

with $T_1$ is a small enough positive number and $T_2$ is an arbitrary positive number such that $T_2 > T_1$. In the sequel, $f_1$ stands for an arbitrary fixed elements of $L^2(D)$ and $f_1^{(i)} = f_1|_{\Omega_i}$, $i = 1,2,3$.

Theorem 4.1 applied to the triangular domain $\Omega_1$, shows that there exists a unique solution $w_1 \in \mathcal{H}^{1,2}(\Omega_1)$ of the problem

$$\begin{cases} \partial_t w_1 - c(t)\partial_x^2 w_1 = f_1^{(1)} \ \text{ a.e. on } \Omega_1, \\ w_1|_{\Gamma_{1,1}} = 0, \\ \partial_x w_1 + \beta_2 w_1|_{\Gamma_{2,1}} = 0, \end{cases} \tag{5.1}$$

where $f_1^{(1)} \in L^2(\Omega_1)$ and $\Gamma_{i,1}$ are the parts of the boundary of $\Omega_1$ where $x = \varphi_i(t)$, $i = 1,2$.

**Lemma 5.1.** *If $w \in \mathcal{H}^{1,2}(]0,T[\times]0,1[)$, then $w|_{t=0} \in H^1(\gamma_0)$, $w|_{x=0} \in H^{\frac{3}{4}}(\gamma_1)$ and $w|_{x=1} \in H^{\frac{3}{4}}(\gamma_2)$, where $\gamma_0 = \{0\} \times ]0,1[$, $\gamma_1 = ]0,T[\times\{0\}$ and $\gamma_2 = ]0,T[\times\{1\}$.*

It is a particular case of Theorem 2.1 ( [14, Vol. 2]). The transformation

$$(t,x) \longmapsto (t',x') = (t, \varphi(t)\, x + \varphi_1(t))$$

leads to the following lemma:

**Lemma 5.2.** *If $w \in \mathcal{H}^{1,2}(\Omega_2)$, then $w|_{\Gamma_{T_1}} \in H^1(\Gamma_{T_1})$, $w|_{x=\varphi_1(t)} \in H^{\frac{3}{4}}(\Gamma_{1,2})$ and $w|_{x=\varphi_2(t)} \in H^{\frac{3}{4}}(\Gamma_{2,2})$, where $\Gamma_{i,2}$ are the parts of the boundary of $\Omega_2$ where $x = \varphi_i(t)$, $i = 1,2$.*

Hereafter, we denote the trace $w_1|_{\Gamma_{T_1}}$ by $\psi_1$ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $w_1 \in \mathcal{H}^{1,2}(\Omega_1)$ (see Lemma 5.2). Now, consider the following problem in $\Omega_2$ :

$$\begin{cases} \partial_t w_3 - c(t)\partial_x^2 w_3 = f_1^{(2)} \text{ a.e. on } \Omega_2, \\ w_3|_{\Gamma_{T_1}} = \psi_1, \\ w_3|_{\Gamma_{1,2}} = 0, \\ \partial_x w_3 + \beta_2 w_3|_{\Gamma_{2,2}} = 0, \end{cases} \tag{5.2}$$

where $f_1^{(2)} \in L^2(\Omega_2)$ and $\Gamma_{i,2}$ are the parts of the boundary of $\Omega_2$ where $x = \varphi_i(t)$, $i = 1,2$. We use the following result, which is a consequence of Theorem 4.3 ( [14, Vol.2]), to solve Problem (5.2).

**Proposition 5.1.** *Let $Q$ be the rectangle $]0,T[\times]0,1[$, $f_1, f_2 \in L^2(Q)$ and $\psi_1, \psi_2 \in H^1(\gamma_0)$. Then, the following problem admits a (unique) solution $u \in \mathcal{H}^{1,2}(Q)$:*

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f_1 \in L^2(Q), \\ u|_{\gamma_0} = \psi_1, \\ u|_{\gamma_1} = 0, \\ \partial_x u + \beta_2 u|_{\gamma_2} = 0, \end{cases}$$

*where $\gamma_0 = \{0\} \times ]0,1[$, $\gamma_1 = ]0,T[ \times \{0\}$ and $\gamma_2 = ]0,T[ \times \{1\}$.*

Thanks to the transformation

$$(t,x) \longmapsto (t,y) = (t, \varphi(t)\, x + \varphi_1(t)),$$

we deduce the following result:

**Proposition 5.2.** *Problem (5.2) admits a (unique) solution $w_3 \in \mathcal{H}^{1,2}(\Omega_2)$.*

Hereafter, we denote the trace $w_3|_{\Gamma_{T_2}}$ by $\Phi_1$ which is in the Sobolev space $H^1(\Gamma_{T_2})$ because $w_3 \in \mathcal{H}^{1,2}(\Omega_2)$ (see Lemma 5.2). Now, consider the following problem in $\Omega_3$ :

$$\begin{cases} \partial_t w_5 - c(t)\partial_x^2 w_5 = f_1^{(3)} \text{ a.e. on } \Omega_3, \\ w_5|_{\Gamma_{T_2}} = \Phi_1, \\ w_5|_{\Gamma_{1,3}} = 0, \\ \partial_x w_5 + \beta_2 w_5|_{\Gamma_{2,3}} = 0, \end{cases} \tag{5.3}$$

where $f_1^{(3)} \in L^2(\Omega_3)$ and $\Gamma_{i,3}$ are the parts of the boundary of $\Omega_3$ where $x = \varphi_i(t)$, $i = 1,2$. By similar arguments like those used previously, we deduce the following result:

**Proposition 5.3.** *Problem (5.3) admits a (unique) solution $w_5 \in \mathcal{H}^{1,2}(\Omega_3)$.*

Finally, the function $u$ defined by

$$u := \begin{cases} w_1 \text{ in } \Omega_1, \\ w_3 \text{ in } \Omega_2, \\ w_5 \text{ in } \Omega_3, \end{cases}$$

is the (unique) solution of Problem (1.1)–(1.2). This ends the proof of Theorem 1.1.

**Remark 5.1.** *Let us consider the following problem: to find a function $v \in \mathcal{H}^{1,2}(D)$ that satisfies the equation*

$$\partial_t v - c(t)\partial_x^2 v = f_2 \ \ a.e.\ \ on\ D \tag{5.4}$$

*and the boundary conditions*

$$v|_{\Gamma_2} = \partial_x v + \beta_1 v|_{\Gamma_1} = 0, \tag{5.5}$$

*where $f_2 \in L^2(D)$ and the coefficient $c$ and the domain $D$ have the same properties as in Problem (1.1),(1.2).*

*By using the same arguments like those used in solving Problem (1.1), (1.2), we can show that Problem (5.4)–(5.5) admits a (unique) solution $v$ belonging to $\mathcal{H}^{1,2}(D)$, under the assumption*

$$\beta_1 < 0 \ \ and\ \ 2c(t)\beta_1 - \varphi_1'(t) \leqslant 0 \ \ \ a.e.\ \ t \in \, ]0, +\infty[\, .$$

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# Глобальные во времени результаты для решения параболического уравнения в непрямоугольных областях

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**Аннотация.** В этой статье рассматривается параболическое уравнение

$$\partial_t w - c(t)\partial_x^2 w = f \text{ in } D,\ D = \left\{(t,x) \in \mathbb{R}^2 : t > 0,\ \varphi_1(t) < x < \varphi_2(t)\right\},$$

где $\varphi_i : [0,+\infty[\to \mathbb{R}, i = 1,\ 2$ и $c : [0,+\infty[\to \mathbb{R}$, удовлетворяя некоторым условиям, задача дополняется граничными условиями типа Дирихле-Робина. Мы изучаем проблему глобальной регулярности в подходящем параболическом пространстве Соболева. В частности, докажем, что для $f \in L^2(D)$ существует единственное решение $w$ такое, что $w$, $\partial_t w$, $\partial^j w \in L^2(D)$, $j = 1, 2$. Обратите внимание, что случай ограниченных непрямоугольных областей изучается в [9]. Доказательство основано на оценках энергии после преобразования задачи в полосовой области в сочетании с некоторым интерполяционным неравенством. Эта работа дополняет результаты, полученные в [19] в случае граничных условий Коши-Дирихле.

**Ключевые слова:** параболические уравнения, уравнение теплопроводности, непрямоугольные области, неограниченные области, анизотропные пространства Соболева.

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# On Limit Theorem for the Number of Vertices of the Convex Hulls in a Unit Disk

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**Abstract.** This paper is devoted to further investigation of the property of a number of vertices of convex hulls generated by independent observations of a two-dimensional random vector with regular distributions near the boundary of support when it is a unit disk. Following P. Groeneboom [4], the Binomial point process is approximated by the Poisson point process near the boundary of support and vertex processes of convex hulls are constructed. The properties of strong mixing and martingality of vertex processes are investigated. Using these properties, asymptotic expressions are obtained for the expectations and variance of the vertex processes that correspond to the results previously obtained by H. Carnal [2]. Further, using the properties of strong mixing of vertex processes, the central limit theorem for a number of vertices of a convex hull is proved.

**Keywords:** convex hull, Poisson point process, Markovian jump process, martingales, Central limit theorem.

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## Introduction

The functionals of convex hulls are complex objects in analytical aspect. Therefore, studying the properties of even the simplest functionals of convex hulls such as the number of vertices or the area, has for a long time remained a difficult task. This explains the fact that such well-known researchers as in [2, 3, 15] and others, limited their interests to studying the average value of the number of vertices, the area, and the perimeter of a random polygon. For many years, due to the lack of valid research methods, the attempts to develop this area have not been successful.

In paper [4] has made a significant progress in this field. He managed for the first time to obtain the limit distribution for the number of vertices of a convex hull in the case when the support of initial uniform distribution is either a convex polygon or an ellipse. His research method is based on the original idea of using the Poisson approximations of a binomial point process near the boundary of the support of initial distribution. Then he applied powerful methods such as martingales, mixing of stationary processes and others. Based on this method, in [1] have established the limiting distribution for the area of the convex hull when the support of initial distribution is a convex polygon. In [6] proved the limit theorems for the area outside a convex hull when the support is a unit disk. These results in a more general form, for the

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joint distribution of the vertex number, area, and perimeter, were obtained by [12] using the idea of [4] on Poisson approximations of a binomial point process near a polygon boundary. In [7] has developed this problem for the case when the convex hull is generated by distributions with exponential tails, including, in particular, the normal distribution.

The approach used in this paper is a modification of the methods proposed by [4, 5, 12] and adapted to a wider class of initial distributions.

## 1. Statement of the problem and results formulation

Let the support of initial distribution $A$ be a unit disk with a center at a point$(0,1)$.

Suppose that random points $(r_i, \alpha_i)$ are given in the polar coordinate system (with pole $(0,1)$) in a disk $A$, where $r_i$ and $\alpha_i$ are independent and $\alpha_i$is uniformly distributed in $[-\pi,\pi]$ and

$$P\left(r_i > 1-x\right) = x^{\beta} L\left(\frac{1}{x}\right), \; 0<x<1, \; \beta \geqslant 1, \tag{1}$$

where $L(x)$ is the slowly varying function in the Karamata sense given by

$$L(u) = \exp\left\{\int_1^u \frac{\varepsilon(t)}{t} dt\right\}, \quad \varepsilon(t) \to 0, \quad t \to \infty.$$

Next, assume that $X_i = r_i \sin\alpha_i$, $1-Y_i = r_i\cos\alpha_i$ and denote by $C_n$ the convex hull generated by random points $(X_1,Y_1), (X_2,Y_2), \ldots, (X_n,Y_n)$, and denote by $\nu_n, s_n$ and $l_n$ the number of vertices, the area and the perimeter of the $C_n$, respectively.

Denoting the largest root of the equation by $b_n$

$$nx^{-(\beta+\frac{1}{2})} L(x) = 1. \tag{2}$$

In this case, in [2] obtained asymptotic expressions for the expectations of $E\nu_n$, $Es_n$ and $El_n$. In the one-dimensional case, in [13] studied the role of the extreme summands in the sums, when the tail of the distribution of the initial random variable is (1) regularly varying. This paper is a continuation of [9, 10, 13] in the multidimensional case. According to P. Groeneboom's remark, we consider $\nu_n$ for the case when $L(x)=1$. Then from (2) we get

$$b_n = n^{\frac{2}{2\beta+1}}. \tag{3}$$

The basic theorem of the present paper is given.

**Theorem 1.** *Let the conditions At $n \to \infty$ the following ratio is true*

$$\frac{\nu_n - a_1(\beta,n) b_n^{\frac{1}{2}}}{a_2(\beta,n) b_n^{\frac{1}{4}}} \overset{d}{\Rightarrow} \mathrm{N}(0,1).$$

*Here $\overset{d}{\Rightarrow}$ means the weak convergence,* $\mathrm{N}(0,1)$ *denotes the standard normal distribution with parameters* $(0,1)$*,* $a_1(\beta,n)$*,* $a_2(\beta,n)$ *are positive constants determined from relations (12) and (13).*

In particular, if $\beta=1$ Groeneboom's result [4] follows.

**Corollary.** If condition (3) is satisfied, then

$$\frac{\nu_n - c_1(\beta) b_n^{\frac{1}{2}}}{c_2(\beta) b_n^{\frac{1}{4}}} \overset{d}{\Rightarrow} \mathrm{N}(0,1),$$

where $c_1(\beta), c_2(\beta)$ have an explicit form, and $c_1(1), c_2(1)$ coincide with the corresponding constants in the [4].

## 2. Preliminaries

In this section we give a modification of the key approach by [4, 12] on the Poisson approximation of binomial point process (b.p.p.) $\mathrm{B}_n(\cdot)$ generated by $n$ random sample from the distribution (1) in the unit disk.

Assume that

$$S_\varepsilon = \left\{(x,y):\ 1-\varepsilon \leqslant \sqrt{x^2+(1-y)^2} \leqslant 1\right\},$$
$$\Lambda_\beta(A) = P\left((X_1,Y_1)\in A\right). \tag{4}$$

Consider a convex hull $C_n'$ generated by a nonhomogeneous Poisson point process (n.h.p.p.p.) $\Pi_n(S_\varepsilon)$ with intensity of $n\Lambda_\beta(\cdot)$.

**Lemma 1.** *Let $\mathrm{B}_n(S_\varepsilon)$ be the n.h.b.p.p. with parameter$(n,\Lambda_\beta(\cdot))$. Then there is the n.h.p.p.p. $\Pi_n(S_\varepsilon)$ with intensity$n\Lambda_\beta(\cdot)$ such that*

$$P\left(\mathrm{B}_n(S_\varepsilon)\neq \Pi_n(S_\varepsilon)\right)\leqslant 2\Lambda_\beta(S_\varepsilon),\quad P\left(C_n\neq C_n'\right)\to 0,$$

*at $n\to\infty$, $\varepsilon\to 0$.*

To formulate Lemma 2, we need some notation.

Let $R(\delta) = \left\{(x,y):\ y<1,\ 1-\dfrac{\delta^2}{2} \leqslant \sqrt{x^2+(1-y)^2}\leqslant 1,\ \left|\dfrac{x}{1-y}\right| < tg\delta\right\},$

$$R^*(\delta) = \left\{(x,y):\ |x|\leqslant\delta,\ \frac{x^2}{2}\leqslant y\leqslant \frac{x^2}{2}+\frac{\delta^2}{2}\right\},$$

where $\delta=\delta_n = c\sqrt{\log n/b_n}$.

For any set of forms

$$A = \left\{(x,y):\ \ a\leqslant x\leqslant b,\ \ \frac{x^2}{2}\leqslant y_1(x)\leqslant y\leqslant y_2(x)\right\}$$

introduce a measure

$$\Lambda^*(A) = \frac{n\beta}{2\pi}\int_a^b dx\int_{y_1(x)}^{y_2(x)}\left(y-\frac{x^2}{2}\right)^{\beta-1}dy,$$

$$\Lambda^*(B)=0,\quad \ if\ \ B\subset\left\{(x,y):\ \ y<\frac{x^2}{2}\right\}.$$

Then assume that

$$\Lambda(B)=0,\ \ at\ \ B\subset\left\{(x,y):\ x^2+(1-y)^2>1\right\}.$$

**Lemma 2.** *There is $\Pi_n^*(S_\varepsilon)$ the n.h.p.p.p. with intensity$\Lambda_n^*(\cdot)$ such that for any $\varepsilon>0$*

$$P\left(\Pi_n^*(S_\varepsilon)\neq\Pi(S_\varepsilon)\right) = o\left(\frac{\log^{\beta+2}n}{b_n^{1-\varepsilon}}\right).$$

We denote by $C_n^*$ the convex hull generated by the realization of n.h.p.p.p. $\Pi_n^*(D)$, where

$$D=\left\{(x,y):\ \ \frac{x^2}{2}\leqslant y\leqslant 1\right\},\quad S^*(\varepsilon)=\left\{(x,y):\ \ \frac{x^2}{2}\leqslant y\leqslant\frac{x^2}{2}+\varepsilon,\ \ y\leqslant 1\right\}.$$

Then assume that

$$R_n = \left\{ (x, y) : \ \frac{x^2}{2b_n} \leqslant y \right\}.$$

Introduce the following measure

$$\Lambda_n^0(A) = \begin{cases} -\dfrac{1}{2\pi\sqrt{b_n}} \displaystyle\iint\limits_A \left( y - \frac{x^2}{2} \right)^{\beta-1} dxdy, \ A \subset R_n, \\ 0, \qquad\qquad\qquad\qquad at \ \ A \not\subset R_n. \end{cases} \tag{5}$$

Then denoting by $\Pi_n(\cdot)$ n.h.p.p.p. with intensity$\Lambda_n^0(\cdot)$ it is easy to see that

$$\Pi_n(\cdot) \stackrel{d}{=} \Pi_n^*(b_n^{-1}\cdot). \tag{6}$$

Now the whole circle is divided by $m_n$parts, where $m_n = \sqrt{b_n}/\log n$. Each section is $2\pi\sqrt{b_n}\log n$ long, with central angle $2\pi \log n/\sqrt{b_n}$. Disk section corresponding to the circumference section

$$\left(\pi\sqrt{b_n}(\log n)(2k-1), \pi\sqrt{b_n}(\log n)(2k+1)\right)$$

is denoted by $I_{k,n}$,$(k = 0, 1, \ldots, m_n - 1)$.

From Lemma 2, Poisson's processes $\Pi_n^*(b_n^{-1}\cdot)$ and $\Pi_n^0(\cdot)$ are almost similar. So consider Poisson's process $\Pi_n^0(\cdot)$ in $I_{k,n}$only.

Following [4], consider the statement of Poisson point process in each sector $I_{k,n}$ separately.

The vertex process $W_n(a) = (X_n(a), Y_n(a))$ for any $a \in (a_-, a_+)$ is such a point $(X_k, Y_k)$ of n.h.p.p.p. realization $\Pi_n^0(\cdot)$, for which $Y_k - aX_k$ takes the minimum value, where $a_- = -\pi \log n/\sqrt{b_n}$, $a_+ = \pi \log n/\sqrt{b_n}$.

It is easy to understand from the definition that, $W_n(a)$ is a non-stationary Markov jump process.

The following lemma gives the types of distributions $W_n(a)$ which correspond to various situations.

**Lemma 3.** *Let* $s = y - ax + a^2 b_n/2$.
*Then*

1)$$P\left(W_n(0) \in (dx, dy)\right) = \frac{\beta}{2\pi\sqrt{b_n}} \exp\left\{ -\frac{y^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1; \frac{1}{2}\right) \right\} \left( y - \frac{x^2}{2b_n} \right)^{\beta-1} dxdy;$$

2)$$P\left(W_n(0) \in (dx, dy)\right) = \frac{\beta}{2\pi\sqrt{b_n}} \exp\left\{ -\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1; \frac{1}{2}\right) \right\} \left( y - \frac{x^2}{2b_n} \right)^{\beta-1} dxdy;$$

3)$$P\left(W_n(a) = W_n(0)/W_n(0) = (x, y)\right) = \\ = \exp\left\{ -\frac{1}{2\pi\sqrt{b_n}} \int_{x-ab_n}^{\sqrt{2b_n s}} \left( s - \frac{u^2}{2b_n} \right)^{\beta} du - \int_x^{\sqrt{2b_n y}} \left( y - \frac{u^2}{2b_n} \right)^{\beta} dy \right\}.$$

*Proof.* Let $v = a(u - x) + y$ be a straight line passing through points $(x, y)$ with angular coefficient $a$, $A(a, x, y)$ is the set of points in the domain bounded by lines $v = a(u - x) + y$ and $v = u^2/(2b_n)$.

It is easy to see that if $u_1$ and $u_2$ are the roots of equation

$$u^2/(2b_n) = a(u - x) + y,$$

then $u_{1,2} = ab_n + \sqrt{2b_n s}$.

Calculate $\Lambda_n^0\left(A(a,x,y)\right)$ (see (5)). Considering

$$y+a(u-x)-\frac{u^2}{2b_n}=y-ax+\frac{a^2b_n}{2}-\frac{(u-ab_n)^2}{2b_n}=s-\frac{(u-ab_n)^2}{2b_n}$$

we get

$$\Lambda_n^0\left(A(a,x,y)\right)=\frac{1}{2\pi\sqrt{b_n}}\int_{u_1}^{u_2}\left[y+a(u-x)-\frac{u^2}{2b_n}\right]^{\beta}du=\frac{1}{2\pi\sqrt{b_n}}\int\limits_{-\sqrt{2b_ns}}^{\sqrt{2b_ns}}\left(s-\frac{u^2}{2b_n}\right)^{\beta}du=$$

$$=\frac{\sqrt{2}s^{\beta+\frac{1}{2}}}{\pi}\int_0^1\left(1-u^2\right)^{\beta}du=\frac{s^{\beta+\frac{1}{2}}}{\pi\sqrt{2}}B\left(\beta+1;\frac{1}{2}\right).$$

Next, let $d=\sqrt{(\Delta x)^2+(\Delta y)^2}$, $v=au+c_-$, $v=au+c_+$ be two straight lines parallel to $v=a(u-x)+y$ and passing at distance$d$ from below and above, respectively.

By $A_d^-(a,x,y)$ and $A_d^+(a,x,y)$denote the sets bounded by lines $v=au+c_-$, $v=u^2/(2b_n)$ and $v=au+c_+$, $v=u^2/(2b_n)$ respectively. Assume that $\Delta_{x,y}=[x,x+\Delta x]\times[y,y+\Delta y]$.

It follows from the definition $W_n(a)$ that if $\overline{\pi}(A)$ is the number of points in $A$ realization of n.h.p.p.p. $\Pi_n(\cdot)$, then

$$P\left(W_n(a)\in\Delta_{x,y}\right)\leqslant P\left(\overline{\pi}(\Delta_{x,y})\geqslant 1,\ \overline{\pi}\left(A_d^-(a,x,y)\right)=0\right). \tag{7}$$

On the other hand, it is easy to see that

$$P\left(W_n(a)\in\Delta_{x,y}\right)\geqslant P\left(\overline{\pi}(\Delta_{x,y})=1,\ \overline{\pi}\left(A_d^+(a,x,y)-\Delta_{x,y}\right)=0\right). \tag{8}$$

Considering the property of the Poisson process (the independence of increments) and from inequalities (7) and (8), using (5) at $d\to 0$ we obtain the first relation of the lemma.

Similarly, the other relations of the lemma are obtained.

Assume that

$$R_n(a)=X_n(a)-ab_n,\quad S_n(a)=Y_n(a)-\frac{X_n^2(a)}{2b_n}+\frac{R_n^2(a)}{2b_n},\quad T_n(a)=\left(R_n(a),S_n(a)\right).$$

Obviously that $T_n(0)=W_n(0)$ a.s. and therefore

$$P\left(T_n(0)\in(dr,ds)\right)=P\left(W_n(0)\in(dr,ds)\right).$$

Lemma 3 leads to the following lemma.

**Lemma 4.** *$T_n(a)$ is a stationary Markov jump process and*

1) $$P\left(T_n(0)\in(dr,ds)\right)=\frac{\beta}{2\pi\sqrt{b_n}}\exp\left\{-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2}\pi}B\left(\beta+1;\frac{1}{2}\right)\right\}\left(s-\frac{r^2}{2b_n}\right)^{\beta-1}drds;$$

2) $$P\left(T_n(a)=(r_1,s_1)/T_n(0)=(r_0,s_0)\right)=$$

$$=\exp\left\{-\frac{1}{\sqrt{2}\pi L\left(b_n\right)}\left[s_1^{\beta+\frac{1}{2}}\int_{\frac{r_1}{\sqrt{2b_ns_1}}}^{1}\left(1-t^2\right)^{\beta}dt-s_0^{\beta+\frac{1}{2}}\int_{\frac{r_0}{\sqrt{2b_ns_0}}}^{1}\left(1-t^2\right)^{\beta}dt\right]\right\},$$

*where* $r_1=r_0-ab_n$, $s_1=s_0-ar_0+a^2b_n/2$*;*

3) $$P\left(T_n(a)\in(dr_1,ds_1)/T_n(0)=(r_0,s_0)\right)=P\left(T_n(a)\in(dr_1,ds_1)\right),$$

*if* $ab_n - \sqrt{2b_n s_1} > \sqrt{2b_n s_0}$;

4) $P\left(T_n(a) \in (dr_2, ds_2)/T_n(0) = (r_1, s_1)\right) =$

$$= \frac{1}{2\pi\sqrt{b_n}} \exp\left\{ - \frac{1}{\sqrt{2\pi}} \left[ s_2^{\beta+\frac{1}{2}} \int\limits_{\frac{s_1-s_2}{a\sqrt{2b_n s_2}}+\frac{ab_n}{\sqrt{2b_n s_2}}}^{1} \left(1-t^2\right)^{\beta} dt - \right.\right.$$

$$\left.\left. - s_1^{\beta+\frac{1}{2}} \int\limits_{\frac{s_1-s_2}{a\sqrt{2b_n s_1}}+\frac{ab_n}{\sqrt{2b_n s_1}}}^{1} \left(1-t^2\right)^{\beta} dt \right]\right\} \left(s_2 - \frac{r_2^2}{2b_n}\right)^{\beta-1} dr_2 ds_2.$$

Here assume that

$$(r_i, s_i) \in D = \left\{(r,s): \ s \geqslant \left(r^2\right)/\left(2b_n\right)\right\}, \quad ab_n - \sqrt{2b_n s_2} \leqslant \sqrt{2b_n s_1},$$

$$s_2 + \left(a^2 b_n\right)/2 + ar_2 \geqslant s_1 \geqslant s_2 - \left(a^2 b_n\right)/2 + ar_1.$$

Consider the following $\sigma$-algebras generated by process $T_n(a)$:

$$\Im_n^0 = \sigma\left\{T_n(c) : c \leqslant 0\right\}, \quad \Im_n^{a+} = \sigma\left\{T_n(c) : c \geqslant a\right\}.$$

From Lemmas 3 and 4 it is easy to prove the properties of strong mixing of the process $T_n(a)$

**Lemma 5.** *For any $A \in \Im_n^0$ and $B \in \Im_n^{a+}$ $\left|P\left(A \bigcap B\right) - P(A)P(B)\right| \leqslant \tau_n(a)$, where*

$$\tau_n(a) \leqslant 4 \exp\left\{ -\frac{1}{\sqrt{2\pi}} \left(\frac{a^2 b_n}{8}\right)^{\beta+\frac{1}{2}} B\left(\beta+1; \frac{1}{2}\right) \right\}.$$

**Lemma 6.** *If $a > \left(a_n \varepsilon_n^*\right)/b_n$, then under the conditions of Lemma 5 we have $\sum\limits_{n=1}^{\infty} \left(\tau_n(a)\right)^{\tau} < \infty$ for any $\tau > 0$, where $a_n = \sqrt{2b_n \log n}$, $\varepsilon_n^* = (\log n)^{-\frac{2\beta-\delta-1}{2(2\beta+1)}}$, $0 < \delta < 1$.*

*Proof.* If $a > \left(a_n \varepsilon_n^*\right)/b_n$, then from Lemma 5 we get

$$\tau_n(a) \leqslant 4 \exp\left\{ -c \left(\log n\right)^{1+\frac{\delta}{3}} \right\}.$$

This immediately implies the statement of Lemma 6. □

Now introduce notations

$$M^{(k)}\left(t; R^2\right) = \frac{\beta}{2\pi\sqrt{b_n}} \int_r^{\sqrt{2b_n s}} (u-r)^k \left(s - \frac{u^2}{2b_n}\right)^{\beta-1} du =$$
$$= \frac{\beta}{2\pi\sqrt{b_n}} \int_0^{\sqrt{2b_n s}-r} u^k \left(s - \frac{(u+r)^2}{2b_n}\right)^{\beta-1} du,$$

where $t = (r, s)$.

**Lemma 7.** *Processes*

$$N(a) - \int_0^a M^{(1)}\left(T(b); R^2\right) db$$

*and*

$$N^2(a) - \int_0^a \left[2N(b)+1\right] M^{(1)}\left(T(b); R^2\right) db$$

*are martingales with respect to $\sigma$-algebra $\Im_{[0,a]} = \sigma\left\{T(c): \ 0 \leqslant c \leqslant a\right\}$.*

*Proof.* We have

$$E\left\{N(a+h)-N(a)/\Im_{[0,a]}\right\}=E\left\{N(a+h)-N(a)/T(a)\right\}$$

Hence, due to stationary nature of the process $T(a)$

$$E\left\{N(a+h)-N(a)/T(a)=(r,s)\right\}=E\left\{N(h)-N(0)/T(0)=(r,s)\right\}\sim$$

$$\sim E\overline{\pi}(A^*(h;r,s))\sim\Lambda^0\left(A^*(h;r,s)\right),$$

where $\overline{\pi}(A^*(h;r,s))$ is defined in the previous paragraph and

$$A^*(h;r,s)=A^0(h;r,s)\bigcup A^1(h;r,s).$$

Further, by the definition $\Lambda^0(\cdot)$of a measure (see (5)), it is easy to show that at small $h$ $\Lambda_n^0\left(A^0(h;r,s)\right)=o(h)$ and

$$\Lambda_n^0\left(A^{(1)}(a,b,x_0,y_0)\right)\leqslant C_2h^{\beta-\varepsilon}\int\limits_{ab_n+\sqrt{2b_ns_0}}^{bb_n+\sqrt{2b_ns_1}}du=O\left(h^{\beta+1-\varepsilon}\right).\tag{9}$$

From the latter, again using definition (5) at small $h$, we have

$$\Lambda_n^0\left(A^*(h;r,s)\right)=\Lambda_n^0\left(A^0(h;r,s)\right)+o(h)=hM^{(1)}\left(t;R^2\right)+o(h).\tag{10}$$

By virtue of (9) and (10), we obtain the proof of the first statement of the lemma. Proceed to the proof of the second statement of Lemma 6.

We have

$$E\left\{N^2(a+h)-N^2(a)/T(a)=(r,s)\right\}=$$

$$=E\left\{(N(a+h)-N(a))\left(N(a+h)-N(a)+2N(a)\right)/T(a)=(r,s)\right\}=$$

$$=E\Big\{(N(a+h)-N(a))^2/T(a)=(r,s)\Big\}+2N(a)E\big\{N(a+h)-N(a)/-T(a)=(r,s)\big\}=$$

$$=E\big\{N(a+h)-N(a)/T(a)=(r,s)\big\}+o(h)+2N(a)E\big\{N(a+h)-N(a)/T(a)=(r,s)\big\}=$$

$$=\left(2N(a)+1\right)E\big\{N(a+h)-N(a)/T(a)=(r,s)\big\}+o(h)=$$

$$=\left(2N(a)+1\right)hM^{(1)}\left(t,R^2\right)+o(h).$$

So, Lemma 7 is completely proved. □

Using these Lemmas, calculate the asymptotic behavior of the moments $N(a)$ and $N^2(a)$at fixed $a$ and at $n\to\infty$.

Let

$$c_0=\left(B\left(2\beta+1,\frac{1}{2}\right)\Big/\sqrt{2}\right)\left(\frac{\sqrt{2}\pi}{B\left(2\beta+1,\frac{1}{2}\right)}\right)^{2-\frac{1}{2\beta+1}}\Gamma\left(3-\frac{1}{2\beta+1}\right),$$

where $B(\,\cdot\,,\,\cdot\,)$ and $\Gamma(\,\cdot\,)$ are the known beta and gamma functions, respectively.

**Lemma 8.** *We have*

$$EN(a)=a\lambda_n^{(1)}\sqrt{b_n},\quad DN(a)=a\lambda_n^{(2)}\sqrt{b_n},\quad as\ n\to\infty,$$

*where* $\lambda_n^{(1)}=\dfrac{c_0}{\sqrt{2\pi}}+o(1),\ \ \lambda_n^{(2)}=c_0+o(1).$

*Proof.* We use Lemma 7. Since the process $T(\cdot)$ is stationary, we have

$$EN(a) = E\int_0^a M^{(1)}\left(T(b); R^2\right) db = aEM^{(1)}\left(T(0); R^2\right). \tag{11}$$

By definition of $M^{(1)}\left(t; R^2\right)$, after some identical transformations we have

$$\begin{aligned} EM^{(1)}\left(T(0); R^2\right) &= \frac{\beta}{2\pi\sqrt{b_n}} E\int_0^{\sqrt{2b_n}s-r} u\left(s - \frac{(u+r)^2}{2b_n}\right)^{\beta-1} du = \\ &= \frac{\beta}{2\pi\sqrt{b_n}} E\left\{\int_0^{\sqrt{2b_n}s-r} u\,\left(s - \frac{(u+r)^2}{2b_n}\right)^{\beta-1} du\right\} = \\ &= \frac{\sqrt{2b_n}\beta^2}{2\pi^2}\left\{\int_0^\infty \exp\left[-\frac{B\left(\beta+1,\frac{1}{2}\right)}{\sqrt{2}\pi}s^{\beta+\frac{1}{2}}\right] s^{2\beta-\frac{1}{2}}ds\right\}\times \\ &\times \int_{-1}^1 \left\{(1-r^2)^{\beta-1}\int_0^{1-r} u\left(1-(u+r)^2\right)^{\beta-1}du\right\}dr = \\ &= \frac{\sqrt{b_n}\beta^2}{2\pi^2(4\beta+1)}\left(\frac{\sqrt{2}\pi}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{\frac{4\beta+1}{2\beta+1}}\cdot\Gamma\left(2-\frac{1}{2\beta+1}\right)\times \\ &\times \int_{-1}^1 (1-r^2)^{\beta-1}dr\int_0^{1-r} u\left(1-(u+r)^2\right)^{\beta-1}du. \end{aligned} \tag{12}$$

It is easy to calculate that

$$\int_{-1}^1 (1-r^2)^{\beta-1}dr\int_0^{1-r} u\left(1-(u+r)^2\right)^{\beta-1}du = \frac{1}{\beta}\int_0^1 \frac{t^{2\beta+1}dt}{\sqrt{1-t}} = \frac{B\left(2\beta,\frac{1}{2}\right)}{\beta}. \tag{13}$$

From relations (11)–(13), the proof of the statement of the first part of Lemma 8 follows.

The second part of Lemma 8 is easy to prove, using Lemma 7 and the first part of Lemma 8, and Lemma 2.6 considering in [4]. □

## 3. Proof of the theorem

Assume that $N^*_{k,m_n}$ is the number of vertices of the convex hull in $I^*_{k,m_n}$ — "big block" and $N^{**}_{k,m_n}$ is the number of vertices of the convex hull in $I^{**}_{k,m_n}$ — "small block", where sectors $I^*_{k,m_n}$ and $I^{**}_{k,m_n}$ correspond to

$$\left(\pi\sqrt{b_n}(\log n)(2k-1)+\xi_n, \pi\sqrt{b_n}(\log n)(2k+1)-\xi_n\right)$$

and

$$\left(\pi\sqrt{b_n}(\log n)(2k+1)-\xi_n, \pi\sqrt{b_n}(\log n)(2k+1)+\xi_n\right)$$

of the part of disk, respectively and

$$\xi_n = \pi\sqrt{b_n}(\log n)^{1-\delta}, \quad 0<\delta<1/\left[2(2\beta+1)\right].$$

Hence

$$\nu_n = \sum_{k=0}^{m_n-1} N^*_{k,m_n} + \sum_{k=0}^{m_n-1} N^{**}_{k,m_n}.$$

By the principle of construction of sectors $I^{*}_{k,m_n}$ and $I^{**}_{k,m_n}$, $N^{**}_{k,m_n}$ is insignificant relative to $N^{*}_{k,m_n}$. Further, from Lemmas 5 and Theorem 17.2.2 in [8], we can apply the classical central limit theorem for the summs of random variables $N^{*}_{0,m_n} + N^{*}_{1,m_n} + \cdots + N^{*}_{m_n-1,m_n}$ and $N^{**}_{0,m_n} + N^{**}_{1,m_n} + \cdots + N^{**}_{m_n-1,m_n}$. Therefore, we get that

$$\frac{\nu_n - m_n(EN^{*}_{k,m_n} + EN^{**}_{k,m_n})}{\sqrt{m_n DN^{*}_{k,m_n}}} = \frac{1}{\sqrt{m_n}} \sum_{k=0}^{m_n-1} \frac{N^{*}_{k,m_n} - EN^{*}_{k,m_n}}{\sqrt{DN^{*}_{k,m_n}}} +$$

$$+\frac{1}{\sqrt{m_n}} \sum_{k=0}^{m_n-1} \frac{N^{**}_{k,m_n} - EN^{**}_{k,m_n}}{\sqrt{DN^{**}_{k,m_n}}} \frac{\sqrt{DN^{**}_{k,m_n}}}{\sqrt{DN^{*}_{k,m_n}}} + o(1) =$$

$$= \frac{1}{\sqrt{m_n}} \sum_{k=0}^{m_n-1} \frac{N^{*}_{k,m_n} - EN^{*}_{k,m_n}}{\sqrt{DN^{*}_{k,m_n}}} + \frac{1}{\sqrt{m_n}} \sum_{k=0}^{m_n-1} \frac{N^{**}_{k,m_n} - EN^{**}_{k,m_n}}{\sqrt{DN^{**}_{k,m_n}}} \cdot o(1) + o(1) \stackrel{d}{\Rightarrow} \mathrm{N}(0,1).$$

The theorem is proved.

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## О предельной теореме для числа вершин выпуклых оболочек в единичном круге

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**Аннотация.** Данная статья посвящена дальнейшему исследованию свойства ряда вершин выпуклых оболочек, порожденных независимыми наблюдениями двумерного случайного вектора с регулярными распределениями вблизи границы носителя, когда он является единичным диском. Следуя П. Гренебуму [4], биномиальный точечный процесс аппроксимируем пуассоновским точечным процессом вблизи границы опоры и строим вершинные процессы выпуклых оболочек. Исследованы свойства сильного перемешивания и мартингальности вершинных процессов. Используя эти свойства, получаем асимптотические выражения для ожиданий и дисперсии вершинных процессов, которые соответствуют результатам, ранее полученным Н. Карнала [2]. Далее, используя свойства сильного перемешивания вершинных процессов, доказываем центральную предельную теорему для ряда вершин выпуклой оболочки.

**Ключевые слова:** выпуклая оболочка, пуассоновский точечный процесс, скачкообразный марковский процесс, мартингальность, центральная предельная теорема.

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# On Some Examples of Systems of Transcendent Equations

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**Abstract.** This article discusses examples of transcendent systems of equations of a general form. The residue integrals are determined over the cycles associated with the system. Formulas are given for their calculation and their relationship with the power sums of the roots of the system is established.

**Keywords:** transcendent systems of equations, residue integrals, power sums of roots.

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For systems of nonlinear algebraic equations in $\mathbb{C}^n$, based on a multidimensional logarithmic residue, formulas were previously obtained for finding power sums of the roots of a system without calculating the roots themselves (see [1–3]). For different types of systems, such formulas have different forms. On this basis, a new method for the study of systems of algebraic equations in $\mathbb{C}^n$ is constructed. It arose in the work of L. A. Aizenberg [1], and its development is continued in monographs [2,4]. Its main idea is to find power sums of roots of systems (for positive powers) and then using one-dimensional or multidimensional recurrent Newton formulas (see [5]). Unlike the classical method of elimination, it is less labor intensive and does not increase the multiplicity of roots. It is based on a formula (see [1]) obtained using the multidimensional logarithmic residue, to find the sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations without finding the roots themselves.

For systems of transcendent equations, formulas for the sum of the values of the roots of the system, as a rule, cannot be obtained, since the number of roots of a system can be infinite and a series of coordinates of such roots can be diverging. Nevertheless, transcendent systems of equations arise, for example, in the problems of chemical kinetics [6, 7]. Thus, the urgent task is to consider such systems.

In the works [8–15] power sums of roots are considered for a negative power for different systems of non-algebraic (transcendent) equations. To compute these power sums, a residue integral is used, the integration of which is carried out over skeletons of polycircles centered at zero. Note that this residue integral is not, generally speaking, a multidimensional logarithmic residue or a Grothendieck residue. For various types of lower homogeneous systems of functions included in the system, formulas are given for finding residue integrals, their relationship with power sums of the roots of the system to a negative degree is established.

Article [16] investigated more complex systems in which the lower homogeneous parts are decomposed into linear factors and integration cycles in residue integrals, and are constructed from these factors.

In [15], a system is studied that arises in the Zel'dovich–Semenov model (see [6,7]) in chemical kinetics.

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The object of this study is transcendent systems of equations in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations: formulas are found for calculating the residue integrals, power sums of roots for a negative power, their relationship with the residue integrals is established. See [16, 17].

Let $f_1(z), \ldots, f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in a multidimensional complex space $\mathbb{C}^n$, $z = (z_1, \ldots, z_n)$.

We expand functions $f_1(z), \ldots, f_n(z)$ into Taylor series in a neighborhood of the origin and consider a system of equations of the form

$$f_i(z) = P_i(z) + Q_i(z) = 0, \quad i = 1, \ldots, n, \tag{1}$$

where $P_i$ is the lower homogeneous part of the Taylor expansion of the function $f_i(z)$. The degree of all monomials (in the totality of variables) included in $P_i$, is $m_i$, $i = 1, \ldots, n$. In functions $Q_i$, the degrees of all monomials are strictly greater than $m_i$.

The expansion of the functions $Q_j$, $P_j$, $j = 1, \ldots, n$, in a neighborhood of zero in Taylor series converging absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\| > m_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\beta\| = m_j} b^j_\beta z^\beta, \tag{3}$$

$$j = 1, \ldots, n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indices i.e. $\alpha_j$ and $\beta_j$ are non-negative integers, $j = 1, \ldots, n$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$, and monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z), \ldots, P_n(z)$ it is non-degenerate, i.e. its common zero is only the point 0, the origin. Consider an open set (special analytic polyhedron) of the form

$$D_P(r_1, \ldots, r_n) = \{z : \ |P_i(z)| < r_i, \ i = 1, \ldots, n\},$$

where $r_1, \ldots, r_n$ are positive numbers. Its *skeleton* has the form

$$\Gamma_P(r_1, \ldots, r_n) = \Gamma_P(r) = \{z : \ |P_i(z)| = r_i, \ i = 1, \ldots, n\}.$$

These sets play an important role in the theory of multidimensional residues (see, for example, [2]).

For sufficiently small $r_i$, the cycles $\Gamma_P$ lie in the domain of holomorphy of functions $f_i$, therefore, the series

$$\sum_{\|\alpha\| > m_i} |a^j_\alpha| r_1^{\alpha_1} \cdots r_n^{\alpha_n}$$

converge, $i = 1, 2, \ldots, n$. Then on the cycle $\Gamma_P(tr) = \Gamma_P(tr_1, tr_2, \ldots, tr_n)$ for sufficiently small $t > 0$, we have

$$|P_i(tr)| = \left| \sum_{\|\beta\| = m_i} b^i_\beta (tr)^\beta \right| = \sum_{\|\beta\| = m_i} t^{\|\beta\|} |b^i_\beta| r^\beta = t^{m_i} \sum_{\|\beta\| = m_i} |b^i_\beta| r^\beta, \quad i = 1, \ldots, n,$$

and

$$|Q_i(tr)| = \left| \sum_{\|\alpha\| > m_i} a^i_\alpha (tr)^\alpha \right| \leqslant \sum_{\|\alpha\| > m_i} t^{\|\alpha\|} |a^i_\alpha| r^\alpha = t^{m_i+1} \sum_{\|\alpha\| > m_i} |a^i_\alpha| r^\alpha t^{\|\alpha\| - (m_i+1)}.$$

Therefore, for sufficiently small $t$ on the cycle $\Gamma_P(tr)$ the inequalities hold

$$|P_i(z)| > |Q_i(z)|, \quad i = 1, 2, \dots, n. \tag{4}$$

Thus,

$$f_i(z) \neq 0 \quad \text{на} \quad \Gamma_P(tr), \quad i = 1, 2, \dots, n.$$

In what follows, we assume that $t = 1$, that is, that the inequality (4) is valid on the cycle $\Gamma_P(r_1, \dots, r_n)$.

We introduce the concept of *residue integral* $J_\gamma$ (see [18]). Define

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z^{\gamma+I}} \cdot \frac{df}{f} = \tag{5}$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n},$$

where $\gamma = (\gamma_1, \dots \gamma_n)$ is a multi-index. This residue integral is defined if $r_1, \dots, r_n$ is chosen so that the inequality (4) holds and the cycle $\Gamma_P$ does not intersect with the coordinate planes. Note that this integral is not a multidimensional logarithmic residue or a Grothendieck residue.

Recall some concepts from the space of the theory of functions $\overline{\mathbb{C}}^n$ which equal to the product of $n$ copies of Riemann spheres $\mathbb{CP}^1$, i.e. $\overline{\mathbb{C}}^n = \mathbb{CP}^1 \times \dots \times \mathbb{CP}^1$.

Let $z_j : w_j$ be homogeneous coordinates in the $j$-th factor of the space $\overline{\mathbb{C}}^n$ and let

$$F_j(z_1, w_1, \dots, z_n, w_n) = 0, \quad j = 1, \dots, n \tag{6}$$

be a system of equations consisting of polynomials $F_j$ homogeneous for each pair of variables $(z_k, w_k)$, $k = 1, \dots, n$. We will consider only those roots $(z_1, w_1, \dots, z_n, w_n)$ systems (6) for which

$$(z_k, w_k) \in \mathbb{C}^2 \setminus \{(0, 0)\}, \quad k = 1, \dots, n.$$

The roots of the system (6) with pairs having proportional coordinates determine one root $(z_1 : w_1, \dots, z_n : w_n)$ in $\overline{\mathbb{C}}^n$.

Let

$$a = (z_1^{(0)} : w_1^{(0)}, \dots, z_n^{(0)} : w_n^{(0)})$$

be the root of the system (6) for which all $w_k^{(0)}$ $neq0$. Then the point $(z_1, 1, \dots, z_n, 1)$ is the root of the system

$$F_j(z_1, 1, \dots, z_n, 1) = 0, \quad j = 1, \dots, n,$$

in $\mathbb{C}^n$. Roots of $a$ for which some $w_j^{(0)}$ are equal to zero correspond to infinitely remote roots in $\overline{\mathbb{C}}^n$.

For a given system of equations of the form (1) for which all $f_j(z)$ are polynomials, then in order to find the infinitely remote roots of this system in $\overline{\mathbb{C}}^n$, you must first go to homogeneous coordinates, substituting the $z_k/w_k$ relationship instead of $z_k$ and discarding the resulting denominator, thereby obtaining a system of type (6). Solving it, we find ordinary roots and infinitely remote roots of the system (1).

We return to the consideration of the system (1). Assume that, in addition to non-degeneracy, the system $P_1(z), \dots, P_n(z)$ does not have infinite roots in the space $\overline{\mathbb{C}}^n$.

We now consider as functions $Q_i(z)$, $i = 1, \dots, n$, polynomials of the form

$$Q_i(z) = \sum_{\|\alpha\| > m_i} a_\alpha^i z^\alpha. \tag{7}$$

Suppose that for each $i$-th equation in (1) the conditions

$$\deg_{z_i} P_i < \deg_{z_i} Q_i, \quad \deg_{z_j} P_i \geqslant \deg_{z_j} Q_i, \quad j \neq i. \tag{8}$$

Here $\deg_{z_i} P(z)$ is the degree of the polynomial $P$ in the variable $z_i$ for the remaining variables We have $\deg P_i = m_i$. Denote $\deg Q_i = s_i$, a $\deg_{z_j} P_i = m_i^j$, $\deg_{z_j} Q_i = s_i^j$. Then $m_i < s_i$, $m_i^i < s_i^i$, $i = 1, \ldots, n$. In addition, $m_i^j \geqslant s_i^j$ for $j \neq i$. Cases when $\sum\limits_{j=1}^{n} m_i^j > m_i$.

In all functions, we write $f_i(z) = P_i(z) + Q_i(z)$, $i = 1, 2, \ldots, n$, and replace $z_i = \dfrac{1}{w_i}$, $i = 1, \ldots, n$, assuming that all $w_i \neq 0$. We get

$$P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \sum_{\|\beta\| = m_i} b_\beta^i \frac{1}{w_1^{\beta_1}} \cdots \frac{1}{w_n^{\beta_n}} = \frac{1}{w_1^{m_i^1}} \cdots \frac{1}{w_n^{m_i^n}} \sum_{\|\beta| = m_i} b_\beta^i w_1^{m_i^1 - \beta_1} \cdots w_n^{m_i^n - \beta_n},$$

and

$$Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \sum_{\|\alpha\| > m_i} a_\alpha^i \frac{1}{w_1^{\alpha_1}} \cdots \frac{1}{w_n^{\alpha_n}} = \frac{1}{w_1^{s_i^1}} \cdots \frac{1}{w_n^{s_i^n}} \sum_{\|\alpha\| > m_i} a_\alpha^i w_1^{s_i^1 - \alpha_1} \cdots w_n^{s_i^n - \alpha_n}.$$

We have

$$\begin{gathered} f_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) + Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \\ = \frac{1}{w_1^{m_i^1} \cdots w_i^{s_i^i} \cdots w_n^{m_i^n}} \cdot \left(\tilde{P}_i(w) + \tilde{Q}_i(w)\right), \end{gathered} \tag{9}$$

where $\tilde{P}_i$ are homogeneous polynomials

$$\begin{gathered} \tilde{P}_i(w_1, \ldots, w_n) = w_1^{m_i^1} \cdots w_i^{s_i^i} \cdots w_n^{m_i^n} \cdot P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \\ = w_i^{s_i^i - m_i^i} \sum_{\|\beta\| = m_i} b_\beta^i w_1^{m_i^1 - \beta_1} \cdots w_n^{m_i^n - \beta_n} = w_i^{s_i^i - m_i^i} \cdot \tilde{\tilde{P}}_i, \end{gathered}$$

and $\tilde{\tilde{P}}_i$ are homogeneous polynomials

$$\tilde{\tilde{P}}_i = \sum_{\|\beta\| = m_i} b_\beta^i w_1^{m_i^1 - \beta_1} \cdot \ldots \cdot w_n^{m_i^n - \beta_n}.$$

In $\tilde{\tilde{P}}_i$, neither $w_1, \ldots,$ nor $w_n$.

The polynomials $\tilde{Q}_i$ have the form

$$\tilde{Q}_i(w_1, \ldots, w_n) = w_1^{m_i^1} \cdots w_i^{s_i^i} \cdots w_n^{m_i^n} \cdot Q_i\left(\frac{1}{w}, \ldots, \frac{1}{w_n}\right) =$$

$$= w_1^{m_i^1} \cdots w_i^{s_i^i} \cdots w_n^{m_i^n} \cdot \frac{1}{w_1^{s_i^1}} \cdots \frac{1}{w_n^{s_i^n}} \sum_{\|\alpha\| > m_i} a_\alpha^i w_1^{s_i^1 - \alpha_1} \cdots w_n^{s_i^n - \alpha_n} =$$

$$= w_1^{m_i^1 - s_i^1} \cdots [w_i] \cdots w_n^{m_i^n - s_i^n} \cdot \sum_{\|\alpha\| > m_i} a_\alpha^i w_1^{m_i^1 - \alpha_1} \cdots w_n^{m_i^n - \alpha_n}.$$

Denote by $\tilde{f}_i$ the functions

$$\tilde{f}_i(w) = \tilde{P}_i(w) + \tilde{Q}_i(w) = w_i^{s_i^i - m_i^i} \cdot \tilde{\tilde{P}}_i + \tilde{Q}_i(w), \quad i = 1, 2, \ldots, n. \tag{10}$$

We have

$$\deg \tilde{P}_i > \deg \tilde{Q}_i, \quad i = 1, \ldots, n. \tag{11}$$

Consider a system of equations of the form (1) with polynomials $Q_i(z)$ satisfying the conditions (8).

Let $\Gamma_{\tilde{P}} = \Gamma_{\tilde{P}}(\varepsilon)$ denote the cycle

$$\Gamma_{\tilde{P}} = \{w \in \mathbb{C}^n : |\tilde{P}_i| = \varepsilon_i, \quad \varepsilon_i > 0, \ i = 1, \ldots, n\}. \tag{12}$$

This cycle does not intersect with the coordinate planes for almost all $\varepsilon_i, \ i = 1, \ldots, n$.

Consider the residue integral $\tilde{J}_\gamma$ of the form

$$\tilde{J}_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_{\tilde{P}}} w^{\gamma+I} \frac{df(1/w)}{f(1/w)},$$

where $w^{\gamma+I} = w_1^{\gamma_1+1} \cdots w_n^{\gamma_n+1}, \ f(1/w) = f_1(1/w_1, \ldots, 1/w_n) \cdots f_n(1/w_1, \ldots, 1/w_n), \ df(1/w) = $ $= df_1(1/w_1, \ldots, 1/w_n) \wedge \ldots \wedge df_n(1/w_1, \ldots, 1/w_n)$.

In fact, $\tilde{J}_\gamma$ is obtained from the integral $J_\gamma$ (5) using the substitution in the integrand $z_j = 1/w_j, \ j = 1, \ldots, n$, and replacing $\Gamma_P$ by $\Gamma_{\tilde{P}}$. But the equality of these integrals needs to be proved.

Since the inequalities (11) hold for functions from the system (10), and the system of functions $\tilde{P}_1(w), \ldots, \tilde{P}_n(w)$ is non-degenerate, the well known Bezout theorem says that the system of equations

$$\tilde{f}_j(w) = 0, \quad j = 1, \ldots, n, \tag{13}$$

has a finite number of roots (counting each root so many times what its multiplicity is) and this number is equal to the product of the degrees of the polynomials $\tilde{P}_j(w)$.

We cite the theorem from [16].

**Theorem 1.** *The following equality holds:*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}} =$$

$$= \sum_{\|\alpha\| \leqslant \|\gamma\|+n} (-1)^{||\alpha||} \int\limits_{\Gamma_{\tilde{P}}} \left[ \tilde{\Delta} \cdot w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{\tilde{Q}_1^{\alpha_1} \cdot \tilde{Q}_2^{\alpha_2} \cdots \tilde{Q}_n^{\alpha_n}}{\tilde{P}_1^{\alpha_1+1} \cdot \tilde{P}_2^{\alpha_2+1} \cdots \tilde{P}_n^{\alpha_n+1}} \right] dw,$$

*where $\tilde{\Delta}$ is the Jacobian of the system (10).*

For what follows, we need a generalized Grothendieck residue transformation formula (see [19], as well as [4, Ch. 2]).

**Theorem 2** ( [19]). *Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$ and $g_j(w)$, $j, k = 1, \ldots, n$, be related by*

$$g_j = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1, 2, \ldots, n,$$

*the matrix $A = \|a_{jk}\|_{j,k=1}^n$ consists of polynomials. Let us consider cycles*

$$\Gamma_f = \{w : \ |f_j(w)| = r_j, \ j = 1, \ldots, n\}, \quad \Gamma_g = \{w : \ |g_j(z)| = r_j, \ j = 1, \ldots, n\},$$

*where all* $r_j > 0$.

*Then the equality is valid:*

$$\int\limits_{\Gamma_f} h(w)\frac{dw}{f^\alpha} = \sum_{K,\sum\limits_{s=1}^{n} k_{sj}=\beta_s} \frac{\beta!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\int\limits_{\Gamma_g} h(w)\frac{\det A \prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}\,dw}{g^\beta}, \tag{14}$$

*where* $\beta! = \beta_1!\beta_2!\ldots\beta_n$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, *the summation in the formula is over all integer non-negative matrices* $K = \|k_{sj}\|_{s,j=1}^n$ *with the conditions that the sum* $\sum\limits_{s=1}^{n} k_{sj} = \alpha_j$, *then* $\beta_j = = \sum\limits_{j=1}^{n} k_{js}$.

*Here* $f^\alpha = f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^\beta = g_1^{\beta_1}\cdots g_n^{\beta_n}$.

From this theorem, a statement is obtained in [16].

**Theorem 3.** *The formulas are valid*

$$\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}} = \frac{(-1)^n}{(2\pi\sqrt{-1})^n}\int\limits_{\Gamma_{\tilde P}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde f_1}{\tilde f_1}\wedge\frac{d\tilde f_2}{\tilde f_2}\wedge\ldots\wedge\frac{d\tilde f_n}{\tilde f_n} =$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n}\frac{(-1)^{n+\|\alpha\|}}{(2\pi\sqrt{-1})^n}\int\limits_{\Gamma_{\tilde P}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{\tilde\Delta\cdot\tilde Q_1^{\alpha_1}\cdot\tilde Q_2^{\alpha_2}\cdots\tilde Q_n^{\alpha_n}\,dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\tilde P_1^{\alpha_1+1}\cdot\tilde P_2^{\alpha_2+1}\cdots\tilde P_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\|\leqslant\|\gamma\|+n}\frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n}k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\tilde\Delta\cdot\det A\cdot Q^\alpha\prod\limits_{s,j=1}^{n}a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n}w_j^{\beta_jN_j+\beta_j+N_j}}\right],$$

*where* $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$, *and the functional* $\mathfrak{M}$ *maps the Laurent polynomial to its free term.*

In fact, in Theorem 3, analogues of the classical Waring formulas for finding power sums of the roots of a system of algebraic equations are obtained.

Note that in [20] general algebraic systems of equations were considered, decompositions of their solutions in hypergeometric series were obtained. In addition, it proves analogues of Waring's formulas for systems of the form

$$y_j^{m_j} + \sum_{\lambda\in\Lambda^{(j)}\cup\{0\}} x_\lambda^{(j)}y^\lambda = 0, \quad \lambda_1+\ldots+\lambda_n < m_j, \quad j=1,\ldots,n,$$

those higher homogeneous parts are monomials. We considered other (more general) systems of equations with functions of the form (10).

Consider a more general situation. Let the functions $f_j$ be meromorphic and have the form

$$f_j(z) = \frac{f_j^{(1)}(z)}{f_j^{(2)}(z)}, \quad j=1,2,\ldots,n, \tag{15}$$

where $f_j^{(1)}(z)$ and $f_j^{(2)}(z)$ are entire functions in $\mathbb{C}^n$ that decompose into infinite products uniformly converging in $\mathbb{C}^n$, $f_j^{(2)}(0)\neq 0$,

$$f_j^{(1)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(1)}(z), \quad f_j^{(2)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(2)}(z),$$

moreover, each of the factors has the form $P_{j,s}(z) + Q_{j,s}(z)$, and $Q_{j,s}(z)$ satisfy conditions (8), $s = 1, 2, \ldots$.

For each set of indices $j_1, \ldots, j_n$, where $j_1, \ldots, j_n \in \mathbb{N}$, and each set of numbers $i_1, \ldots, i_n$, where $i_1, \ldots, i_n$ are equal 1 or 2, systems of nonlinear equations

$$f^{(i_1)}_{1,j_1}(z) = 0, \quad f^{(i_2)}_{2,j_2}(z) = 0, \ \ldots, \ f^{(i_n)}_{n,j_n}(z) = 0, \tag{16}$$

have a finite number of roots not lying on coordinate planes.

The roots of all such systems (not lying on the coordinate planes) are no more than a countable set. Renumber them (taking into account multiplicities):

$$z_{(1)}, \ z_{(2)}, \ldots, z_{(l)}, \ldots .$$

Denote by $\sigma_{\beta+I}$ the expression

$$\sigma_{\beta+I} = \sum_{l=1}^{\infty} \frac{\varepsilon_l}{z^{\beta_1+1}_{1(l)} \cdot z^{\beta_2+1}_{2(l)} \cdots z^{\beta_n+1}_{n(l)}}. \tag{17}$$

Here $\beta_1, \ldots, \beta_n$, as before, are non-negative integers, and the sign $\varepsilon_l$ is $+1$, if in a system of the form (16), the root which is $z_{(l)}$, includes an even number of functions $f^{(2)}_{j_s}$; and is equal to $-1$ if in a system of the form (16), the root which is $z_{(l)}$, includes an odd number of functions $f^{(2)}_{j_s}$.

For a system (16) composed of functions of the form (15), the points $z_{(l)}$ are roots or singular points (poles). All functions $f_j$ are holomorphic in a neighborhood of zero and are defined for them integrals $J_\beta$, since they have the form (1).

**Theorem 4.** *For a system of equations with meromorphic functions (15) the series (17) absolutely converges, and*

$$J_\beta = (-1)^n \sigma_{\beta+I}.$$

**Example 1.**

Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = z_1 - z_2 + az_1^2 + bz_1^3 = 0, \\ f_2(z_1, z_2) = 1 + cz_2 = 0. \end{cases}$$

We make the change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1^2 w_2 - w_1^3 + aw_1 w_2 + bw_2 = 0, \\ \widetilde{f}_2 = w_2 + c = 0. \end{cases} \tag{18}$$

The Jacobian of the system (18) $\widetilde{\Delta}$ is

$$\widetilde{\Delta} = \begin{vmatrix} 2w_1w_2 - 3w_1^2 + aw_2 & w_1^2 + aw_1 + b \\ 0 & 1 \end{vmatrix} = 2w_1w_2 - 3w_1^2 + aw_2.$$

It is clear that

$$\begin{cases} \widetilde{Q}_1 = aw_1w_2 + bw_2, \\ \widetilde{Q}_2 = c. \end{cases}$$

$$\begin{cases} \widetilde{P}_1 = w_1^2 w_2 - w_1^3, \\ \widetilde{P}_2 = w_2. \end{cases}$$

Since

$$w_1^3 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$
$$w_2 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

it is easy to show that the elements $a_{ij}$ of the matrix $A$ are equal

$$a_{11} = -1,\ a_{12} = w_1^2,$$

$$a_{21} = 0,\ a_{22} = 1.$$

Thus, $\det A = -1$.

By Theorem 3

$$J_{(0,0)} = \sum_{\|K\| = k_{11}+k_{12}+k_{21}+k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times \mathfrak{M}\left[\frac{(3w_1^2 - 2w_1w_2 - aw_2) \cdot (aw_1w_2 + bw_2)^{k_{11}+k_{21}} \cdot c^{k_{12}+k_{22}} \cdot (-1)^{k_{11}} \cdot (w_1^2)^{k_{12}} \cdot 0^{k_{21}} \cdot 1^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1} \cdot w_2^{(k_{21}+k_{22})-1}}\right].$$

Simple calculations give that

$$J_{(0,0)} = c^2.$$

Recall the well-known decomposition of the sine function into an infinite product:

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right),$$

which uniformly and absolutely converge on the complex plane and has a growth order of 1.

Consider the system of equations

$$\begin{cases} f_1(z_1, z_2) = z_1 - z_2 + az_1^2 + bz_1^3 = 0, \\ f_2(z_1, z_2) = \dfrac{\sin z_2}{z_2} = 0. \end{cases}$$

Using the formula obtained above and the known sum, we obtain that the integral $J_{(0,0)}$ is equal to the sum of the series

$$J_{(0,0)} = 2\sum_{s=1}^{\infty} \frac{1}{\pi^2 s^2} = \frac{1}{3}.$$

**Example 2.** Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = z_1z_2 + b_1z_1 + b_2z_2 = 0, \\ f_2(z_1, z_2) = 1 + a_1z_1 + a_2z_2 = 0. \end{cases} \tag{19}$$

We make the change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = 1 + b_2w_1 + b_1w_2 = 0, \\ \widetilde{f}_2 = w_1w_2 + a_2w_1 + a_1w_2 = 0. \end{cases} \tag{20}$$

The Jacobian of the system (24) $\widetilde{\Delta}$ is

$$\widetilde{\Delta} = \begin{vmatrix} b_2 & b_1 \\ w_2 + a_2 & w_1 + a_1 \end{vmatrix} = b_2w_1 - b_1w_2 + (a_1b_2 - a_2b_1).$$

Notice that

$$\begin{cases} \widetilde{Q}_1 = 1, \\ \widetilde{Q}_2 = a_1 w_2 + a_2 w_1. \end{cases} \tag{21}$$

$$\begin{cases} \widetilde{P}_1 = b_1 w_2 + b_2 w_1, \\ \widetilde{P}_2 = w_1 w_2. \end{cases} \tag{22}$$

We calculate $\det A$ :

Since

$$w_1^2 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$

$$w_2^2 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

where $\widetilde{P}_1 = b_1 w_2 + b_2 w_1$, $\widetilde{P}_2 = w_1 w_2$.

Therefore, the elements of $a_{ii}$ are equal

$$a_{11} = \frac{w_1}{b_2},\ a_{12} = -\frac{b_1}{b_2},$$

$$a_{21} = \frac{w_2}{b_1},\ a_{22} = -\frac{b_2}{b_1}.$$

Hence,

$$\det A = \frac{w_2}{b_2} - \frac{w_1}{b_1} = \frac{w_2 b_1 - w_1 b_2}{b_1 b_2}.$$

Notice that

$$\tilde{Q}_1 = 1, \quad \tilde{Q}_2 = 1.$$

Carrying out the same calculations as in the previous example, we obtain

$$J_{(0,0)} = -\frac{2(a_1 + b_2)}{\bar{\Delta}}.$$

**Example 3.**

Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = a_1 z_1 - a_2 z_2 + z_1^2 = 0, \\ f_2(z_1, z_2) = b_1 z_1 + b_2 z_2 + z_2^2 = 0. \end{cases} \tag{23}$$

It satisfies the conditions (8) on $Q_j(z)$. We assume that $a_1 b_2 + a_2 b_1 \neq 0$, i.e. the system of lower homogeneous polynomials is non-degenerate.

We make the change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \tilde{f}_1 = -a_2 w_1^2 + a_1 w_1 w_2 + w_2 = 0, \\ \tilde{f}_2 = b_2 w_1 w_2 + b_1 w_2^2 + w_1 = 0. \end{cases} \tag{24}$$

This system has 4 roots, on the coordinate planes there is one root, (0,0).

The Jacobian $\tilde{\Delta}$ of the system (24) is equal to

$$\tilde{\Delta} = \begin{vmatrix} -2a_2 w_1 + a_1 w_2 & a_1 w_1 + 1 \\ b_2 w_2 + 1 & 2b_1 w_2 + b_2 w_1 \end{vmatrix} = -2a_2 b_2 w_1^2 - 4a_2 b_1 w_1 w_2 + 2a_1 b_1 w_2^2 - a_1 w_1 - b_2 w_2 - 1.$$

Notice that

$$\tilde{Q}_1 = w_2, \quad \tilde{Q}_2 = w_1. \tag{25}$$

$$\tilde{P}_1 = -a_2 w_1^2 + a_1 w_1 w_2, \quad \tilde{P}_2 = b_2 w_1 w_2 + b_1 w_2^2. \tag{26}$$

To find the matrix $A$, we use Example 8.3 from [4].

We introduce the matrix

$$\mathrm{Res} = \begin{pmatrix} -a_2 & a_1 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 & 0 \\ 0 & 0 & b_2 & b_1 \end{pmatrix}.$$

The determinant $\Delta$ of the matrix Res is equal to $\Delta = a_2 b_1 (a_2 b_1 + a_1 b_2)$.

We calculate some minors according to Example 8.3 from [4]:

$$\tilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_2 b_1^2 - a_1 b_1 b_2, \qquad \tilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_1 b_1^2,$$

$$\tilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2 b_1, \qquad \tilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \end{vmatrix} = 0.$$

$$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2 b_2^2,$$

$$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2 b_2, \qquad \Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2 b_1 + a_1 a_2 b_2.$$

Therefore, the elements $a_{ij}$ of the matrix $A$ are equal

$$a_{11} = \frac{1}{\Delta}\left(\tilde{\Delta}_1 w_1 + \tilde{\Delta}_2 w_2\right) = \frac{1}{\Delta}\left((-a_2 b_1^2 - a_1 b_1 b_2) w_1 - a_1 b_1^2 w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\tilde{\Delta}_3 w_1 + \tilde{\Delta}_4 w_2\right) = \frac{a_1^2 b_1 w_1}{\Delta}, \quad a_{21} = \frac{1}{\Delta}\left(\Delta_1 w_1 + \Delta_2 w_2\right) = \frac{-a_2 b_2^2 w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3 w_1 + \Delta_4 w_2\right) = \frac{1}{\Delta}\left(-a_2^2 b_2 w_1 + (a_2^2 b_1 + a_1 a_2 b_2) w_2\right).$$

Then, it is easy to verify that

$$w_1^3 = a_{11}\tilde{P}_1 + a_{12}\tilde{P}_2, \quad w_2^3 = a_{21}\tilde{P}_1 + a_{22}\tilde{P}_2.$$

We calculate $\det A$ :

$$\det A = \frac{1}{\Delta}\left(a_2 b_2 w_1^2 - a_2 b_1 w_1 w_2 - a_1 b_1 w_2^2\right).$$

By Theorem 3

$$J_{(0,0)} = \sum_{\|K\| \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times \mathfrak{M}\left[\frac{\tilde{\Delta} \cdot \det A \cdot \tilde{Q}_1^{k_{11}+k_{21}} \cdot \tilde{Q}_2^{k_{12}+k_{22}} \cdot a_{11}^{k_{11}} \cdot a_{12}^{k_{12}} \cdot a_{21}^{k_{21}} \cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1} \cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

Denote $\bar{\Delta} = a_2 b_1 + a_1 b_2$. Cumbersome but simple calculations (using the definition of the functional $\mathfrak{M}$) give that

$$J_{(0,0)} = \frac{1}{\bar{\Delta}} - \frac{2a_1 b_2}{a_2 b_1 \bar{\Delta}} + \frac{6a_1^2 b_2^2}{a_2 b_1 \bar{\Delta}^2} - \frac{b_2^3}{b_1 \bar{\Delta}^2} + \frac{a_1^3}{a_2 \bar{\Delta}^2} + \frac{8a_1 b_2}{\bar{\Delta}^2} - \frac{4}{a_2 b_1} = \frac{a_1^3}{a_2 \bar{\Delta}^2} - \frac{a_1 b_2}{\bar{\Delta}^2} - \frac{3a_2 b_1}{\bar{\Delta}^2} - \frac{b_2^3}{b_1 \bar{\Delta}^2}.$$

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# О некоторых примерах систем трансцендентных уравнений

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**Аннотация.** В данной статье рассматриваются примеры трансцендентных систем уравнений общего вида. Вычетные интегралы определяются по циклам, связанным с системой. Приведены формулы для их расчета, и установлена связь со степенными суммами корней системы.

**Ключевые слова:** трансцендентные системы уравнений, вычетные интегралы, степенные суммы корней.

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# Colorings of the Graph $K_2^m + K_n$

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**Abstract.** In this paper, we characterize chromatically unique, determine list-chromatic number and characterize uniquely list colorability of the graph $G = K_2^m + K_n$. We shall prove that $G$ is $\chi$-unique, $\mathrm{ch}(G) = m + n$, $G$ is uniquely 3-list colorable graph if and only if $2m + n \geqslant 7$ and $m \geqslant 2$.

**Keywords:** chromatic number, list-chromatic number, chromatically unique graph, uniquely list colorable graph, complete r-partite graph.

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## 1. Introduction and preliminaries

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of $v$ (resp., the degree of $v$ with respect to $W$), denoted by $\deg(v)$ (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of $G$ induced by $W \subseteq V(G)$ is denoted by $G[W]$. The independent sets and complete graphs of order $n$ are denoted by $O_n$ and $K_n$, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called *r-partite graph* if $V$ admits a partition into $r$ classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of $G$ induced by $V_i$, $i = 1, \ldots, r$, is independent set. An $r$-partite graph in which every two vertices from different partition classes are adjacent is called complete $r$-partite graph and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$ . The complete $r$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by $K_s^r$

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. We call $G_1$ and $G_2$ *isomorphic*, and write $G_1 \cong G_2$, if there exists a bijection $f : V_1 \to V_2$ with $uv \in E_1$ if and only if $f(u)f(v) \in E_2$ for all $u, v \in V_1$.

Let $G = (V, E)$ be a graph and $\lambda$ is a positive integer.

A $\lambda$-*coloring* of $G$ is a bijection $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-coloring is called the *chromatic number* of $G$ and is denoted by $\chi(G)$. We say that a graph $G$ is *n-chromatic* if $n = \chi(G)$.

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Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, two $\lambda$-colorings $f$ and $g$ are considered different if and only if $f(v_k) \neq g(v_k)$ for some $k = 1, 2, \ldots, n$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of distinct $\lambda$-colorings of $G$. It is well-known that for any graph $G$, $P(G, \lambda)$ is a polynomial in $\lambda$, called the *chromatic polynomial* of $G$. The notion of chromatic polynomials was first introduced by Birkhoff [3] in 1912 as a quantitative approach to tackle the four-color problem. Two graphs $G$ and $H$ are called *chromatically equivalent* or in short *$\chi$-equivalent*, and we write in notation $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is called *chromatically unique* or in short *$\chi$-unique* if $G' \cong G$ (i.e., $G'$ is isomorphic to $G$) for any graph $G'$ such that $G' \sim G$. For examples, all cycles are $\chi$-unique [8]. The notion of $\chi$-unique graphs was first introduced and studied by Chao and Whitehead [4] in 1978. The readers can see the surveys [8,9] and [12] for more informations about $\chi$-unique graphs. Recently, Ngo Dac Tan and Le Xuan Hung characterized chromatically unique split graphs [12] (A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs of $G$ induced by $I$ and $K$ are independent sets and complete graphs, respectively).

Let $(L_v)_{v \in V}$ be a family of sets. We call a coloring $f$ of $G$ with $f(v) \in L_v$ for all $v \in V$ is a *list coloring from the lists $L_v$*. We will refer to such a coloring as an $L$-coloring. The graph $G$ is called *$\lambda$-list-colorable*, or *$\lambda$-choosable*, if for every family $(L_v)_{v \in V}$ with $|L_v| = \lambda$ for all $v$, there is a coloring of $G$ from the lists $L_v$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-choosable is called the *list-chromatic number*, or *choice number* of $G$ and is denoted by $\mathrm{ch}(G)$. In [7], we characterized list-chromatic number for split graphs, we have proved that if $G$ is a split graphs then $\mathrm{ch}(G) = \chi(G)$.

Let $G$ be a graph with $n$ vertices and suppose that for each vertex $v$ in $G$, there exists a list of $k$ colors $L_v$, such that there exists a unique $L$-coloring for $G$, then $G$ is called a *uniquely $k$-list colorable graph* or a U$k$LC graph for short. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [6] and by Mahmoodian and Mahdian [10] (Mahmoodian and Mahdian have obtained some results on the uniquely $k$-list colorable complete multipartite graphs, for example, they proved that graph $G = O_m + K_n$ is U3LC when $(m, n) \in \{(4, 6), (5, 5), (6, 4)\}$).

Finding a general result for the problems raised above is a difficult task, requiring a lot of time and effort for mathematicians. There have been many interesting and insightful research results on these issues for different graph classes. However, these are still issues that have not been resolved thoroughly, so much more attention is needed. In this paper, we shall characterize chromatically unique, determine list-chromatic number and characterize uniquely list colorability of the graph $G = K_2^m + K_n$. Namely, we shall prove that $G$ is $\chi$-unique (Section 2), $\mathrm{ch}(G) = m+n$ (Section 3), $G$ is U3LC if and only if $2m + n \geqslant 7$ and $m \geqslant 2$ (Section 4). These results contribute to solving the coloring problem for a complete multipartite graph.

## 2. Chromatic uniqueness

We need the following Lemmas 1–4 to prove our results.

**Lemma 1** ([2])**.** *If $K_n$ is a complete graph on $n$ vertices then $\chi(K_n) = n$.*

**Lemma 2.** *If $G = K_{n_1, n_2, \ldots, n_r}$ is a complete $r$-partite graph then $\chi(G) = r$.*

**Lemma 3** ([11])**.** *Let $G$ and $H$ be two $\chi$-equivalent graphs. Then*

*(i) $|V(G)| = |V(H)|$;*

*(ii) $|E(G)| = |E(H)|$;*

*(iii) $\chi(G) = \chi(H)$;*

*(iv) $G$ is connected if and only if $H$ is connected;*

*(v) $G$ is 2-connected if and only if $H$ is 2-connected.*

**Lemma 4.** *Let $G = (V_1 \cup V_2 \cup \ldots \cup V_{m+n}, E)$ be a $(m+n)$-partite graph with $m \geqslant 1$, $n \geqslant 1$, $|V_1| \geqslant |V_2| \geqslant \ldots \geqslant |V_{m+n}|$ and $|V_1| + |V_2| + \ldots + |V_{m+n}| = 2m + n$. Then*

$$|E| \leqslant \frac{(2m+n)^2 - 4m - n}{2}.$$

*In particular,*

$$|E| = \frac{(2m+n)^2 - 4m - n}{2}$$

*if and only if $G$ is a complete $(m+n)$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_{m+n}|}$ with*

$$|V_1| = |V_2| = \ldots = |V_m| = 2,\ |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1.$$

*Proof.* We prove the lemma by induction on $t = m+n$. For $t = 2$ the assertion holds, so let $t > 2$ and assume the assertion for smaller values of $t$. If $|V_{m+n}| \geqslant 2$ then $|V_1| + |V_2| + \ldots + |V_{m+n}| \geqslant 2m + 2n > 2m + n$, a contradiction. So, $|V_{m+n}| = 1$. If $|V_m| \geqslant 3$ then $|V_1| + |V_2| + \ldots + |V_{m+n}| \geqslant 3m + n > 2m + n$, a contradiction. Therefore, $|V_m| \leqslant 2$. Now we consider separately two cases.

*Case 1: There exists $i \in \{1, 2, \ldots, m\}$ such that $|V_i| = 2$.*

Set $G' = G - V_i$. It is clear that $G'$ is a $(m+n-1)$-partite graph

$$(V_1 \cup V_2 \cup \ldots \cup V_{i-1} \cup V_{i+1} \cup \ldots \cup V_{m+n}, E').$$

By the induction hypothesis,

$$|E'| \leqslant \frac{(2(m-1)+n)^2 - 4(m-1) - n}{2}.$$

We have

$$\begin{aligned} |E| &\leqslant |E'| + |V_i|(|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots + |V_{m+n}|) \leqslant \\ &\leqslant \frac{(2(m-1)+n)^2 - 4(m-1) - n}{2} + 2(2m+n-2) = \\ &= \frac{(2m+n)^2 - 4m - n}{2}. \end{aligned}$$

It is not difficult to see that

$$|E| = \frac{(2m+n)^2 - 4m - n}{2}$$

if and only if $G$ is a complete $(m+n)$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_{m+n}|}$ with

$$|V_1| = |V_2| = \ldots = |V_m| = 2, \quad |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1.$$

*Case 2: $|V_i| \neq 2$ for every $i = 1, 2, \ldots, m$.*

In this case, $|V_1| \geqslant 3$. Let $h \in \{1, 2, \ldots, m\}$ such that $|V_h| = 1$ and $|V_{h-1}| \geqslant 3$. Let $G_1 = K_{p_1,p_2,\ldots,p_{m+n}}$ be a complete $(m+n)$-partite graph such that $p_h = |V_h| + 1 = 2$, $p_{h-1} = |V_{h-1}| - 1$ and $p_i = |V_i|$ for every $i \in \{1, 2, \ldots, m+n\} \setminus \{h-1, h\}$. By Case 1,

$$|E(G_1)| \leqslant \frac{(2m+n)^2 - 4m - n}{2}.$$

We have

$$|E(G_1)| = \sum_{1 \leqslant i < j \leqslant m+n} p_i p_j =$$

$$\begin{aligned}
&= \sum_{i,j\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} p_ip_j + \sum_{i\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} p_ip_{h-1} + \\
&+ \sum_{i\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} p_ip_h + p_{h-1}p_h = \\
&= \sum_{i,j\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} |V_i||V_j| + \sum_{i\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} |V_i|(|V_{h-1}|-1) + \\
&+ \sum_{i\in\{1,\ldots,m+n\}\setminus\{h-1,h\}} |V_i|(|V_h|+1) + (|V_{h-1}|-1)(|V_h|+1) = \\
&= \sum_{1\leqslant i<j\leqslant m+n} |V_i||V_j| + |V_{h-1}| - |V_h| - 1 \geqslant \\
&\geqslant |E| + 1.
\end{aligned}$$

It follows that

$$|E| < \frac{(2m+n)^2 - 4m - n}{2}.$$

□

Now we characterize chromatically unique for the graph $G = K_2^m + K_n$.

**Theorem 5.** *The graph $G = K_2^m + K_n$ is $\chi$-unique.*

*Proof.* It is clear that $G$ is a complete $(m+n)$-partite graph $K_{p_1,p_2,\ldots,p_{m+n}}$ with

$$p_1 = p_2 = \ldots = p_m = 2, \quad p_{m+1} = p_{m+2} = \ldots = p_{m+n} = 1.$$

Let $G' = (V', E')$ is a graph such that $G' \sim G$. Since Lemma 2 and (iii) of Lemma 3 we have

$$\chi(G') = \chi(G) = m + n.$$

Let $G'$ has a coloring $f$ using $m+n$ colors $1, 2, \ldots, m+n$. Set

$$V_i' = \{u \in V' \mid f(u) = i\}.$$

for every $i = 1, 2, \ldots, m+n$. It follows that $G'$ is a $(m+n)$-partite graph $(V_1' \cup V_2' \cup \ldots \cup V_{m+n}', E')$. By (i) and (ii) of Lemma 3 we have

$$|V(G')| = |V(G)| = 2m + n, \quad |E(G')| = |E(G)| = \frac{(2m+n)^2 - 4m - n}{2}.$$

Without loss of generality we may

$$|V_1'| \geqslant |V_2'| \geqslant \ldots \geqslant |V_{m+n}'|.$$

By Lemma 4, we have

$$|V_1'| = |V_2'| = \ldots = |V_m'| = 2, \quad |V_{m+1}'| = |V_{m+2}'| = \ldots = |V_{m+n}|' = 1.$$

It follows that $G' \cong G$. Thus $G$ is $\chi$-unique. □

## 3. List-chromatic number

We need the following Lemmas 6–8 to prove our results.

**Lemma 6** ([5])**.** *If $G$ is a graph then* $\mathrm{ch}(G) \geqslant \chi(G)$.

**Lemma 7** ([5])**.** *If $G_1$ is a subgraph of $G_2$ then* $\mathrm{ch}(G_1) \leqslant \mathrm{ch}(G_2)$.

We determine list-chromatic number for complete graphs.

**Lemma 8.** *If $K_n$ is a complete graph on $n$ vertices then* $\mathrm{ch}(K_n) = n$.

Now we determine list-chromatic number for the graph $G = K_2^r$.

**Theorem 9.** *List-chromatic number of $G = K_2^r$ is*

$$\mathrm{ch}(G) = r.$$

*Proof.* By Lemma 2 and Lemma 6, we have $\mathrm{ch}(G) \geqslant r$. Now we prove $\mathrm{ch}(G) \leqslant r$ by induction on $r$. For $r = 1$ the assertion holds, so let $r > 1$ and assume the assertion for smaller values of $r$.

Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ is a partition of $V(G)$ such that for every $i = 1, \ldots, r$, $|V_i| = 2$ and the subgraphs of $G$ induced by $V_i$, is independent set. Set

$$V_i = \{v_{i1}, v_{i2}\}$$

for every $i = 1, \ldots, r$. Let $L_{v_{ij}}$ be the lists of colors of $v_{ij}$ such that $|L_{v_{ij}}| = r$ for every $i = 1, 2, \ldots, r; j = 1, 2$. Now we consider separately two cases.

*Case 1: There exists $i \in \{1, 2, \ldots, r\}$ such that $L_{v_{i1}} \cap L_{v_{i2}} \neq \emptyset$.*

Without loss of generality we may assume that $L_{v_{11}} \cap L_{v_{12}} \neq \emptyset$ and $a \in L_{v_{11}} \cap L_{v_{12}}$. set $G' = G - V_1$. It is clear that $G'$ is a graph $K_2^{r-1}$. Again set

$$L'_{v_{ij}} \subseteq L_{v_{ij}} \setminus \{a\}$$

such that $|L'_{v_{ij}}| = r - 1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

By the induction hypothesis, there exists $(r-1)$-choosable $g$ of $G'$ with the lists of colors $L'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

Let $f$ be the coloring of $G$ such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2$,

$f(v_{1j}) = a$ for every $j = 1, 2$.

Then $f$ is a $r$-choosable for $G$, ie., $\mathrm{ch}(G) \leqslant r$.

*Case 2: $L_{v_{i1}} \cap L_{v_{i2}} = \emptyset$ for every $i = 1, 2, \ldots, r$.*

Let $b \in L_{v_{11}}$. Set $G' = G - V_1 = K_2^{r-1}$ and

$$L'_{v_{ij}} \subseteq L_{v_{ij}} \setminus \{b\}$$

such that $|L'_{v_{ij}}| = r - 1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

By the induction hypothesis, there exists $(r-1)$-choosable $g$ of $G'$ with the lists of colors $L'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$. Since $|L_{v_{11}} \cup L_{v_{12}}| = 2r$ and $|V(G'| = 2(r-1)$, it follows that

$$|(L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))| \geqslant 2.$$

We again divide this case into two subcases.

*Subcase 2.1:* $((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}} \neq \emptyset$.

Let $c \in ((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}}$. Let $f$ be the coloring of $G$ such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2$,

$f(v_{11}) = b$, $f(v_{12}) = c$.

Then $f$ is a $r$-choosable for $G$, ie., $\mathrm{ch}(G) \leqslant r$.

*Subcase 2.2:* $((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}} = \emptyset$.

By $|(L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))| \geqslant 2$, there exists $d \in (L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))$, $d \neq b$. It is clear that $b, d \in L_{v_{11}}$. Since $|L_{v_{12}}| = r$ and $|g(V(G'))| \leqslant 2(r-1)$, there exists $i \in \{2, 3, \ldots, r\}$ such that $g(v_{i1}), g(v_{i2}) \in L_{v_{12}}$. Without loss of generality we may assume that $g(v_{21}), g(v_{22}) \in L_{v_{12}}$. Let $e \in (L_{v_{21}} \cup L_{v_{22}}) \setminus g(V(G'))$. First assume that $e \in L_{v_{21}}$. If $e \neq b$ then coloring $f$ of $G$ such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = b$, $f(v_{12}) = g(v_{21})$.

is a $r$-choosable for $G$. If $e = b$ then coloring $f$ of $G$ such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = d$, $f(v_{12}) = g(v_{21})$.

is a $r$-choosable for $G$. By symmetry, we can show that $\mathrm{ch}(G) \leqslant r$ if $e \in L_{v_{22}}$. □

**Theorem 10.** *List-chromatic number of $G = K_2^m + K_n$ is*

$$\mathrm{ch}(G) = m + n.$$

*Proof.* It is clear that $G = K_2^m + K_n$ is a complete $(m+n)$-partite graph. By Lemma 2 and Lemma 6, we have $\mathrm{ch}(G) \geqslant m+n$. Now we prove $\mathrm{ch}(G) \leqslant m+n$. It is not difficult to see that $G$ is a subgraph of $K_2^{m+n}$. By Lemma 7 and Theorem 9, $\mathrm{ch}(G) \leqslant m+n$. Thus, $\mathrm{ch}(G) = m+n$. □

# 4. Uniquely list colorability

If a graph $G$ is not uniquely $k$-list colorable, we also say that $G$ has property $M(k)$. So $G$ has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size $k$, either there is no list coloring for $G$ or there exist at least two list colorings. The least integer $k$ such that $G$ has the property $M(k)$ is called the *m-number* of $G$, denoted by $m(G)$. This conception was originally introduced by Mahmoodian and Mahdian in [10].

We need the following Lemmas 11–16 to prove our results.

**Lemma 11** ([10])**.** *A connected graph $G$ has the property $M(2)$ if and only if every block of $G$ is either a cycle, a complete graph, or a complete bipartite graph.*

**Lemma 12** ([10])**.** *For every graph $G$ we have $m(G) \leqslant |E(\overline{G})| + 2$.*

**Lemma 13** ([10])**.** *Every UkLC graph has at least $3k - 2$ vertices.*

**Lemma 14.** *If $2m + n = 7$ and $m \geqslant 2$ then $G = K_2^m + K_n$ is U3LC.*

*Proof.* It is clear that $G = K_2^m + K_n$ is a complete $(m+n)$-partite graph. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{m+n}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_m| = 2, |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1$ and for every $i = 1, \ldots, m$ the subgraphs of $G$ induced by $V_i$, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, m$ and $V_{m+i} = \{v_i\}$ for every $i = 1, \ldots, n$. Now we consider separately two cases.

*Case 1: $m = 2$ and $n = 3$.*

We assign the following lists for the vertices of this graph:

$$L_{u_{11}} = \{1,2,3\}, \quad L_{u_{12}} = \{1,4,5\}, \quad L_{u_{21}} = \{1,2,3\}, \quad L_{u_{22}} = \{2,4,5\},$$

$$L_{v_1} = \{1,2,5\}, \quad L_{v_2} = \{1,2,4\}, \quad L_{v_3} = \{1,2,3\}.$$

A unique coloring $f$ exists from the assigned lists:

$$f(u_{11}) = 1, \; f(u_{12}) = 1, \; f(u_{21}) = 2, \; f(u_{22}) = 2,$$

$$f(v_1) = 5,\ f(v_2) = 4,\ f(v_3) = 3.$$

*Case 2: $m = 3$ and $n = 1$.*

We assign the following lists for the vertices of this graph:

$$L_{u_{11}} = \{1,4,5\},\quad L_{u_{12}} = \{2,4,5\},\quad L_{u_{21}} = \{1,2,3\},\quad L_{u_{22}} = \{3,4,5\},$$

$$L_{u_{31}} = \{1,2,4\},\quad L_{u_{32}} = \{3,4,5\},\quad L_{v_1} = \{3,4,5\}.$$

A unique coloring $f$ exists from the assigned lists:

$$f(u_{11}) = 1,\ f(u_{12}) = 2,\ f(u_{21}) = 3,\ f(u_{22}) = 3,$$

$$f(u_{31}) = 4,\ f(u_{32}) = 4,\ f(v_1) = 5.$$

□

**Lemma 15.** *If $m = 2$ and $n \geqslant 3$ then $G = K_2^m + K_n$ is U3LC.*

*Proof.* We prove $G$ is U3LC by induction on $n$. If $n = 3$ then by Lemma 14, $G$ is U3LC. So let $n > 3$ and assume the assertion for smaller values of $n$.

Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{n+2}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = 2$, $|V_3| = |V_4| = \ldots = |V_{n+2}| = 1$ and for every $i = 1,2$ the subgraphs of $G$ induced by $V_i$, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1,2$ and $V_{i+2} = \{v_i\}$ for every $i = 1,\ldots,n$. Set $G' = G - v_n$. By the induction hypothesis, for each vertex $v$ in $G'$, there exists a list of 3 colors $L'_v$, such that there exists a unique $f'$ for $G'$.

We assign the following lists for the vertices of $G$:

$$L_{u_{11}} = L'_{u_{11}},\quad L_{u_{12}} = L'_{u_{12}},\quad L_{u_{21}} = L'_{u_{21}},\quad L_{u_{22}} = L'_{u_{22}},$$

$$L_{v_1} = L'_{v_1},\quad \ldots,\ L_{v_{n-1}} = L'_{v_{n-1}},\quad L_{v_n} = \{f'(v_1), f'(v_2), t\},$$

with $t \notin L'_{u_{11}} \cup L'_{u_{12}} \cup L'_{u_{21}} \cup L'_{u_{22}} \cup L'_{v_1} \cup \ldots \cup L'_{v_{n-1}}$.

A unique coloring $f$ of $G$ exists from the assigned lists: $f(v) = f'(v)$ if $v \in V(G')$, $f(v_n) = t$.

□

**Lemma 16.** *If $m = 3$ and $n \geqslant 1$ then $G = K_2^m + K_n$ is U3LC.*

*Proof.* We prove $G$ is U3LC by induction on $n$. If $n = 1$ then by Lemma 14, $G$ is U3LC. So let $n > 1$ and assume the assertion for smaller values of $n$.

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{n+3}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = |V_3| = 2$, $|V_4| = |V_5| = \ldots = |V_{n+3}| = 1$ and for every $i = 1,2,3$ the subgraphs of $G$ induced by $V_i$, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1,2,3$ and $V_{i+3} = \{v_i\}$ for every $i = 1,\ldots,n$. Set $G' = G - v_n$. By the induction hypothesis, for each vertex $v$ in $G'$, there exists a list of 3 colors $L'_v$, such that there exists a unique $f'$ for $G'$.

We assign the following lists for the vertices of $G$:

$$L_{u_{11}} = L'_{u_{11}},\quad L_{u_{12}} = L'_{u_{12}},\quad L_{u_{21}} = L'_{u_{21}},\quad L_{u_{22}} = L'_{u_{22}},\quad L_{u_{31}} = L'_{u_{31}},\quad L_{u_{32}} = L'_{u_{32}}$$

$$L_{v_1} = L'_{v_1},\ \ldots,\ L_{v_{n-1}} = L'_{v_{n-1}},\quad L_{v_n} = \{f'(v_1), f'(v_2), t\},$$

with $t \notin L'_{u_{11}} \cup L'_{u_{12}} \cup L'_{u_{21}} \cup L'_{u_{22}} \cup L'_{u_{31}} \cup L'_{u_{32}} \cup L'_{v_1} \cup \ldots \cup L'_{v_{n-1}}$.

A unique coloring $f$ of $G$ exists from the assigned lists: $f(v) = f'(v)$ if $v \in V(G')$, $f(v_n) = t$.

□

**Theorem 17.** *$G = K_2^m + K_n$ is U3LC if and only if $2m + n \geqslant 7$ and $m \geqslant 2$.*

*Proof.* Firrst we prove the necessity. Suppose that $G = K_2^m + K_n$ is U3LC. By Lemma 13, $|V(G)| = 2m + n \geqslant 7$. If $m = 1$ then $|E(\overline{G})| = 1$, by Lemma 12, $m(G) \leqslant |E(\overline{G})| + 2 = 3$, a contradiction. Therefore, $m \geqslant 2$.

Now we prove the sufficiency. We prove $G$ is U3LC by induction on $m$. If $m = 2$ then by Lemma 15, $G$ is U3LC. If $m = 3$ then by Lemma 16, $G$ is U3LC. So let $m > 3$ and assume the assertion for smaller values of $m$.

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{m+n}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = = |V_m| = 2$, $|V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1$ and for every $i = 1, 2, \ldots, m$ the subgraphs of $G$ induced by $V_i$, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, m$ and $G' = G - V_m$. By the induction hypothesis, for each vertex $v$ in $G'$, there exists a list of 3 colors $L'_v$, such that there exists a unique $f'$ for $G'$.

We assign the following lists for the vertices of $G$: $L_{u_{m1}} = L_{u_{m2}} = \{f'(u_{11}), f'(u_{21}), t\}$ with $t \notin f'(G')$, $L_v = L'_v$ if $v \in V(G')$.

A unique coloring $f$ of $G$ exists from the assigned lists: $f(u_{m1}) = f(u_{m2}) = t$, $f(v) = f'(v)$ if $v \in V(G')$. □

# 5. Conclusion

The coloring problem, including the list coloring problem, has always been much researched in graph theory because it has many applications in computer science. The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [13]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [13]. The research of list coloring consists of two parts: the choosability and the unique list colorability.

The main results of the paper have identified the list-chromatic number (Theorem 10), characterized chromatically unique (Theorem 5) and characterized uniquely list colorability (Theorem 17) of the graph $G = K_2^m + K_n$. The desire in the future will achieve deeper results on the issues raised in this article.

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# Раскраски графа $K_2^m + K_n$

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**Аннотация.** В этой статье мы характеризуем хроматически уникальное хроматическое число в списке и однозначно характеризуем окрашиваемость графа списка $K_2^m + K_n$. Мы докажем, что $G$ $\chi$ единственно, $\mathrm{ch}(G) = m + n$, $G$ является однозначным трехцветным графом раскраски тогда и только тогда, когда $2m + n \geqslant 7$ и $m \geqslant 2$.

**Ключевые слова:** хроматическое число, хроматический номер списка, хроматически уникальный граф, однозначный список раскрашиваемого графа, полный r-раздельный граф.

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# Exact Solution of 3D Navier–Stokes Equations

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**Abstract.** Procedure for constructing exact solutions of 3D Navier–Stokes equations for an incompressible fluid flow is proposed. It is based on the relations representing the previously obtained first integral of the Navier–Stokes equations. A primary generator of particular solutions is proposed. It is used to obtain new classes of exact solutions.

**Keywords:** incompressible fluid, motion, equation, integral, primary generator of solutions, exact solution.

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## Introduction

The Navier–Stokes equations describe the motion of fluid and gaseous media in the presence of viscosity. These equations are widely used for solving practical problems in various fields. These fields traditionally include hydraulic engineering, oceanology, shipbuilding, aircraft engineering, tribology and cardiology.

The simplest version of the equations corresponds to the case of incompressible fluid motion. In this case the density and all other physical characteristics of the fluid are constant and unknowns are the components of velocity vector $u$, $v$, $w$ and pressure $p$ [1, 2]. In this case the Navier–Stokes equations in dimensionless variables and in the presence of the potential of external forces can be represented as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial (p+\Phi)}{\partial x} + \frac{1}{Re}\Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial (p+\Phi)}{\partial y} + \frac{1}{Re}\Delta v, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial (p+\Phi)}{\partial z} + \frac{1}{Re}\Delta w, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

where $\Delta$ is the 3D Laplace operator with respect to spatial coordinates: $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, $\Phi$ is the potential of external forces, $Re$ is the Reynolds number.

The study of equations (1–4) is one of the directions of modern mathematical physics [3, 4]. However, at present many issues are not fully clarified and they require additional research. One

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of the main problems is the lack of a general constructive method of solution. How to construct solutions of $3D$ Navier–Stokes equations with all non-linear terms? There is no answer to this question yet but practice needs resolution of this issue.

An important step along this path is the construction of exact solutions. Some solutions are known [5–8]. Now broad classes of solutions are of particular interest. Each class of exact solutions introduces new understanding of general laws and to some extent creates a basis for developing methods to construct exact solutions.

## 1. Integral of the Navier–Stokes equations

The procedure for constructing an integral of equations (1–4) was proposed by author [9, 10]. So, the integral is represented by nine relations. In the most simple notation they are

$$p + \Phi + \frac{U^2}{2} + d + d_t = p_0, \tag{5}$$

$$\begin{aligned} u^2 - v^2 + \frac{2}{Re}\Big( - \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) = -\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{10}}{\partial y^2} - \frac{\partial^2 \Psi_{11}}{\partial z^2} - \frac{\partial^2 \Psi_{12}}{\partial z^2} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} + \\ + \frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t}\Big( - \frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_3}{\partial y} + \frac{\partial (\Psi_5 + \Psi_6)}{\partial z}\Big), \end{aligned} \tag{6}$$

$$\begin{aligned} v^2 - w^2 + \frac{2}{Re}\Big( - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) = \frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{11}}{\partial x^2} - \frac{\partial^2 \Psi_{12}}{\partial y^2} + \frac{\partial^2 \Psi_{12}}{\partial z^2} - \frac{\partial^2 \Psi_{13}}{\partial x \partial y} - \\ - \frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t}\Big( \frac{\partial (\Psi_1 + \Psi_2)}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z}\Big), \end{aligned} \tag{7}$$

$$\begin{aligned} uv - \frac{1}{Re}\Big( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) = -\frac{\partial^2 \Psi_{10}}{\partial x \partial y} + \frac{1}{2}\frac{\partial}{\partial z}\Big( - \frac{\partial \Psi_{15}}{\partial x} + \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{13}}{\partial z}\Big) + \\ + \frac{1}{2}\frac{\partial}{\partial t}\Big( - \frac{\partial \Psi_3}{\partial x} - \frac{\partial \Psi_1}{\partial y} - \frac{\partial (\Psi_8 + \Psi_9)}{\partial z}\Big), \end{aligned} \tag{8}$$

$$\begin{aligned} uw - \frac{1}{Re}\Big( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) = \frac{\partial^2 \Psi_{11}}{\partial x \partial z} + \frac{1}{2}\frac{\partial}{\partial y}\Big( - \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{14}}{\partial y} - \frac{\partial \Psi_{13}}{\partial z}\Big) + \\ + \frac{1}{2}\frac{\partial}{\partial t}\Big( - \frac{\partial \Psi_5}{\partial x} + \frac{\partial (\Psi_9 - \Psi_7)}{\partial y} + \frac{\partial \Psi_2}{\partial z}\Big), \end{aligned} \tag{9}$$

$$\begin{aligned} vw - \frac{1}{Re}\Big( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) = -\frac{\partial^2 \Psi_{12}}{\partial y \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\Big( \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{13}}{\partial z}\Big) + \\ + \frac{1}{2}\frac{\partial}{\partial t}\Big( \frac{(\partial \Psi_7 + \Psi_8)}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z}), \end{aligned} \tag{10}$$

$$u = \frac{1}{2}\bigg( \frac{\partial}{\partial y}\Big( - \frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_7}{\partial z}\Big) + \frac{\partial}{\partial z}\Big( - \frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_8}{\partial y} - \frac{\partial \Psi_2}{\partial z}\Big)\bigg), \tag{11}$$

$$v = \frac{1}{2}\bigg( \frac{\partial}{\partial x}\Big( \frac{\partial \Psi_3}{\partial x} - \frac{\partial \Psi_1}{\partial y} - \frac{\partial \Psi_7}{\partial z}\Big) + \frac{\partial}{\partial z}\Big( \frac{\partial \Psi_9}{\partial x} + \frac{\partial \Psi_6}{\partial y} - \frac{\partial \Psi_4}{\partial z}\Big)\bigg), \tag{12}$$

$$w = \frac{1}{2}\bigg( \frac{\partial}{\partial x}\Big( \frac{\partial \Psi_5}{\partial x} - \frac{\partial \Psi_8}{\partial y} + \frac{\partial \Psi_2}{\partial z}\Big) + \frac{\partial}{\partial y}\Big( - \frac{\partial \Psi_9}{\partial x} - \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z}\Big)\bigg). \tag{13}$$

Functions $\Psi_j$ denote new unknowns that arise in the process of integration. In the case being considered there are fifteen functions and they complete the system of unknowns. The term "stream pseudo-function" was introduced for them [9, 10]. Thus, a total of nineteen unknowns are introduced, namely, four major unknowns and fifteen associated unknowns.

Relation (5) contains additional terms $p_0$, $\frac{U^2}{2}$, $d$ and $d_t$. The first one is an additive pressure constant. Another three terms represent combinations of unknowns defined in a special way. Value $\frac{U^2}{2}$ is dimensionless velocity

$$\frac{U^2}{2} = \frac{u^2 + v^2 + w^2}{2}.$$

Values $d$ and $d_t$ are dissipative terms defined as

$$d = -\frac{U^2}{6} - \frac{1}{3}\left(\Delta_{xy}\Psi_{10} - \Delta_{xz}\Psi_{11} + \Delta_{yz}\Psi_{12} + \frac{\partial^2\Psi_{13}}{\partial x\partial y} - \frac{\partial^2\Psi_{14}}{\partial x\partial z} + \frac{\partial^2\Psi_{15}}{\partial y\partial z}\right), \tag{14}$$

$$d_t = \frac{1}{3}\frac{\partial}{\partial t}\left(\frac{\partial(\Psi_2 - \Psi_1)}{\partial x} + \frac{\partial(\Psi_4 - \Psi_3)}{\partial y} + \frac{\partial(\Psi_6 - \Psi_5)}{\partial z}\right). \tag{15}$$

Symbols $\Delta_{yz}$, $\Delta_{xz}$, $\Delta_{xy}$ in (14) denote the incomplete Laplace operators with respect to spatial coordinates

$$\Delta_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{xz} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Relations (5–13) include the major unknowns $u$, $v$, $w$, $p$, the associated unknowns $\Psi_j$, given potential function of external forces $\Phi$ and the Reynolds number $Re$. The order of derivatives for major unknowns is one. It is less than their order in original equations (1–4). Relations (5–13) represent the first integral of the Navier – Stokes equations (1–4).

The integral of equations (1–4) in the form (5–13) allows us to construct exact solutions in a new way.

## 2. Primary generator of solutions

The primary generator of solutions allows us to construct the set of solutions of original equations (1–4). One such primary generator is presented below.

Let us briefly analyze relations (5–13) that represent the first integral of the Navier–Stokes equations. Relations (5) and (11–13) give expressions for the major unknowns $u, v, w, p$ in terms of associated unknowns $\Psi_j$, where $j = 1, 2, \ldots, 15$. It is fair to conclude that these four relations determine general structure of solutions for equations (1–4).Let us note that unknowns $u, v, w$ defined by (11–13) satisfy continuity equation (4). Relation (5) is special because it contains the unknown $p$. In the way of practical solution of equations this relation should be used at the last stage when all other unknowns have been already found.

When considering relations (6–13) in general, the following features attract attention [11]. In the right-hand sides of (11–13) there are derivatives of only the first nine associated unknowns $\Psi_k$, $k = 1, 2, \ldots 9$ but there are fifteen associated unknowns in total. Unknowns $\Psi_k$ with $k = 10, 11, \ldots, 15$ do not appear in relations (11–13). These unknowns are present in relations (6–10) in the form of linear combinations of second derivatives. It is possible to exclude these unknowns from (6–7) and to obtain general relations. The procedure for constructing such relations is briefly described below.

Let us denote the sums of all terms in (6–10) that are independent of unknowns $\Psi_k$, $k = 1, 2, \dots, 9$, by $f_j$ $(j = 2, 3, \dots 6)$. So we have

$$
\begin{aligned}
f_2 &= u^2 - v^2 + \frac{2}{Re}\Big(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial t}\Big(\frac{\partial \Psi_1}{\partial x} - \frac{\partial \Psi_3}{\partial y} - \frac{\partial \Psi_5}{\partial z} - \frac{\partial \Psi_6}{\partial z}\Big),\\
f_3 &= v^2 - w^2 + \frac{2}{Re}\Big(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) - \frac{\partial}{\partial t}\Big(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z}\Big),\\
f_4 &= uv - \frac{1}{Re}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) + \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_8}{\partial z} + \frac{\partial \Psi_9}{\partial z}\Big),\\
f_5 &= uw - \frac{1}{Re}\Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) + \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_7}{\partial y} - \frac{\partial \Psi_9}{\partial y} - \frac{\partial \Psi_2}{\partial z}\Big),\\
f_6 &= vw - \frac{1}{Re}\Big(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) - \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial \Psi_7}{\partial x} + \frac{\partial \Psi_8}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z}\Big).
\end{aligned} \tag{16}
$$

As a result, five non-linear equations (6–10) are represented in the form

$$
-\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{10}}{\partial y^2} - \frac{\partial^2 \Psi_{11}}{\partial z^2} - \frac{\partial^2 \Psi_{12}}{\partial z^2} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} + \frac{\partial^2 \Psi_{14}}{\partial x \partial z} = f_2, \tag{17}
$$

$$
\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{11}}{\partial x^2} - \frac{\partial^2 \Psi_{12}}{\partial y^2} + \frac{\partial^2 \Psi_{12}}{\partial z^2} - \frac{\partial^2 \Psi_{13}}{\partial x \partial y} - \frac{\partial^2 \Psi_{14}}{\partial x \partial z} = f_3, \tag{18}
$$

$$
-\frac{\partial^2 \Psi_{10}}{\partial x \partial y} + \frac{1}{2}\frac{\partial}{\partial z}\Big(-\frac{\partial \Psi_{15}}{\partial x} + \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{13}}{\partial z}\Big) = f_4, \tag{19}
$$

$$
\frac{\partial^2 \Psi_{11}}{\partial x \partial z} + \frac{1}{2}\frac{\partial}{\partial y}\Big(-\frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{14}}{\partial y} - \frac{\partial \Psi_{13}}{\partial z}\Big) = f_5, \tag{20}
$$

$$
-\frac{\partial^2 \Psi_{12}}{\partial y \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\Big(\frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{13}}{\partial z}\Big) = f_6. \tag{21}
$$

Let us eliminate terms with unknowns $\Psi_k$ for $k = 10, 11, \dots, 15$. To do this we take term by term derivatives of (17–21) with respect to coordinates and then select the necessary linear combinations to exclude terms with the specified unknown. As a result, terms with unknowns $\Psi_k$ at $k = 10, 11, \dots, 15$ are excluded from (17–18). Then we obtain two equations [11]

$$
\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_4}{\partial x^2} + \frac{\partial^2 f_4}{\partial y^2} + \frac{\partial^2 f_5}{\partial y \partial z} - \frac{\partial^2 f_6}{\partial x \partial z} = 0, \tag{22}
$$

$$
\frac{\partial^2 f_3}{\partial y \partial z} + \frac{\partial^2 f_4}{\partial x \partial z} - \frac{\partial^2 f_5}{\partial x \partial y} - \frac{\partial^2 f_6}{\partial y^2} + \frac{\partial^2 f_6}{\partial z^2} = 0. \tag{23}
$$

Taking into account (16), it is clear that only nine unknowns are present in equations (22–23). These unknowns are $\Psi_k$ with $k = 1, 2, \dots, 9$. This fact is obvious since $u, v, w$ are expressed in terms of these unknowns, according to (11–13). So, system of two equations (22–23) can be considered as primary generator of solutions of $3D$ Navier–Stokes equations (1–4). Any set of functions $\Psi_1$, $\Psi_2, \dots,$ $\Psi_9$ that satisfy this system allows one to determine all other unknowns including the major ones. Firstly $u$, $v$, $w$ are found according to (11–13). Then using (16), $f_j$ are defined for $j = 2, 3, \dots, 6$. Next, six unknowns $\Psi_{10}$, $\Psi_{11}$, $\dots$, $\Psi_{15}$ are determined with the help of (17–21). Finally, using relation (5) and taking into account (14–15), we determine unknown $p$. As a result, all major unknowns are determined and the main problem is solved.

# 3. Method implementation

As an example of the implementation of the described approach we construct a set of solutions that correspond to a cascade of plane waves into deep water provided $\Phi = 0$. Let us assume that unknowns $u, v, w$ are represented in complex variables as linear combinations of plane waves. General structure for $u$, $v$, $w$ is defined by relations (11–13). Let us assume that

$$\begin{aligned} &\frac{\partial \Psi_1}{\partial y} - \frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_7}{\partial z} = A(t) e^{i(n_1 x + m_1 y + l_1 z)}, \\ &-\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_8}{\partial y} - \frac{\partial \Psi_2}{\partial z} = B(t) e^{i(n_2 x + m_2 y + l_2 z)}, \\ &\frac{\partial \Psi_9}{\partial x} + \frac{\partial \Psi_6}{\partial y} - \frac{\partial \Psi_4}{\partial z} = C(t) e^{i(n_3 x + m_3 y + l_3 z)}, \end{aligned} \tag{24}$$

where $i$ is the imaginary unit, $n_k, m_k, l_k,\ (k = 1, 2, 3)$ are some constants and $A(t)$, $B(t)$, $C(t)$ are some functions of time.

Taking into account (11–13), we obtain the following expressions

$$\begin{aligned} u &= \frac{i}{2}(A m_1 e^{i(n_1 x + m_1 y + l_1 z)} + B l_2 e^{i(n_2 x + m_2 y + l_2 z)}), \\ v &= \frac{i}{2}(-A n_1 e^{i(n_1 x + m_1 y + l_1 z)} + C l_3 e^{i(n_3 x + m_3 y + l_3 z)}), \\ w &= \frac{i}{2}(-B n_2 e^{i(n_2 x + m_2 y + l_2 z)} - C m_3 e^{i(n_3 x + m_3 y + l_3 z)}). \end{aligned} \tag{25}$$

So, $u$, $v$, $w$ are defined by (25), where $n_k, m_k, l_k$ for $k = 1, 2, 3$ are still unknown wave numbers and $A(t)$, $B(t)$, $C(t)$ are indeterminate functions of time.

Let us consider primary generator of solutions (22–23) and find the restrictions imposed on these equations.

Substituting (16) into (22–23) and taking into account (24) and (25), we obtain the following results. Components of two kinds are present in (22–23). Components of the first kind are linear combinations of quantities $e^{i(n_1 x + m_1 y + l_1 z)}$, $e^{i(n_2 x + m_2 y + l_2 z)}$, $e^{i(n_3 x + m_3 y + l_3 z)}$. Components of the second kind are quadratic combinations of quantities $e^{i(n_1 x + m_1 y + l_1 z)}$, $e^{i(n_2 x + m_2 y + l_2 z)}$, $e^{i(n_3 x + m_3 y + l_3 z)}$. Components of the first kind are mutually reduced if functions $A(t), B(t), C(t)$ satisfy the ordinary differential equations of the first order

$$\begin{aligned} \frac{dA}{dt} &= -\frac{A}{Re}\big(n_1^2 + m_1^2 + l_1^2\big), \\ \frac{dB}{dt} &= -\frac{B}{Re}\big(n_2^2 + m_2^2 + l_2^2\big), \\ \frac{dC}{dt} &= -\frac{C}{Re}\big(n_3^2 + m_3^2 + l_3^2\big). \end{aligned} \tag{26}$$

Components of the second kind are also mutually reduced and equations (22–23) are identically satisfied if wave numbers $n_k$, $m_k$, $l_k$ satisfy the following system of six algebraic equations

$$\begin{aligned}
&2n_1l_3(n_1+n_3)(m_1+m_3)-m_1l_3(n_1+n_3)^2+m_1l_3(m_1+m_3)^2-\\
&\qquad -m_1m_3(m_1+m_3)(l_1+l_3)-n_1m_3(n_1+n_3)(l_1+l_3)=0,\\
&-m_3l_2(m_2+m_3)(l_2+l_3)+n_2l_3(n_2+n_3)(l_2+l_3)-l_2l_3(n_2+n_3)^2+l_2l_3(m_2+m_3)^2=0,\\
&2m_1l_2(n_1+n_2)(m_1+m_2)+n_1l_2(n_1+n_2)^2-n_1l_2(m_1+m_2)^2-\\
&\qquad -m_1n_2(m_1+m_2)(l_1+l_2)-n_1n_2(n_1+n_2)(l_1+l_2)=0,\\
&-n_1l_2(n_1+n_2)(l_1+l_2)+m_1n_2(n_1+n_2)(m_1+m_2)-\\
&\qquad -n_1n_2(m_1+m_2)^2+n_1n_2(l_1+l_2)^2=0,\\
&-2n_2m_3(m_2+m_3)(l_2+l_3)+n_2l_3(m_2+m_3)^2-n_2l_3(l_2+l_3)^2+\\
&\qquad +l_2l_3(n_2+n_3)(l_2+l_3)+m_3l_2(n_2+n_3)(m_2+m_3)=0,\\
&-2n_1l_3(m_1+m_3)(l_1+l_3)+m_1l_3(n_1+n_3)(l_1+l_3)+m_1m_3(n_1+n_3)(m_1+m_3)-\\
&\qquad -n_1m_3(m_1+m_3)^2+n_1m_3(l_1+l_3)^2=0.
\end{aligned}\tag{27}$$

Solutions of equations (26) are easy to find. They are defined by expressions

$$A(t)=A(0)e^{-\frac{(n_1^2+m_1^2+l_1^2)t}{Re}},\quad B(t)=B(0)e^{-\frac{(n_2^2+m_2^2+l_2^2)t}{Re}},$$
$$C(t)=C(0)e^{-\frac{(n_3^2+m_3^2+l_3^2)t}{Re}},\tag{28}$$

where $A(0)$, $B(0)$, $C(0)$ are arbitrary constants.

Preliminary analysis of system (27) shows that it has many real and complex solutions . Each set of numbers that satisfy (27) generates a solution of Navier–Stokes equations (1–4). Some special cases are presented below. Each of them can be considered as an implementation of the above approach.

# 4. Special cases

**4.1. Solution 1.** The simplest solution corresponds to the case when the wave vectors are collinear. In this case $n_3, m_3, l_3$ are arbitrary and not all equal to zero. In addition, the following proportionality relations are fulfilled $n_1=\mu n_3,\ m_1=\mu m_3,\ l_1=\mu l_3,\ n_2=\xi n_3,\ m_2=\xi m_3,$ $l_2=\xi l_3$, where $\mu$ and $\xi$ have arbitrary but not equal to zero values. In this case all six equations (27) are identically satisfied.

According to (25) and (28), expressions for velocities are

$$\begin{aligned}
u&=\frac{i}{2}\Big(A(0)\mu m_3e^{-\frac{\mu^2(n_3^2+m_3^2+l_3^2)}{Re}t+i\mu(n_3x+m_3y+l_3z)}+B(0)\xi l_3e^{-\frac{\xi^2(n_3^2+m_3^2+l_3^2)}{Re}t+i\xi(n_3x+m_3y+l_3z)}\Big),\\
v&=\frac{i}{2}\Big(-A(0)\mu n_3e^{-\frac{\mu^2(n_3^2+m_3^2+l_3^2)}{Re}t+i\mu(n_3x+m_3y+l_3z)}+C(0)l_3e^{-\frac{(n_3^2+m_3^2+l_3^2)}{Re}t+i(n_3x+m_3y+l_3z)}\Big),\\
w&=\frac{i}{2}\Big(-B(0)\xi n_3e^{-\frac{\xi^2(n_3^2+m_3^2+l_3^2)}{Re}t+i\xi(n_3x+m_3y+l_3z)}-C(0)m_3e^{-\frac{(n_3^2+m_3^2+l_3^2)}{Re}t+i(n_3x+m_3y+l_3z)}\Big).
\end{aligned}\tag{29}$$

According to (5), the unknown $p$ is

$$p=p_0.\tag{30}$$

**4.2. Solution 2.** Analysis of algebraic equations (27) leads to the conclusion that system admits the following solution $n_1 = n_2 = 0,\ n_3 = \sqrt{3},\ m_1 = m_2 = m_3 = \dfrac{1}{\sqrt{2}},\ l_1 = l_2 = -2,\ l_3 = 1.$ In this case unknowns $u$, $v$, $w$ are defined as

$$u = \frac{i}{2}\left(\frac{1}{\sqrt{2}}A(0) - 2B(0)\right)e^{-\frac{9}{2Re}t + i\left(\frac{1}{\sqrt{2}}y - 2z\right)}, \quad v = \frac{i}{2}C(0)e^{-\frac{9}{2Re}t + i\left(\sqrt{3}x + \frac{1}{\sqrt{2}}y + z\right)},$$
$$w = -\frac{i}{2\sqrt{2}}C(0)e^{-\frac{9}{2Re}t + i\left(\sqrt{3}x + \frac{1}{\sqrt{2}}y + z\right)}. \tag{31}$$

According to (5), the unknown $p$ is defined as

$$p = p(0) + \frac{1}{4}C(0)\left(\frac{\sqrt{3}}{2}A(0) - \sqrt{6}B(0)\right)e^{-\frac{9}{2Re}t + i(\sqrt{3}x + \sqrt{2}y - z)}. \tag{32}$$

**4.3. Solution 3.** Equations (27) are also satisfied if $n_1 = n_2 = 0$, $n_3 = i\sqrt{3},\ m_1 = m_2 = m_3 = i\sqrt{3},\ l_1 = l_2 = 1,\ l_3 = 2.$

The velocities in this case are defined as

$$u = \frac{i}{2}\left(i\sqrt{3}A(0) + B(0)\right)e^{\frac{2}{Re}t - \sqrt{3}y + iz}, \quad v = iC(0)e^{\frac{2}{Re}t - \sqrt{3}x - \sqrt{3}y + 2iz},$$
$$w = \frac{1}{2}\sqrt{3}C(0)e^{\frac{2}{Re}t - \sqrt{3}x - \sqrt{3}y + 2iz}. \tag{33}$$

For pressure we have the following expression

$$p = p(0) + \frac{1}{4}C(0)\left(i\sqrt{3}A(0) + B(0)\right)e^{\frac{4}{Re}t - \sqrt{3}x - 2\sqrt{3}y + 3iz}. \tag{34}$$

# Conclusion

As a result of the implementation of the proposed approach new complex solutions of $3D$ Navier–Stokes equations (1–4) are obtained. They are defined by expressions (29–34).

Let us pay attention to the qualitative differences of the obtained solutions. Let us consider coefficients at the time $t$ in Solution 1 and Solution 2. The following inequalities are true for these coefficients: $-\dfrac{\mu^2(n_3^2 + m_3^2 + l_3^2)}{Re} < 0$, $-\dfrac{\xi^2(n_3^2 + m_3^2 + l_3^2)}{Re} < 0$ for Solution 1 and $-\dfrac{9}{Re2} < 0$ for Solution 2. For Solution 3 we have $\dfrac{2}{Re} > 0$. Then Solution 1 and Solution 2 decay exponentially with time. On the contrary, Solution 3increases exponentially with time. This pattern holds for both pressure and the magnitude of velocity.

The following fact is also worth attention.The pressure increases in half the time by comparison to the magnitude of velocity.

If we compare expressions for pressure (30), (32) for Solution 1 and Solution 2 then there is also a qualitative difference. The pressure does not depend on coordinates for Solution 1 whereas the pressure depends on $x$, $y$ and $z$ for Solution 2.

Let us pay attention to another interesting feature of the proposed method for constructing solutions. The above relations allow us to construct purely real solutions of the Navier–Stokes equations. To do this, let us consider relations (24–26) and assume that $n_k = -iN_k$, $m_k = -iM_k$, $l_k = -iL_k$, where $k = 1, 2, 3,$ $i$ is the imaginary unit and $N_k, M_k, L_k$ are real numbers. In this case algebraic equations (27) retain their form but we should take $N_k, M_k, L_k$ instead of $n_k, m_k, l_k$. Any set of real numbers $(N_1, M_1, L_1),\ (N_2, M_2, L_2),\ (N_3, M_3, L_3)$ that satisfies equations (27) allows us to construct purely real solutions of Navier–Stokes equations (1–4).

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## Точные решения 3D-уравнений Навье–Стокса

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**Аннотация.** В работе предложена процедура построения точных решений 3D-уравнений Навье–Стокса для несжимаемой жидкости. За основу принимаются соотношения, представляющие первый интеграл уравнений Навье–Стокса, ранее полученные автором. Построен первичный генератор частных решений, и с его помощью найдены новые классы точных решений.

**Ключевые слова:** несжимаемая жидкость, движение, уравнение, интеграл, первичный генератор решений, точное решение.

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# Enumeration of Bi-Commutative–AG-groupoids

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**Abstract.** In this paper, we introduce (left, right) bi-commutative AG-groupoids and provide a simple method to test whether an arbitrary AG-groupoid is bi-commutative AG-groupoid or not. We also explore some of the general properties of these AG-groupoids. Further we introduce and study some properties of ideals in these AG-groupoids and decompose left commutative AG-groupoids by defining some congruences on these AG-groupoids.

**Keywords:** AG-groupoid, Bi-Commutative–AG-groupoids, (left, right) commutative AG-groupoids.

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## 1. Introduction and preliminaries

An AG-groupoid $S$ is in general a non-associative groupoid that satisfies the left invertive law,

$$(ab)c = (cb)a \quad \forall a, b, c \in S. \tag{1.1}$$

It is called medial if satisfies the medial property, $(ab)(cd) = (ac)(bd) \, \forall a, b, c, d \in S$. It is easy to prove that every AG-groupoid is medial [1]. An AG-grouoid is called an AG-monoid if it contains the left identity element. Every AG-monoid is paramedial [2], i.e., it satisfies the identity, $(ab)(cd) = (db)(ca)$. Recently many new classes of AG-groupoids have been introduced by various researchers [3–9]. These new classes are studied in a variety of papers like for instance [10–15]. AG-groupoid is a vast field of algebra that can have almost all concepts of other algebraic structures with different characteristics and properties. A rapid research in this area can be seen on various aspects in a couple of years. AG-groupoids have a range of applications in flocks theory [1], geometry [16], topology [17], matrices [18] and in finite mathematics [19]. The structure of AG-groupoid has been strengthen by AG-rings [20, 21]. Recently many varieties of ideals, Γ-ideals, bi-ideals prime ideals, semiprime ideals and quasiprime ideals have also been

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defined and investigated by various researchers [22–25]. Fuzzification of AG-groupoids [26,27] and other relevant concepts have also made the field interesting and valuable. All this have attracted a considerable researchers to investigate and enhance the area.

A groupoid $G$ is called left (resp. right) commutative groupoid if $G$ satisfies the identity $(ab)c = (ba)c$ (resp. $a(bc) = a(cb)$) $\forall\, a,b,c \in G$ [28]. In this article, we extend the concept of these groupoids to introduce new classes of AG-groupoids as left commutative AG-groupoid or shortly an LC-AG-groupoid, a right commutative AG-groupoid or an RC-AG-groupoid, and a Bi-commutative AG-groupoid or BC-AG-groupoid. In Section 2, we properly define these notions and list some non-associative examples of these AG-groupoids to show their existence. In Subsection 2.1 and 2.2 we provide a method to verify an arbitrary AG-groupoid for LC-AG-groupoid and RC-AG-groupoid. We use the GAP software [29] and the relevant data of [29] to enumerate these new classes of AG-groupoids up to order 6. We discuss the enumeration of these AG-groupoids in Section 3. In Section 4, we define and characterize ideals of these AG-groupoids, while in Section 5, we investigate some basic properties of these AG-groupoids and establish their relations with some of the already known AG-groupoids. We list these known subclasses of AG-groupoids with their defining identities in Tab. 1, that arise in various papers like, [16,18,19] and are used in the rest of this article.

Table 1. AG-groupoid with their defining identities

| AG-groupoid | satisfying identity |
|---|---|
| Left nuclear square AG-groupoid | $a^2 \cdot bc = a^2 b \cdot c$ |
| Middle nuclear square AG-groupoid | $ab^2 \cdot c = a \cdot b^2 c$ |
| Right nuclear square AG-groupoid | $ab \cdot c^2 = a \cdot bc^2$ |
| $T^1$-AG-groupoid | $ab = cd \Rightarrow ba = dc$ |
| Medial AG-groupoid | $ab \cdot cd = ac \cdot bd$ |
| Paramedial AG-groupoid | $ab \cdot cd = db \cdot ca$ |
| Flexible -AG-groupoid | $ab \cdot a = a \cdot ba$ |
| AG-3-band | $a \cdot aa = aa \cdot a = a$ |
| Left alternative AG-groupoid | $aa \cdot b = a \cdot ab$ |
| Self-dual AG-groupoid | $a \cdot bc = c \cdot ba$ |
| AG*-groupoid | $ab \cdot c = b \cdot ac$ |
| AG**-groupoid | $a \cdot bc = b \cdot ac$ |

## 2. Bi-commutative-AG-groupoids and Bi-commutative AG-test

We extend the concept of bi-commutativity of groupoid [28] to AG-groupoid and introduce the following subclasses of AG-groupoids.

**Definition 1.** *An AG-groupoid $S$ is called*

*1. – a left commutative AG-groupoid (LC-AG-groupoid) if $\forall a,b,c \in S$,*

$$(ab)c \quad = \quad (ba)c \tag{2.1}$$

*2. – a right commutative AG-groupoid (RC-AG-groupoid) if $\forall a,b,c \in S$,*

$$a(bc) \quad = \quad a(cb) \tag{2.2}$$

*3. – a bi-commutative AG-groupoid (BC-AG-groupoid) if it is both LC-AG-groupoid and an RC-AG-groupoid.*

**Example 1.** *Let $S = \{1, 2, 3, 4\}$. Then one can easily verify that:*

*(i)* $(S, \cdot)$ *in table* $(i)$ *is an LC-AG-groupoid of order* 4 *and satisfies Equation 2.1,*

*(ii)* $(S, *)$ *in table* $(ii)$ *is an RC-AG-groupoid of order* 4 *that satisfies Equation 2.2 and*

*(iii)* $(S, \circ)$ *in table* $(iii)$ *is BC-AG-groupoid of order* 4 *and satisfies both the properties of (2.1) and (2.2).*

| $\cdot$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 3 |
| 2 | 1 | 1 | 4 | 3 |
| 3 | 3 | 3 | 1 | 1 |
| 4 | 3 | 3 | 1 | 1 |

| $*$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 3 |
| 3 | 1 | 1 | 1 | 1 |
| 4 | 2 | 2 | 2 | 1 |

| $\circ$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 2 |
| 2 | 2 | 1 | 1 | 1 |
| 3 | 2 | 1 | 1 | 1 |
| 4 | 3 | 1 | 1 | 1 |

The procedure of testing a groupoid for an AG-groupoid has been explained by P. V. Protic and N. Stevanovic [24]. Here we also present a similar method to verify an arbitrary AG-groupoid for LC and RC-AG-groupoids.

## 1. Left Commutative AG-groupoid Test

We describe a procedure to test whether an arbitrary AG-groupoid $(G, \cdot)$ is an LC-AG-groupoid or not. For this we define the following binary operations:

$$a \circ b = (ab)x \tag{2.3}$$
$$a \star b = (ba)x \tag{2.4}$$

Now (2.1) holds if

$$a \circ b = a \star b \tag{2.5}$$

or

$$a \circ b = b \circ a \tag{2.6}$$

To test whether an arbitrary AG-groupoid is an LC-AG-groupoid, it is necessary and sufficient to check if the operation " $\circ$ " and " $\star$ " coincide $\forall x \in G$. To this end we check the validity of Identity (2.1) or $a \circ b = a \star b$. In other words it is enough to check whether the operation $\circ$ is commutative i.e. $a \circ b = b \circ a$. The tables of the operation " $\circ$ " for any fixed $x \in G$ is obtained by multiplying a fixed element $x \in G$ by the elements of the " $\cdot$ " table row-wise. It further gives the tables of the operation " $\star$ " if these are symmetric along the main diagonal. Hence it could easily be checked whether an arbitrary AG-groupoid is left commutative AG-groupoid or not. We illustrate this procedure with the following example.

**Example 2.** *Check the following AG-groupoids $(G_1, \cdot)$ and $(G_2, \cdot)$ for an LC-AG-groupoid.*

Table 2

| $\cdot$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 |

Table 3

| $\cdot$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 3 | 1 | 2 |
| 3 | 2 | 3 | 1 |

We extend Tab. 2 in the way as described above. It is obvious that the tables constructed for the operation " $\circ$ " on the right of the original table are symmetric about the main diagonal and thus coincide with the " $\star$ " tables as required. Hence $(G_1, \cdot)$ is an LC-AG-groupoid. While in extended table for Tab. 3 is not symmetric about the main diagonal and thus $(G_2, \cdot)$ is not an LC-AG-groupoid.

(i)

| $\cdot$ | 1 | 2 | 3 | 1 | | | 2 | | | 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Extended table for $(G_1, \cdot)$

(ii)

| $\cdot$ | 1 | 2 | 3 | 1 | | | 2 | | | 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 1 | 3 | 2 | 2 | 1 | 3 | 3 | 2 | 1 |
| 2 | 3 | 1 | 2 | 2 | 1 | 3 | 3 | 2 | 1 | 1 | 3 | 2 |
| 3 | 2 | 3 | 1 | 3 | 2 | 1 | 1 | 3 | 2 | 2 | 1 | 3 |

Extended table for $(G_2, \cdot)$

## 2. Right Commutative AG-groupoid Test

Now, we discuss a procedure to check an AG-groupoid $(G, \cdot)$ for RC-AG-groupoid, for this we define the following two binary operations:

$$a \heartsuit b = a(bx) \tag{2.7}$$
$$a \diamondsuit b = a(xb) \tag{2.8}$$

Equation (2.2) holds if,

$$a \heartsuit b = a \diamondsuit b \tag{2.9}$$

For any fixed $x \in G$, re-writing $x$-row of the " $\cdot$ " table as an index row of the new table and multiplying it by the elements of the index column to construct table of operation "$\diamondsuit$". These extended tables are given to the right of the original table in the following example. Similarly the table for the operation "$\heartsuit$" for any fixed $x \in G$ is obtained by taking the elements of $x$-column of the " $\cdot$ " table as an index row of the new table and multiplying it by the elements of the index column of the original table to construct tables for the operation "$\heartsuit$", which are given downward in the extended table of the following example. If the tables for the operation "$\heartsuit$" and "$\diamondsuit$" coincides for all $x \in G$, then Equation (2.9) holds and the AG-groupoid is right commutative-AG-groupoid in this case.

**Example 3.** *Check the following AG-groupoid for RC-AG-groupoid.*

| $\cdot$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 |

Extend the above table in the way as described we get the extended form as follows:

| · | 1 | 2 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

It is clear from the extended table that the tables for the operations "$\heartsuit$" and "$\diamondsuit$" coincide for every $x \in G$, so $(G, \cdot)$ is an RC-AG-groupoid.

## 3. Enumeration of BC-AG-groupoids

Enumeration and classification of various mathematical entries is a well worked area of research in finite and pure mathematics. In abstract algebra the classification of algebraic structure is an important pre-requisite for their construction. The classification of finite simple groups is considered as one of the major intellectual achievement of twentieth century. Enumeration results can be obtained by a variety of means like; combinatorial or algebraic consideration. Non-associative structures, quasigroup and loops have been enumerated up to size 11 using combinatorial consideration and bespoke exhaustive generation software [30]. FINDER (Finite domain enumeration) [31] has been used for enumeration of IP loops up to size 13 [32]. Associative structures, semigroups and monoids have been enumerated up to size 9 and 10 respectively by constraint satisfaction techniques implemented in the Minion constraint solver with bespoke symmetry breaking provided by the computer algebra system GAP [29]. The third author of this article has implemented the same techniques under the guidance of A. Distler (the author of [33–35]) to deal the enumeration of AG-groupoids using the constraint solving techniques developed for semigroups and monoids.

Further, they provided a simple enumeration of the structures by the constraint solver and obtained a further division of the domain into a subclass of AG-groupoids using the computer algebra system GAP and were able to enumerate all AG-groupoids up to isomorphism up to size 6. They also presented enumeration for various other subclasses of AG-groupoids.

It is worth mentioning that most of the data presented in [36] has been verified by one of the reviewers of the said article with the help of Mace4 and Isofilter as has been mentioned in the acknowledgement of the said article. All this validate the enumeration and classification results for our bi-commutative AG-groupoids, as the same technique and relevant data of [36] has been used for the purpose. Further, all the tables of size up to 3 have been verified manually for our BC-AG-groupoids. In the following we describe the used algorithms with GAP commands for enumeration of our subclasses of AG-groupoid.

**Algorithm 1.** ***GAP Function for Testing if $S$ is an LC-AG-groupoid***

```
InstallMethod (IsLCAGGroupoidTable, "for matrix,"
[IsMatrix]
function (ls),
local i, j, k;
if not IsAGGroupoidTable (ls) then
return false;
fi;
for i in [1..Length (ls)]do
      for j in [1..Length (ls)] do
            for k in [1..Length (ls)] do
if ls[ls[i][j]][k] <> ls[ls[j][i]][k] then return false;
fi;
```

```
od;
    od;
        od;
return true;
end );
```

**Algorithm 2.** ***GAP Function for Testing if $S$ is an RC-AG-groupoid***

```
InstallMethod (IsRCAGGroupoidTable, "for matrix,"
[IsMatrix]
function (ls),
local i, j, k;
if not IsAGGroupoidTable (ls) then
return false;
fi;
for i in [1..Length(ls)] do
    for j in [1..Length(ls)] do
        for k in [1..Length(ls)] do
if ls[i][ls[j][k]] <> ls[i][ls[k][j]] then return false;
fi;
od;
    od;
        od;
return true;
end );
```

Tab. 4 presents the enumeration of BC-AG-groupoids of order 3 to 6.

Table 4. Enumeration of BC-AG-groupoids up to order 6

| Order | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| Total AG-groupoids | 8 | 269 | 31467 | 40097003 |
| LC-AG-groupoids | 6 | 194 | 22276 | 34845724 |
| RC-AG-groupoids | 2 | 52 | 1800 | 170977 |
| BC-AG-groupoids | 2 | 47 | 1558 | 150977 |

# 4. Ideals in LC-AG-groupoids and RC-AG-groupoids

In this section, we investigate ideals for LC and RC-AG-groupoids. We also characterize LC and RC-AG-groupoids by the properties of their minimal ideals. We start with the following definition and list some observations regarding ideals for LC and RC-AG-groupoids.

A subset $A$ of the AG-groupoid $S$ is a left (right) ideal of $S$ if,

$$SA \subseteq A(AS \subseteq A), \tag{4.1}$$

$A$ is a two sided ideal or simply an ideal of $S$ if it is both left and right ideal of $S$.

**Remark 1** ([24]). *If $S$ is an AG-groupoid and $a \in S$, then by the Identity (1.1), it follows that:*

$$(aS)S = \underset{x,y\in S}{\cup}(ax)y = \underset{x,y\in S}{\cup}(yx)a \subseteq Sa.$$

*From this we conclude that $(AS)S \subseteq SA$.*

Further we have the following remarks.

**Remark 2.** *If $S$ is an AG-groupoid with left identity $e$ and $a \in S$, then by the medial property and Identity (2.2), it follows that:*

$$S(aS) = \underset{x,y\in S}{\cup} x(ay) = \underset{x,y\in S}{\cup}(ex)(ay) = \underset{x,y\in S}{\cup}(ea)(xy) \subseteq aS.$$

*In general for any $A \subseteq S$ we conclude that $S(AS) \subseteq AS$.*

**Remark 3.** *If $S$ is an LC-AG-groupoid and $a \in S$, then by left invertive law and (2.1), it follows that:*

$$(Sa)S = \underset{x,y\in S}{\cup}(xa)y = \underset{x,y\in S}{\cup}(ax)y = \underset{x,y\in S}{\cup}(yx)a \subseteq Sa.$$

*Thus for any $A \subseteq S$ we conclude that $(SA)S \subseteq SA$.*

**Remark 4.** *If $S$ is an RC-AG-groupoid with left identity $e$ and $a \in S$, then by left invertive law and (2.2), it follows that:*

$$\begin{aligned} S(Sa) &= \underset{x,y\in S}{\cup} x(ya) = \underset{x,y\in S}{\cup}(ex)(ya) = \\ &= \underset{x,y\in S}{\cup}(ex)(ay) = \underset{x,y\in S}{\cup}(ea)(xy) \subseteq aS. \end{aligned}$$

*Hence in general $S(SA) \subseteq AS$ for $A \subseteq S$.*

**Remark 5.** *If $S$ is an RC-AG-groupoid with left identity $e$ and $a \in S$, then by medial law and by Identity (2.1), it follows that:*

$$\begin{aligned} (Sa)S &= \underset{x,y\in S}{\cup}(xa)y = \underset{x,y\in S}{\cup}(xa)(ey) = \underset{x,y\in S}{\cup}(xa)(ye) = \\ &= \underset{x,y\in S}{\cup}(xy)(ae) = \underset{x,y\in S}{\cup}(xy)(ea) \subseteq Sa. \end{aligned}$$

*Thus $(SA)S \subseteq SA$ for $A \subseteq S$.*

**Definition 2** ([24])**.** *Let $S$ be an AG-groupoid and $A, B \subseteq S$, than $A$ and $B$ are right (left) connected sets if $AS \subseteq B$ and $BS \subseteq A$ ($SA \subseteq B$ & $SB \subseteq A$).*

**Example 4.** *Let $S = \{1, 2, 3, 4\}$ be an AG-groupoid given by the following table*

| $\cdot$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 2 | 1 | 1 | 1 |
| 4 | 2 | 1 | 2 | 1 |

Now and $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$ be two subsets of $S$. Then clearly $AS \subseteq B$ and $BS \subseteq A$ also $SA \subseteq B$ and $SB \subseteq A$. Thus $A$ and $B$ are left and right connected and hence are connected.

**Remark 6.** *If $L$ is a left and $R$ is a right ideal of an LC-AG-groupoid $S$, then by left invertive law and Identity (4.1), we have*

$$(LR)S = (SR)L = (RS)L \subseteq RL \text{ and } (RL)S = (SL)R \subseteq LR.$$

*It follows that $LR$ and $RL$ are right connected sets.*

**Proposition 1.** *Let $S$ be an LC-AG-groupoid. Then for each $a \in S$ the set $a \cup aS$ and $aS \cup Sa$ are right connected sets.*

*Proof.* If $a \in S$, then by Remarks (1) and (3), we have

$$(a \cup Sa)\, S = aS \cup (Sa)S \subseteq aS \cup Sa,$$

also,

$$(aS \cup Sa)\, S = (aS)S \cup (Sa)S \subseteq Sa \cup Sa \subseteq a \cup Sa.$$

Hence the result follows. □

**Theorem 1.** *Let $S$ be an LC-AG-groupoid. Then for each $a \in S$ the set $a \cup aS \cup Sa$ is right ideal of $S$.*

*Proof.* Let $a \in S$, then by Remarks 1–3, we have

$$\begin{aligned}(a \cup aS \cup Sa)\, S &= aS \cup (aS)S \cup (Sa)S \subseteq aS \cup Sa \cup Sa \\ &\subseteq aS \cup Sa \subseteq a \cup aS \cup Sa.\end{aligned}$$

Hence $(a \cup aS \cup Sa)S$ is right ideal of $S$. □

**Theorem 2.** *Let $S$ be an RC-AG-groupoid with left identity $e$. Then for each $a \in S$ the set $J(a) = a \cup aS \cup Sa$ is the minimal (two sided) ideal of $S$ containing $a$.*

*Proof.* By Remarks (2) and (4), we have

$$S\,(a \cup aS \cup Sa) = Sa \cup S(aS) \cup S(Sa) \subseteq Sa \cup aS \cup aS \subseteq Sa \cup aS \subseteq a \cup aS \cup Sa.$$

Thus $J(a)$ is a left ideal. Now again by Remarks (1) and (3), we have

$$(a \cup aS \cup Sa)S = aS \cup (aS)S \cup (Sa)S \subseteq aS \cup Sa \cup Sa \subseteq aS \cup Sa \subseteq a \cup aS \cup Sa.$$

Thus $J(a)$ is a right ideal, and hence it is a two sided ideal or simply an ideal of $S$. If $J$ is an ideal of $S$ and $a \in J$, then

$$J(a) = a \cup (aS \cup Sa) \subseteq J \cup (JS \cup SJ) \subseteq J \cup (J \cup J) \subseteq J \Rightarrow J(a) \subseteq J.$$

Hence the result follows. □

**Theorem 3.** *If $S$ is an RC-AG-groupoid with left identity $e$, then for $a \in S$ the sets $a(Sa)$ and $(aS)a$ are ideals of $S$. If $a \in a(Sa)$ (resp. $a \in (aS)a$), then $a(Sa)$ (resp. $(aS)a$) is a minimal ideal generated by $a$. Further if $a \in (a(Sa) \cap (aS)a)$, then $(aS)a = a(Sa)$ and it is minimal ideal generated by $a$.*

*Proof.* If $a \in S$, then by the medial law, left invertive law and (2.2), we have

$$\begin{aligned}S\,(a(Sa)) &= \underset{x,y\in S}{\cup} x(a(ya)) = \underset{x,y\in S}{\cup} (ex)(a(ya)) = \underset{x,y\in S}{\cup} (ea)(x(ya)) = \\ &= \underset{x,y\in S}{\cup} a(x(ay)) = \underset{x,y\in S}{\cup} a((ay)x) = \underset{x,y\in S}{\cup} a((xy)a) \subseteq a(Sa).\end{aligned}$$

Similarly, by paramedial, medial, left invertive laws and (2.2), we have

$$\begin{aligned}(a(Sa))\,S &= \underset{x,y\in S}{\cup} (a(xa))y = \underset{x,y\in S}{\cup} (y(xa))a = \underset{x,y\in S}{\cup} (y(ax))a = \\ &= \underset{x,y\in S}{\cup} (y(ax))(ea) = \underset{x,y\in S}{\cup} (y(ax))(ae) = \underset{x,y\in S}{\cup} (e(a)x)(ay) = \\ &= \underset{x,y\in S}{\cup} (ea)((ax)y) = \underset{x,y\in S}{\cup} a((yx)a) \subseteq a(Sa).\end{aligned}$$

Hence, $a(Sa)$ is an ideal of $S$. Now, again using the paramedial and left invertive laws and the Identity (2.2), we have

$$\begin{aligned} S(((aS)a)) &= \bigcup_{x,y\in S} x((ay)a) = \bigcup_{x,y\in S} x(a(ay)) = \bigcup_{x,y\in S} (ex)(a(ay)) = \\ &= \bigcup_{x,y\in S} (ea)(x(ay)) = \bigcup_{x,y\in S} (ea)((ay)x) = \bigcup_{x,y\in S} (ea)((xy)a) = \\ &= \bigcup_{x,y\in S} (aa)((xy)e) = \bigcup_{x,y\in S} (a(xy))(ae) = \bigcup_{x,y\in S} (a(xy))(ea) = \\ &= \bigcup_{x,y\in S} (a(xy))a \subseteq (aS)a \Rightarrow S(((aS)a)) \subseteq (aS)a. \end{aligned}$$

Similarly,

$$\begin{aligned} ((aS)a)\,S &= \bigcup_{x,y\in S} ((ax)a)y = \bigcup_{x,y\in S} (ya)(ax) = \bigcup_{x,y\in S} (ya)(xa) = \bigcup_{x,y\in S} (yx)(aa) = \\ &= \bigcup_{x,y\in S} (e(yx))(aa) = \bigcup_{x,y\in S} (a(yx))(ae) = \bigcup_{x,y\in S} (a(yx))(ea) \\ &\subseteq (aS)a \Rightarrow ((aS)a)\,S \subseteq (aS)a. \end{aligned}$$

Hence $(aS)a$ and $a(Sa)$ are ideals of $S$. If $A$ is an ideal on $S$, then for every $a \in A$ we have $(aS)a \subseteq A$ and $a(Sa) \subseteq A$, clearly. If $a \in A \cap (aS)a$ (resp. $a \in A \cap a(Sa)$), then $(aS)a$ (resp. $a(Sa)$) is a minimal ideal generated by $a$. If $a \in A \cap (aS)a \cap a(Sa)$, then by minimality, it follows that $(aS)a = a(Sa)$. Clearly, for each $a \in S$ it holds that $(aS)a \subseteq Sa$ and $a(Sa) \subseteq Sa$. □

# 5. Characterization of BC-AG-groupoids

In this section, we discuss the relations of BC-AG-groupoid with some already known subclasses of AG-groupoids. We start with the following results which proves that every AG$^*$-groupoid is RC-AG-groupoid, but the converse is not always true as illustrated in Example 5. The Example 6 also shows that every LC-AG-groupoid may not be an AG$^*$-groupoid.

**Theorem 4.** *Every AG$^*$-groupoid is RC-AG-groupoid.*

*Proof.* Let $S$ be an AG$^*$-groupoid, and $a, b, c \in S$. Then

$$a(bc) = (ba)c = (ca)b = a(cb) \Rightarrow a(bc) = a(cb).$$

Hence $S$ is RC-AG-groupoid. □

**Example 5.** *Let $S = \{1, 2, 3\}$ be an RC-AG-groupoid with the following table. Then $(S, *)$ is not an AG$^*$-groupoid since $(1 * 1) * 1 \neq 1 * (1 * 1)$.*

| $*$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 | 2 | 2 |
| 2 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 |

**Example 6.** *Let $S = \{1, 2, 3, 4, 5, 6\}$. Then it is easy to verify that $S$ is an AG$^*$-groupoid, but not an LC-AG-groupoid as clearly, $(1 * 2) * 1 \neq (2 * 1) * 1$.*

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 3 | 4 | 5 | 5 | 5 | 5 |
| 2 | 3 | 4 | 6 | 6 | 5 | 5 |
| 3 | 5 | 5 | 5 | 5 | 5 | 5 |
| 4 | 6 | 6 | 5 | 5 | 5 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 5 | 5 | 5 | 5 | 5 | 5 |

Now, we prove the following:

**Theorem 5.** *Every LC-AG*-groupoid is a semigroup.*

*Proof.* Let $S$ be an LC-AG*-groupoid, then for every $a, b, c \in S$

$$ab \cdot c = ba \cdot c = a \cdot bc \Rightarrow ab \cdot c = a \cdot bc.$$

Thus $S$ is semigroup. □

The converse of the above theorem is not true as shown in the following example.

**Example 7.** *Let $S = \{1, 2, 3, 4, 5, 6\}$. Then $(S, *)$ with the given table is a semigroup. Clearly $S$ is neither LC-AG-groupoid nor AG*-groupoid as,* $(0*1)*0 \neq 1*(0*0)$ *and* $(0*1)*0 \neq (1*0)*0$.

| * | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 2 | 3 | 2 | 2 | 5 | 2 |
| 1 | 4 | 2 | 2 | 2 | 2 | 2 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 5 | 2 | 2 | 2 | 2 | 2 |
| 4 | 2 | 2 | 2 | 2 | 2 | 2 |
| 5 | 2 | 2 | 2 | 2 | 2 | 2 |

An element $a$ of an AG-groupoid $S$ is called left cancellative if $ab = ac \Rightarrow b = c$, right cancellative and cancellative elements are defined analogously.

**Theorem 6.** *Let $S$ be an LC-AG-groupoid. Then $S$ is a commutative semigroup if it has a right cancellative element.*

*Proof.* Let $x$ be a a right cancellative element of an LC-AG-groupoid $S$, and $a, b \in S$. Then

$$(ab)x = (ba)x \Rightarrow ab = ba.$$

Thus $S$ is commutative, but commutativity implies associativity in AG-groupoids. Hence $S$ is a commutative semigroup. □

**Theorem 7.** *Every LC-AG-groupoid is paramedial AG-groupoid.*

*Proof.* Let $S$ be a LC-AG-groupoid, and $a, b, c, d \in S$. Then

$$\begin{aligned} (ab)(cd) &= (ba)(cd) = ((cd)a)b = ((dc)a)b = ((ac)d)b = \\ &= ((ca)d)b = (bd)(ca) = (db)(ca) \\ \Rightarrow (ab)(cd) &= (db)(ca). \end{aligned}$$

Hence $S$ is a paramedial AG-groupoid. □

**Example 8.** *The following is an example of RC-AG-groupoid of order 4 that is not paramedial AG-groupoid.*

| * | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 3 | 1 | 1 |
| 2 | 4 | 4 | 4 | 4 |
| 3 | 1 | 3 | 1 | 1 |
| 4 | 3 | 1 | 3 | 3 |

**Theorem 8.** *Every LC-AG-groupoid is left nuclear square AG-groupoid.*

*Proof.* Let $S$ be a LC-AG-groupoid, and $a, b, c \in S$. Then

$$\begin{aligned} a^2(bc) &= (aa)(bc) = ((bc)a)a = ((cb)a)a = ((ab)c)a = \\ &= (c(ab))a = (a(ab))c = (ab)a)c = ((ba)a)c = \\ &= ((aa)b)c = (a^2b)c \Rightarrow a^2(bc) = (a^2b)c. \end{aligned}$$

Hence $S$ is left nuclear square AG-groupoid. □

The following counterexample shows that neither AG**-groupoid nor BC-AG-groupoid is nuclear square AG-groupoid. However, both these properties jointly gives the desired relation as given in the following theorem.

**Example 9.**

*(i) AG**-groupoid that is not a nuclear square AG-groupoid as* $(3 * 3) * 3^2 \neq 3 * (3 * 3^2)$.

*(ii) BC-AG-groupoid that is not nuclear square AG-groupoid as* $(3 * 3) * 3^2 \neq 3 * (3 * 3^2)$.

| * | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |
| 3 | 1 | 2 | 2 |

(i)

| * | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 |
| 3 | 2 | 2 | 1 |

(ii)

**Theorem 9.** *Let S be a BC-AG**-groupoid. Then the following assertions are equivalent.*

*(i) S is left nuclear square AG-groupoid;*

*(ii) S is middle nuclear square AG-groupoid;*

*(iii) S is right nuclear square AG-groupoid.*

*Proof.* Let $S$ be a BC-AG**-groupoid. Then
$(i) \Rightarrow (ii)$. Assume $(i)$ holds, let $a, b, c \in S$. Then

$$a(b^2c) = b^2(ac) = b^2(ca) = (b^2c)a = (cb^2)a = (ab^2)c \Rightarrow a(b^2c) = (ab^2)c.$$

Thus $S$ is middle nuclear square AG-groupoid.
$(ii) \Rightarrow (iii)$. Assume $(ii)$ holds, let $a, b, c \in S$. Then

$$a(bc^2) = b(ac^2) = b(c^2a) = (bc^2)a = (c^2b)a = (ab)c^2 \Rightarrow a(bc^2) = (ab)c^2.$$

Finally we show,
$(iii) \Rightarrow (i)$. Assume $(iii)$ holds, and $a, b, c \in S$. Then

$$a^2(bc) = b(a^2c) = b(ca^2) = (bc)a^2 = (cb)a^2 = (a^2b)c \Rightarrow a^2(bc) = (a^2b)c.$$

which proves $(i)$. Hence the theorem is proved. □

Now, we give an example of left alternative AG-groupoid and BC-AG-groupoid that are not flexible AG-groupoid.

**Example 10.** *(i) The AG-groupoid in Table (i) below, is left alternative but not flexible AG-groupoid.*

*(ii) The AG-groupoid in Table (ii) is BC- AG-groupoid, but not flexible AG-groupoid.*

| * | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 3 |
| 3 | 1 | 4 | 1 | 1 |
| 4 | 1 | 1 | 2 | 1 |

(i)

| * | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 2 | 1 |
| 4 | 2 | 2 | 1 | 1 |

(ii)

However, we have the following:

**Theorem 10.** *Every BC-AG-groupoid is left alternative AG-groupoid if and only if it is flexible AG-groupoid.*

*Proof.* Let $S$ be a BC-AG-groupoid satisfying the left alternative AG-groupoid property, and let $a, b \in S$. Then

$$(ab)a = (ba)a = (aa)b = a(ab) = a(ba) \Rightarrow (ab)a = a(ba).$$

Hence $S$ is flexible AG-groupoid.

Conversely let $S$ be a BC-AG-groupoid satisfying the flexible AG-groupoid property, then for $a, b \in S$, we have

$$(aa)b = (ba)a = (ab)a = a(ba) = a(ab) \Rightarrow (aa)b = a(ab).$$

Hence $S$ is left alternative AG-groupoid. □

One can easily verify in the following tables that neither $T^1$-AG-groupoid nor BC-AG-groupoid is self-dual AG-groupoid.

**Example 11.** *1. The AG-groupoid given in Table (i) is $T^1$-AG-groupoid but not self-dual AG-groupoid, while the AG-groupoid in Table (ii) is BC- AG-groupoid but not self-dual AG-groupoid.*

| * | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 2 | 1 |

(i)

| * | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 2 |
| 4 | 1 | 1 | 1 | 4 |

(ii)

However, we prove the following:

**Theorem 11.** *Every $T^1$-AG-groupoid $S$ is self-dual AG-groupoid, if any of the following holds.*

(i) $S$ is LC-AG-groupoid.

(ii) $S$ is RC-AG-groupoid.

*Proof.* Let $S$ be a $T^1$-AG-groupoid, and let $a, b, c \in S$.

(i) If $S$ is LC-AG-groupoid, then

$$(bc)a = (cb)a = (ab)c = (ba)c \Rightarrow a(bc) = c(ba).$$

(ii) Again, if $S$ is RC-AG-groupoid, then

$$a(bc) = a(cb) \Rightarrow (bc)a = (cb)a = (ab)c \Rightarrow a(bc) = c(ab) = c(ba).$$

Hence $S$ is self- dual AG-groupoid in each case and the theorem is proved. □

**Theorem 12.** *Every BC-AG-3-band is commutative semigroup.*

*Proof.* Let $S$ be BC-AG-3-band, and let $a, b \in S$. Then

$$\begin{aligned} ab &= (a(aa))(b(bb)) = (ab)((aa)(bb)) = (ba)((aa)(bb)) = \\ &= (ba)((ab)(ab)) = (ba)((ba)(ab)) = (ba)((ba)(ba)) = \\ &= (ba)((bb)(aa)) = (b(bb))(a(aa)) = ba \Rightarrow ab = ba. \end{aligned}$$

Thus $S$ is commutative and hence is associative. Equivalently $S$ is commutative semigroup. □

# 6. Congruences on LC-AG-groupoid

Congruence on various subclasses of AG-groupoids are defined in various papers [23, 37, 38]. In this section we discuss some congruence on LC-AG-groupoids. It is observed that if $S$ is an LC-AG-groupoid and $E_S \neq \emptyset$, where $E_S$ is the collection of all idempotents of $S$ then by medial law, definition of LC-AG-groupoid and repeated use of the left invertive law, $E(S)$ is semilattice, that is for any $e, f \in E_S \neq \emptyset$ :

$$\begin{aligned} &ef = (ee)(ff) = (ef)(ef) = (fe)(ef) = ((ef)e)f = \\ &= ((fe)e)f = ((ee)f)f = (ff)(ee) = fe \Rightarrow ef = fe. \end{aligned}$$

This implies that $E_S$ is commutative. Moreover,$\forall a, b \in S$ and $e \in E_S$, we have

$$\begin{aligned} e(ab) &= (ea)(eb) = ((eb)a)e = ((be)a)e = ((ae)b)e = ((ea)b)e = (b(ea))e = \\ &= (e(ea))b = ((ae)e)b = ((ee)a)b = (ea)b \Rightarrow e(ab) = (ea)b. \end{aligned}$$

Thus $\forall a, b \in S$ and $e \in E_S$, as a consequences of the above we have the following.

**Proposition 2.** *Let $S$ be an LC-AG-groupoid . Then $E_S$ is a semilattice.*

**Example 12.** *LC-AG-band of order 4 that is a semilattice.*

| * | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 |
| 3 | 1 | 2 | 3 | 3 |
| 4 | 1 | 2 | 3 | 4 |

Furthermore, in Example 13, $E_S = \{1, 3\} \subseteq S$ is a semilattice.

**Theorem 13.** *Let $(S, \cdot)$ be an LC-AG-groupoid such that $E_S \neq \phi$, and $\eta$ be a relation defined on $S$ as*

$$\eta = \{(a, b) \in S,\ (xe)a = (ye)b \text{ for every } e \in E_S \text{ and } x, y \in (S, \cdot)\}.$$

*Then $\eta$ is a congruence on $S$.*

*Proof.* Given that $S$ is an LC-AG-groupoid and $E_S$ is the set of all idempotent elements in $S$. A relation $\eta$ is defined on $S$ as,

$$\eta = \{(a, b) \in S,\ (xe)a = (ye)b \text{ for every } e \in E_S\}.$$

First, we show that $\eta$ is equivalence relation on $S$, for this we show that $\eta$ is reflexive, symmetric and transitive relation. Clearly $\eta$ is reflexive as for any $a, x \in S$ $(xe)a = (xe)a \Rightarrow a\eta b$. Let $a\eta b$, then $(xe)a = (ye)b \Leftrightarrow (ye)b = (xe)a \Leftrightarrow b\eta a$. Hence $\eta$ is symmetric. Now, for transitivity let $a\eta b$ and $b\eta c$. Then $(xe)a = (ye)b$ and $(ye)b = (ze)c$ for some $x, y, z \in S \Leftrightarrow (xe)a = (ye)b = (ze)c \Leftrightarrow (xe)a = (ze)c \Leftrightarrow a\eta c$. Hence $\eta$ is transitive. Therefore $\eta$ is an equivalence relation on $S$. Now, we show that $\eta$ is compatible. First we show that $\eta$ is right compatible, for this let $a, b, c$ and $x, y, z$ are elements of $S$, then using left invertive, medial laws, definition of LC-AG-groupoid and Theorem (7), we get

$$\begin{aligned} a\eta b \Rightarrow (xe)a = (ye)b &\Rightarrow ((xe)a)c = ((ye)b)c \\ \Rightarrow (ca)(xe) = (cb)(ye) &\Rightarrow (cx)(ae) = (cy)(be) \\ \Rightarrow (ex)(ac) = (ey)(bc) &\Rightarrow (xe)(ac) = (ye)(bc) \\ \Rightarrow a\eta b &\Rightarrow ac\eta bc. \end{aligned}$$

Therefore $\eta$ is right compatible. Similarly, it is easy to show that $\eta$ is left compatible. Hence $\eta$ is compatible and therefore $\eta$ is a congruence on $S$. □

**Theorem 14.** *Let $S$ be an LC-AG-groupoid and $E_S \neq \emptyset$. Let $\rho$ be a relation on $S$ defined as, $\rho = \{(a, b) \in S,\ ea = eb \text{ for every } e \in E_S\}$. Then $\rho$ is a congruence on $S$.*

*Proof.* Let $S$ be an LC-AG-groupoid and $E_S$ denotes the set of all idempotent elements in $S$. Assume that $E_S \neq \emptyset$. A relation $\rho$ is defined on $S$ as,

$$\rho = \{(a, b) \in S,\ ea = eb \text{ for every } e \in E_S\}.$$

Now, we show that $\rho$ is an equivalence relation on $S$. Obviously $\rho$ is reflexive, as for any $a \in S$ and $e \in E_S$ we have $ea = ea \Rightarrow a\rho b$. Let $a\rho b \Leftrightarrow ea = eb \Leftrightarrow eb = ea \Leftrightarrow b\rho a$. Hence $\rho$ is symmetric. Now, let $a\rho b$ and $b\rho c\Leftrightarrow ea = eb$ and $eb = ec$ for some $a, b \in S \Leftrightarrow ea = eb = ec \Leftrightarrow ea = ec \Leftrightarrow a\rho c$. Hence $\rho$ is transitive. Therefore $\rho$ is an equivalence relation on $S$. Next we show that $\rho$ is compatible.

$\rho$ is right compatible:

$$\begin{aligned} a\rho b \Leftrightarrow ea &= eb \Leftrightarrow (ea)c = (eb)c \\ \Leftrightarrow (ca)e &= (cb)e \Leftrightarrow (ca)(ee) = (cb)(ee) \\ \Leftrightarrow (ac)(ee) &= (bc)(ee) \Leftrightarrow (ae)(ce) = (be)(ce) \\ \Leftrightarrow (ea)(ce) &= (eb)(ce) \Leftrightarrow ((ce)a)e = ((ce)b)e \\ \Leftrightarrow ((ec)a)e &= ((ec)b)e \Leftrightarrow ((ac)e)e = ((bc)e)e \\ \Leftrightarrow (ee)(ac) &= (ee)(bc) \Leftrightarrow e(ac) = e(bc) \\ a\rho b &\Leftrightarrow ac\rho bc. \end{aligned}$$

Hence $\rho$ is right compatible. Similarly $\rho$ is left compatible. Hence $\rho$ is compatible and therefore is a congruence on $S$. □

**Example 13.** *Let $S = \{1, 2, 3, 4\}$. Then $(S, \cdot)$ with the following table is LC-AG-groupoid.*

| $\cdot$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 3 | 1 |
| 4 | 1 | 2 | 1 | 1 |

and $E_S = \{1, 3\}$ define a relation $\rho$ as $a\rho b \Leftrightarrow ea = eb$, $\forall a, b \in S$ and $e \in E_S$, we have

$$\rho = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,3), (4,1), (4,2), (4,4)\}.$$

Clearly $\rho$ is equivalence relation and also left and right compatible, hence is a congruence on $S$.

## Conclusion

In this article, we have introduced some new classes of AG-groupoids that are RC-AG-groupoid, LC-AG-groupoid and BC-AG-groupoid. We have provided various examples generated by GAP for the existence of these subclasses. Enumeration of these classes has also been done up to order 6. We also introduced a procedure to verify an arbitrary AG-groupoid for these classes and proved some basic results for these newly introduced classes like; every AG$^*$-groupoid is RC-AGgroupoid, every LC-AG$^*$-groupoid is semigroup and in general LC-AG-groupoid is semigroup only if, it has a right cancellative element or has a left identity element. We have investigated that BC-AG$^*$-groupoid is nuclear square AG-groupoid and is left alternative if and only if it is flexible. We also investigated ideals in these classes. Some congruences have also been defined on these subclasses.

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## Перечень би-коммутативно–AG-группоидов

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**Аннотация.** В этой статье мы вводим (слева, справа) бикоммутативные AG-группоиды и предлагаем простой метод проверки, является ли произвольный AG-группоид бикоммутативным AG-группоидом или нет. Мы также исследуем некоторые общие свойства этих AG-группоидов. Далее вводим и изучаем некоторые свойства идеалов в этих AG-группоидах и разлагаем левые коммутативные AG-группоиды, определяя некоторые конгруэнции на этих AG-группоидах.

**Ключевые слова:** AG-группоид, бикоммутативно-AG-группоиды, (слева, справа) коммутативные AG-группоиды.

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# On Construction of Positive Closed Currents with Prescribed Lelong Numbers

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**Abstract.** We establish that a sequence $(X_k)_{k\in\mathbb{N}}$ of analytic subsets of a domain $\Omega$ in $\mathbb{C}^n$, purely dimensioned, can be released as the family of upper-level sets for the Lelong numbers of some positive closed current. This holds whenever the sequence $(X_k)_{k\in\mathbb{N}}$ satisfies, for any compact subset $L$ of $\Omega$, the growth condition $\sum\limits_{k\in\mathbb{N}} C_k \operatorname{mes}(X_k \cap L) < \infty$. More precisely, we built a positive closed current $\Theta$ of bidimension $(p,p)$ on $\Omega$, such that the generic Lelong number $m_{X_k}$ of $\Theta$ along each $X_k$ satisfies $m_{X_k} = C_k$. In particular, we prove the existence of a plurisubharmonic function $v$ on $\Omega$ such that, each $X_k$ is contained in the upper-level set $E_{C_k}(dd^c v)$.

**Keywords:** closed positive current, plurisubharmonic function, potential, analytic set, Lelong number.

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## Introduction

We consider respectively, a domain $\Omega$ in $\mathbb{C}^n$, a fixed integer $1 \leqslant p \leqslant n$, a sequence of positive real numbers $(C_k)_{k\in\mathbb{N}}$ and a sequence $(X_k)_{k\in\mathbb{N}}$ of analytic subsets of $\Omega$, purely dimensioned such that

$$\text{for all} \quad k \in \mathbb{N}, \quad \operatorname{Codim}(X_k) = n - d_k \geqslant n - p.$$

We study the existence of a closed positive current $\Theta$, such that for all $k \in \mathbb{N}$, the generic Lelong number of $\Theta$ along $X_k$ satisfies $m_{X_k} = C_k$. The existence of a solution will be shown under an appropriate growth condition on the family $(X_k)_k$ of the form

$$\sum_{k\in\mathbb{N}} C_k \operatorname{mes}(X_k \cap L) = \sum_{k\in\mathbb{N}} C_k \int_L [X_k] \wedge \beta^{d_k} < \infty, \tag{0.1}$$

for any compact subset $L \subset \Omega$.

With the above data, we state the main result of the paper as follows:

**Theorem 0.1.** *Let $(C_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers and $(X_k)_{k\in\mathbb{N}}$ be a sequence of analytic subsets, purely dimensioned in a domain $\Omega$ of $\mathbb{C}^n$. Assume we have for any open ball $B(a,r) \Subset \Omega$ the condition $\sum\limits_{k\in\mathbb{N}} C_k\, mes(X_k \cap B(a,r)) < \infty$. Then there exists a positive closed $(n-p, n-p)$-current $\Theta$ on $\Omega$, such that for all $k \in \mathbb{N}$, $X_k \subset E_{C_k}(\Theta)$.*

In particular, for plurisubharmonic functions on $\Omega$, we prove the following similar result:

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**Theorem 0.2.** *Let $(C_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers and $(X_k)_{k\in\mathbb{N}}$ be a sequence of analytic subsets purely dimensioned in a domain $\Omega$ of $\mathbb{C}^n$. Assume we have for any open ball $B(a,r)\Subset\Omega$ the condition $\sum\limits_{k\in\mathbb{N}} C_k\, mes(X_k\cap B(a,r))<\infty$. Then there exists a plurisubharmonic function $v$ on $\Omega$, such that for all $k\in\mathbb{N}$, $X_k\subset E_{C_k}(dd^c v)$.*

# 1. Problem statement and description of the objective of the paper

The study presented in this paper is motivated by J.-P. Demailly during a visit of the author to the Fourier Institute.

Let $1\leqslant p\leqslant n$ be a fixed natural number and $(X_k)_k$ be a sequence of analytic subsets of $\Omega$ such that for all $k\in\mathbb{N}$, $X_k$ is of pure dimension $d_k$ and $Codim(X_k)=n-d_k\geqslant n-p$. If $(C_k)_k$ is a given sequence of strictly positive numbers, then we would like to know the possibility to find a positive and closed current $\Theta$, of bidimension $(p,p)$, globally defined on $\Omega$, such that the generic Lelong number $m_{X_k}$ of $\Theta$ along each $X_k$ satisfies $m_{X_k}=C_k$?

For convenience, to solve this problem we take essentially the situation where the sequence $(\mathrm{mes}(X_k\cap L))_k$ is lower bounded for any compact subset $L$ with empty interior in $\Omega$.

Indeed, if a given analytic subset $X_k$ intersects some compact $L$, then the area of its intersection with another compact $L'$ contained in the interior of $L$ can not has a zero limit.

Furthermore, if for any compact subset $L$ of $\Omega$ we have $\lim_{k\to\infty}\mathrm{mes}(X_k\cap L)=0$ then the analytic subsets $(X_k)_k$ escape to the frontier of $\Omega$. So we may suppose that there exists some compact subset $K$ of $\Omega$ and a constant $\gamma_K>0$ such that

$$\mathrm{mes}(X_k\cap K)\geqslant\gamma_K,\quad \text{for all } k\in I,$$

where $I\subset\mathbb{N}$ is an infinite subset of positive natural numbers. Hence, the sequence $(C_k)_{k\in I}$ will have to be convergent to 0.

On the other hand, the presented problem has a solution, as well as the sequence $(X_k)_k$ satisfies, for any open ball $B(a,r)\subset\Omega$, the following growth condition

$$\sum_k C_k\,\mathrm{mes}(X_k\cap B(a,r))=\sum_{k\in\mathbb{N}} C_k\int_{B(a,r)}[X_k]\wedge\beta^{d_k}<\infty.$$

Eventually, The founded solution $\Theta$ will satisfy for all $k\in\mathbb{N}$, $C_k=m_{X_k}$.

We may note that Theorem 0.1 holds true if the codimension $d_k$ of each analytic subset $X_k$, is taken such that $d_k=p$. Indeed, in this case, we may take

$$\Theta=\sum_{k\in\mathbb{N}} C_k[X_k].$$

The main tools of the proof of Theorem 0.1, will be based on a beautiful result due to Ben Messaoud [1] about the intermediate currents associated to a given positive closed current. Such a result improved an anterior result due to H. Skoda [2] and P. Lelong [3] which is the basis of the proof of Theorem 0.2.

The originality of our constructions that will appear in the proofs of the main theorems of this paper, lies in the fact that both results of [1] and [2] can be adapted and applied to each $(n-d_k,n-d_k)$-current of integration over $X_k$, $k\in\mathbb{N}$. The methods of [1] and [2] will offer practical approaches to construct the current $\Theta$ of Theorem 0.1 and the function $v$ of Theorem 0.2.

The results of [1] and [2] are expressed as follows:

**Theorem** (H. Ben Messaoud [1]). *Let $T$ be a closed positive $(n-p,n-p)$-current on an open set $\Omega$ in $\mathbb{C}^n$. Then for any $\varepsilon > 0$ and any natural number $1 \leqslant l \leqslant n-p$, there exists a closed positive $(l,l)$-current $T_l$ such that*

1. *For any point $z \in B(0,r)$, we have $\nu(T_l,z) = \nu(T,z)$.*
2. *$\nu(T_l,r) \leqslant A\nu(T,r+\varepsilon r)\log^{2l} r$, where $A = A(\varepsilon) > 0$ is a positive constant.*

**Theorem** (H. Skoda [2]). *Let $T$ be a closed positive $(n-p,n-p)$-current on an open set $\Omega$ in $\mathbb{C}^n$. Then for any $\varepsilon > 0$, there exists a plurisubharmonic function $V$ on $\mathbb{C}^n$ such that for any point $z \in B(0,r)$:*

1. *$\nu\Big(\dfrac{i}{2\pi}\partial\bar\partial V, z\Big) = \nu(T,z)$.*
2. *$V(z) \leqslant A\nu(T,r+\varepsilon r)\log^2 r$, where $A = A(\varepsilon) > 0$ is a positive constant.*

This paper is organized as follows:

- in Section 2: some preliminaries;
- in Section 3: the proof of Theorem 0.1;
- in Section 4: the proof of Theorem 0.2.

## 2. Preliminaries

In this section we shall present some basic notions needed for the rest of this paper. For more informations related to differential geometry and pluripotential theory, we left the reader to consulte for examples [1, 4, 5]. We take $\Omega$ such that $\Delta^n \Subset \Omega$, where $\Delta^n$ is the polydisc in $\mathbb{C}^n$. For integers $p$ and $q$ such that $1 \leqslant p,\ q \leqslant n$, we denote $\mathscr{D}_{(p,q)}(\Omega)$ the space of smooth compactly supported-differential forms of bidegree $(p,q)$ on $\Omega$. The dual $\mathscr{D}'_{(p,q)}(\Omega)$ is the space of currents of bidimension $(p,q)$ or of bidegree $(n-p,n-q)$ on $\Omega$. A current $T$ of bidimension $(p,p)$ on $\Omega$, is said to be positive if for all $\gamma_1,\dots,\gamma_p$ in $\mathscr{D}_{(1,0)}(\Omega)$, the distribution

$$T \wedge i\gamma_1 \wedge \bar\gamma_1 \wedge \cdots \wedge i\gamma_p \wedge \bar\gamma_p$$

is a positive measure. The current $T$ is said to be closed if $dT = 0$ where $d = \partial + \bar\partial$, $d^c = \dfrac{i}{2}(\bar\partial - \partial)$ and $dd^c = i\partial\bar\partial$. We denote respectively $\beta(z) = dd^c|z|^2$ the Kähler form on $\mathbb{C}^n$ and $\alpha = dd^c \log|z|$. Following [3] (see also [4]), a well known fact that the coefficients $\Theta_{I,J}$ of every positive current $\Theta = \sum\limits_{|I|=|J|=n-p} \Theta_{I,J} dz_I \wedge d\bar z_J$ are complex measures and satisfy $\Theta_{I,J} = \overline{\Theta}_{J,I}$ for multi-indices $|I| = |J| = n-p$. Moreover, the diagonal coefficients $\Theta_{I,I}$ are positive measures and the absolute values $|\Theta_{I,J}|$ of the measures $\Theta_{I,J}$ are such that

$$\lambda_I\lambda_J|\Theta_{I,J}| \leqslant 2^p \sum_M \lambda_M^2 \Theta_{I,I}, \quad I\cap J \subset M \subset I \cup J \tag{2.2}$$

where $\lambda_k$ are arbitrary positive coefficients and $\lambda_I = \prod\limits_{\lambda\in I} \lambda_k$.

The positive measure $\sigma_\Theta = \Theta \wedge \dfrac{\beta^p}{p!}$ is called the trace measure of $\Theta$ and satisfies the inequality

$$\sigma_\Theta \leqslant C||\Theta||$$

where $||\Theta|| = \sum_{I,J} |\Theta_{I,J}|$ is the mass measure of $\Theta$ and $C$ is a positive constant.

The limit, as $r \to 0$, of the ratio

$$\nu_\Theta(z,r) = \frac{\sigma_\Theta(B(z,r))}{\frac{\pi^p}{p!}r^{2p}} = \frac{1}{\frac{\pi^p}{p!}r^{2p}} \int_{B(z,r)} \Theta \wedge \beta^p$$

is called the Lelong number of $\Theta$ at point $z$ and is denoted $\nu(\Theta, z)$. At point $z = 0$, we put

$$\nu_\Theta(r) = \frac{\sigma_\Theta(B(0,r))}{\frac{\pi^p}{p!}r^{2p}}.$$

For any $c > 0$ the upper-level sets for the Lelong numbers of $\Theta$ are denoted by

$$E_c(\Theta) = \{z \in \Omega,\, \nu(\Theta, z) \geqslant c\}.$$

According to [5], the sets $E_c(\Theta)$ are analytic subsets. Therefore, following [4], $E_c(\Theta)$ are closed sets of locally finite $\mathscr{H}_{2p}$ Hausdorff measure in $\mathbb{C}^n$.

Finally, if $A$ is an irreducible analytic subset of $\Omega$, we set

$$m_A = \inf\{\nu(\Theta, z), z \in A\}$$

and call $m_A$ the generic Lelong number of $\Theta$ along $A$.

# 3. Proof of Theorem 0.1

The proof of Theorem 0.1 follows from several steps and will be complete after proving some fondamental propositions. We begin by proceeding locally in a neighborhood of a given point $z_0 \in \Omega$ such that $r < d(z_0, \complement\Omega)$ where $r$ is a positive real number. Let $\eta$ be a smooth positive function equal to 1 on $B(z_0, \frac{r}{2})$ and has a compact support in $B(z_0, r)$. We may assume that $z_0 = 0$. Denote by $h_k$ the kernel given by

$$h_k(x) = -\frac{1}{(n + d_k - p - 1)\pi^{n+d_k-p}}|x|^{-2(n+d_k-p-1)}. \tag{3.3}$$

For each current of integration over $X_k$ it is associated an $(n-p-1, n-p-1)$ differential form denoted $U_{\eta,[X_k]}$ given by the following integral expression

$$U_{\eta,[X_k]}(z) = \int_{\xi\in\mathbb{C}^n} \eta(\xi)h_k(z-\xi)\beta^{n+d_k-p-1}(z-\xi) \wedge [X_k](\xi). \tag{3.4}$$

Since the kernel given by (3.3) lies in $L^1_{loc}(\mathbb{C}^n)$, then it is clear that the potential given by (3.4) has locally integrable coefficients in $\mathbb{C}^n$. Let $\chi_k(x)$ denote the negative current of bidegree $(n + d_k - p - 1, n + d_k - p - 1)$ given by the expression

$$\chi_k(x) = h_k(x)\beta^{n+d_k-p-1}(x). \tag{3.5}$$

Therefore, the current $\chi_k$ defined by (3.5) has locally integrable coefficients.

**Proposition 3.1.** *There are currents respectively denoted $T_{\eta,k}$, $J_1(\eta[X_k])$, $J_2(\eta[X_k])$ and $J_3(\eta[X_k])$ such that in the weak sense of currents, we have*

$$\frac{i}{2}\partial\overline{\partial}U_{\eta,[X_k]} = T_{\eta,k} + J_1(\eta[X_k]) + J_2(\eta[X_k]) + J_3(\eta[X_k]).$$

*Proof.* Consider the mappings defined by $p_1 : (x,z) \mapsto x$ and $p_2 : (x,z) \mapsto z$, they are respectively the first and the second projections from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$. Let $\tau$ denote the mapping from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$ defined by $\tau(x,z) = z - x$. The current $U_{\eta,[X_k]}$ can be expressed as a direct image on $\mathbb{C}^n \times \mathbb{C}^n$ as the following

$$U_{\eta,[X_k]}(z) = p_{2*}\left((\tau^*\chi_k) \wedge p_1^*(\eta[X_k])\right). \tag{3.6}$$

The representation given by (3.6) makes the computation of $\frac{i}{2}\partial\overline{\partial}U_{\eta,[X_k]}$, in the weak sense of current, easy and gives

$$\begin{aligned}
\frac{i}{2}\partial\overline{\partial}U_{\eta,[X_k]} \quad &= \quad \underbrace{p_{2*}\Big(\tau^*\Big(\frac{i}{2}\partial\overline{\partial}\chi_k\Big) \wedge p_1^*(\eta[X_k])\Big)}_{T_{\eta,k}} + \underbrace{p_{2*}\Big(\tau^*\Big(\frac{i}{2}\partial\chi_k\Big) \wedge p_1^*(\overline{\partial}\eta \wedge [X_k])\Big)}_{J_1(\eta[X_k])} - \\
&- \quad \underbrace{p_{2*}\Big(\tau^*\Big(\frac{i}{2}\overline{\partial}\chi_k\Big) \wedge p_1^*(\partial\eta \wedge [X_k])\Big)}_{J_2(\eta[X_k])} + \underbrace{p_{2*}\Big(\tau^*\chi_k \wedge p_1^*\Big(\frac{i}{2}\partial\overline{\partial}\eta \wedge [X_k]\Big)\Big)}_{J_3(\eta[X_k])}.
\end{aligned}$$

□

**Proposition 3.2.** *The current $J_1(\eta[X_k])$ has an integral expression as follows*

$$J_1(\eta[X_k]) = \int_{x\in\mathbb{C}^n} K_1(x,z) \wedge \partial\eta(x) \wedge [X_k](x).$$

*The currents $J_2(\eta[X_k])$ and $J_3(\eta[X_k])$ have similar representations.*

*Proof.* If $K_1(x,z)$ denote the component of bidegree $(n-p, n-p)$ in $z$ and $(d_k, d_k - 1)$ in $x$ of the form $\tau^*\Big(\frac{i}{2}\partial\chi_k\Big)$, then we get

$$J_1(\eta[X_k]) = \int_{x\in\mathbb{C}^n} K_1(x,z) \wedge \partial\eta(x) \wedge [X_k](x). \tag{3.7}$$

If $K_2(x,z)$ denote the component of bidegree $(n-p, n-p)$ in $z$ and $(d_k - 1, d_k)$ in $x$ of the form $\tau^*\Big(\frac{i}{2}\partial\chi_k\Big)$, then we get

$$J_2(\eta[X_k]) = \int_{x\in\mathbb{C}^n} K_2(x,z) \wedge \overline{\partial}\eta(x) \wedge [X_k](x). \tag{3.8}$$

Finally, if $K_3(x,z)$ denote the component of bidegree $(n-p, n-p)$ in $z$ and $(d_k - 1, d_k - 1)$ in $x$ of the form $\tau^*\Big(\frac{i}{2}\partial\chi_k\Big)$, then we get

$$J_3(\eta[X_k]) = \int_{x\in\mathbb{C}^n} K_3(x,z) \wedge \partial\overline{\partial}\eta(x) \wedge [X_k](x). \tag{3.9}$$

□

**Proposition 3.3.** *The sum of the modulus of the coefficients of the form $J_1(z) + J_2(z) + J_3(z)$, denoted $||J_1(z) + J_2(z) + J_3(z)||$, is such that*

$$||J_1(z) + J_2(z) + J_3(z)|| \leqslant A \int_{\xi\in\mathbb{C}^n} \left( \frac{||\partial\eta|| + ||\overline{\partial}\eta||}{|z-\xi|} + ||\partial\overline{\partial}\eta|| \right) |h_k|(z-\xi)d\sigma_{[X_k]}(\xi) \tag{3.10}$$

*where $A = A(n,p)$ is a strictly positive constante.*

*Proof.* Since $\frac{i}{2}\partial\chi_k = \sum\limits_{j=1}^{n} |x|^{-2(n-p+d_k)}\overline{x}_j dx_j \wedge \beta^{n-p+d_k-1}$ then we get

$$\tau^*\Big(\frac{i}{2}\partial\chi_k\Big) = \sum_{j=1}^{n} |z-x|^{-2(n-p+d_k)}(\overline{z}_j - \overline{x}_j)(dz_j - dx_j) \wedge \tau^*\beta^{n-p+d_k-1}. \tag{3.11}$$

In addition, by (2.2), the measures coefficients of the current $[X_k]$ are dominated by the trace measure $\sigma_{[X_k]}$. So the representation (3.7) implies the following estimate

$$||J_1(z)|| \leqslant A_1(n,p)\int_{\mathbb{C}^n} |z-x|^{-2(n-p+d_k)+1}||\partial\eta(x)||d\sigma_{[X_k]}(x) \tag{3.12}$$

where $||J_1(z)||$ is the sum of the modulus of the coefficients of the forme $J_1(\eta[X_k])$ and $||\overline{\partial}\eta(x)||$ is the sum of the modulus of the coefficients of $\overline{\partial}\eta(x)$. Similar procedures for $J_2(\eta[X_k])$ and $J_3(\eta[X_k])$ given by representations (3.8) and (3.9) yield similar estimates as in (3.12). These estimates provide the following

$$||J_1(z)+J_2(z)+J_3(z)|| \leqslant A_2\int_{\xi\in\mathbb{C}^n}\left(\frac{||\partial\eta||+||\overline{\partial}\eta||}{|z-\xi|} + ||\partial\overline{\partial}\eta||\right)|h_k|(z-\xi)d\sigma_{[X_k]}(\xi) \tag{3.13}$$

where $A_2 = A_2(n,p)$ is a positive constant. The estimate (3.13) allows to measure the default of positivity of the current $\frac{i}{2}\partial\overline{\partial}U_{\eta,[X_k]}$. □

**Proposition 3.4.** *The positive current $T_{\eta,k}$ is such that*

$$T_{\eta,k} = p_{2*}\left(\tau^*(\alpha^{n+d_k-p}) \wedge p_1^*(\eta[X_k])\right). \tag{3.14}$$

*Furthermore, up to a positive constant, $T_{\eta,k}$ has the same Lelong numbers as the current $\eta[X_k]$.*

*Proof.* To prove the equality (3.14), it suffices to observe that $\frac{i}{2}\partial\overline{\partial}\chi_k$ is a closed positive $(n+d_k-p, n+d_k-p)$-form on $\mathbb{C}^n \smallsetminus \{0\}$ which is invariant under the action of the unitary group $U(n,\mathbb{C})$, then following [4], in the weak sense of currents, we get

$$\tau^*\Big(\frac{i}{2}\partial\overline{\partial}\chi_k\Big) = \tau^*(\alpha^{n+d_k-p}),$$

where $\tau^*\Big(\frac{i}{2}\partial\chi_k\Big)$ is given by (3.11). For the second assertion (equality of the Lelong numbers), a detailed proof of this fact was given in [1]. Notice that the current $T_{\eta,k}$ didn't need to be closed. However, if we assume that $T_{\eta,k}$ is closed, we propose the following proof. We may prove the equality of the Lelong numbers at point $z=0$. Since $T_{\eta,k}$ is assumed to be closed, according to [4], we have

$$\frac{1}{r^{2p}}\int_{|z|<r} T_{\eta,k}\wedge\beta^p = \int_{|z|<r} T_{\eta,k}\wedge\alpha^p. \tag{3.15}$$

On the other hand, the equality (3.14) means that

$$T_{\eta,k}(z) = \int_{x\in\mathbb{C}^n}\alpha^{n+d_k-p}(z-x)\wedge\eta(x)[X_k](x).$$

Then, if we put $I(r) = \int\limits_{|z|<r} T_{\eta,k}\wedge\alpha^p$, then the equality (3.15) will be as follows

$$I = \int\limits_{(x,z)\in\mathbb{C}^n\times\{|z|<r\}} \alpha^{n+d_k-p}(z-x)\wedge\eta(x)[X_k](x)\wedge\alpha^p(z). \tag{3.16}$$

By the change of variables $(x', z') = (x, z - x)$, the equality (3.16) can be written as

$$I(r) = \int_{\{(x',z')\in\mathbb{C}^n\times\mathbb{C}^n,\ |z'-x'|<r\}} \alpha^{n+d_k-p}(z') \wedge \eta(x')[X_k](x') \wedge \alpha^p(z'-x'). \tag{3.17}$$

By the Fubini theorem and the change of variables $\xi = x' - z'$, the equality (3.17) can be transformed as

$$\begin{aligned} I(r) &= \int_{x'\in\mathbb{C}^n}\left[\int_{|z'-x'|<r} \eta(x'-z')[X_k](x'-z') \wedge \alpha^{n+d_k-p}(z') \wedge \alpha^p(z'-x')\right] = \\ &= \int_{x'\in\mathbb{C}^n}\left[\int_{|\xi|<r} \eta(\xi)[X_k](\xi) \wedge \alpha^{n+d_k-p}(x'-\xi) \wedge \alpha^p(\xi)\right]. \end{aligned} \tag{3.18}$$

Since only components of bidegree $(d_k, d_k)$ in $\xi$ and of bidegree $(n-p, n-p)$ in $x'$ of the form $\alpha^{n+d_k-p}(x'-\xi) \wedge \alpha^p(\xi)$ are useful and since by [4], $\int_{x'\in\mathbb{C}^n} \alpha^n(x') = 1$ because $\int_{|\xi-a|<r} (dd^c \log|\xi - a|)^n = 1$ for all $r > 0$ and all $a \in \mathbb{C}^n$. Then, by letting $r \to 0$ in (3.18), we find that $\nu(T_{\eta,k}, 0) = C\nu(\eta[X_k], 0)$ where $C = C(n,p)$ is a positive constant. $\square$

From now, we will proceed globally.

**Proposition 3.5.** *For all $k \in \mathbb{N}$, there exist global currents respectively denoted $U_{[X_k]}, T_k, \Phi_{1,k}, \Phi_{2,k}, \Phi_{3,k}, \Phi_{4,k}$ such that $\frac{i}{2}\partial\overline{\partial}U_{[X_k]}$ is decomposed into*

$$\frac{i}{2}\partial\overline{\partial}U_{[X_k]} = T_k + \Phi_{1,k} + \Phi_{2,k} + \Phi_{3,k} + \Phi_{4,k}.$$

*Proof.* There exists an open cover $(\Omega_j)_j$ for $\Omega$, by relatively compact open subsets such that $\Omega_j \subset \Omega_{j+1}$ and $\Omega = \bigcup_j \Omega_j$. It is clear that $(\Omega'_j)_j$ such that $\Omega'_j = \Omega_{j+1} \smallsetminus \overline{\Omega}_j$, is a locally finite open cover for $\Omega$ that is subordinate to $(\Omega_j)_j$. Let $(\omega_j)_j$ such that $\omega_j \Subset \Omega_{j+1} \smallsetminus \overline{\Omega}_j$ and $(\omega_j)_j$ still cover $\Omega$. Consider $\{(\rho_j, \Omega'_j), j \in \mathbb{N}\}$ a partition of unity on $\Omega$ such that $\mathrm{Supp}\rho_j \subset \Omega'_j$ and $\rho_j = 1$ on $\overline{\omega}_j$. By sticking the currents $U_{\eta_j,[X_k]}$, we can now construct a global potential defined by

$$U_{[X_k]}(z) = \sum_{j=1}^{\infty} \rho_j(z) U_{\eta_j,[X_k]}(z), \quad k \in \mathbb{N}. \tag{3.19}$$

An easy computation, using equality (3.19), in the weak sens of currents, yields

$$\begin{aligned} \frac{i}{2}\partial\overline{\partial}U_{[X_k]} &= \sum_j \rho_j \frac{i}{2}\partial\overline{\partial}U_{\eta_j,[X_k]} + \\ &+ \sum_j \frac{i}{2}\partial\rho_j \wedge \overline{\partial}U_{\eta_j,[X_k]} - \\ &- \sum_j \frac{i}{2}\overline{\partial}\rho_j \wedge \partial U_{\eta_j,[X_k]} + \\ &+ \sum_j \frac{i}{2}\partial\overline{\partial}\rho_j \wedge U_{\eta_j,[X_k]}. \end{aligned} \tag{3.20}$$

Since $\frac{i}{2}\partial\overline{\partial}U_{\eta_j,[X_k]} = T_{\eta_j,k} + J_1(\eta_j[X_k]) + J_2(\eta_j[X_k]) + J_3(\eta_j[X_k])$, then by taking in account equality (3.20), the current $\frac{i}{2}\partial\overline{\partial}U_{[X_k]}$ can be decomposed into

$$\begin{aligned}\frac{i}{2}\partial\overline{\partial}U_{[X_k]} &= \underbrace{\sum_j \rho_j T_{\eta_j,k}}_{T_k} + \underbrace{\sum_j \rho_j (J_1(\eta_j[X_k]) + J_2(\eta_j[X_k]) + J_3(\eta_j[X_k]))}_{\Phi_{1,k}} + \\ &+ \underbrace{\sum_j \frac{i}{2}\partial\rho_j \wedge \overline{\partial}U_{\eta_j,[X_k]}}_{\Phi_{2,k}} - \underbrace{\sum_j \frac{i}{2}\overline{\partial}\rho_j \wedge \partial U_{\eta_j,[X_k]}}_{\Phi_{3,k}} + \underbrace{\sum_j \frac{i}{2}\partial\overline{\partial}\rho_j \wedge U_{\eta_j,[X_k]}}_{\Phi_{4,k}}.\end{aligned}$$

□

**Proposition 3.6.** *For all $k$, there exists a positive closed current $\Theta_k$ of bidimension $(p,p)$ on $\Omega$, such that for any $\varepsilon > 0$ and any open ball $B(a,r) \Subset \Omega$, we have*

$$\sigma_{\Theta_k}(B(a,r)) \leqslant A_3(\varepsilon, r, n, p)\sigma_{[X_k]}(B(a, r + \varepsilon r))$$

*where $A_3 = A_3(\varepsilon, r, n, p)$ is a positive constant.*

*Proof.* We may choice the functions $(\eta_j)_j$ such that for all $j$, $\mathrm{Supp}(\eta_j) \subset \Omega_j$. This makes that each form $\Phi_{s,k},\ s \in \{1,2,3,4\}$, is smooth in a neighborhood of $\mathrm{Supp}(\rho_j)$. Furthermore, following [1], for any $\varepsilon > 0$ and any point $z \in \mathrm{Supp}(\rho_j)$, the sum of the modulus of the coefficients of the form

$$\Phi_k(z) = \Phi_{1,k}(z) + \Phi_{2,k}(z) + \Phi_{3,k}(z) + \Phi_{4,k}(z),$$

denoted $||\Phi_k(z)||$, satisfies the following estimate

$$||\Phi_k(z)|| \leqslant A_4(1 + |z|)^{-2(n-p)}\nu_{[X_k]}((1 + 5\varepsilon)(1 + |z|)) \tag{3.21}$$

where $A_4 = A_4(\varepsilon, n, p)$ is a positive constant. The estimate (3.21) is a default of positivity of the current $\frac{i}{2}\partial\overline{\partial}U_{[X_k]}$. By adding to $\frac{i}{2}\partial\overline{\partial}U_{[X_k]}$ a smooth and closed form sufficiently positive and of course having zero Lelong number every where, of the form $\left(\frac{i}{2}\partial\overline{\partial}w_k\right)^{n-p}$ where $w_k$ is a smooth strictly plurisubharmonic function (see [6]), the potential $U_{[X_k]}$ provides an $(n-p, n-p)$-positive current defined by

$$\Theta_k = \frac{i}{2}\partial\overline{\partial}U_{[X_k]} + \left(\frac{i}{2}\partial\overline{\partial}w_k\right)^{n-p} = T_k + \Phi_k + \left(\frac{i}{2}\partial\overline{\partial}w_k\right)^{n-p}. \tag{3.22}$$

In addition, following [1], the current $\Theta_k$ given by (3.22), satisfies for any $\varepsilon > 0$ and any open ball $B(a,r) \Subset \Omega$

$$\nu_{\Theta_k}(a,r) \leqslant A_5(\varepsilon)(\log r)^{2(n-p)}\nu_{[X_k]}(a, r + \varepsilon r). \tag{3.23}$$

Consequently, (3.22) provides the following

$$\sigma_{\Theta_k}(B(a,r)) \leqslant A_6(\varepsilon)\frac{\pi^{p-d_k}d_k!}{(1+\varepsilon)^{2d_k}p!}(\log r)^{2(n-p)}r^{2(p-d_k)}\sigma_{[X_k]}(B(a, r + \varepsilon r)). \tag{3.24}$$

Since for all $k \in \mathbb{N}$, we have $\frac{\pi^{p-d_k}d_k!}{(1+\varepsilon)^{2d_k}p!} \leqslant \pi^p$ and $r^{2(p-d_k)} \leqslant \max(1, r^{2p})$, the estimation (3.24) can be written as

$$\sigma_{\Theta_k}((B(a,r)) \leqslant A_7(\varepsilon, r, n, p)\sigma_{[X_k]}(B(a, r + \varepsilon r)) \tag{3.25}$$

where $A_7(\varepsilon, r, n, p) = \pi^p A_6(\varepsilon)\max(1, r^{2p})(\log r)^{2n-2p}$. □

**Proposition 3.7.** *There exists a closed positive current $\Theta$ of bidimension $(p,p)$ on $\Omega$, such that for all $k \in \mathbb{N}$ and any point $z \in X_k$, we have $\nu(\Theta, z) \geqslant C_k$.*

*Proof.* Put $\Gamma_N = \sum_{1\leqslant k\leqslant N} C_k\Theta_k$. using the estimates (3.23), (3.24), (3.25) and the condition (0.1), the mass of the current $\Gamma_N$ over a given ball $B(a,r) \Subset \Omega$, satisfies, for any $\varepsilon > 0$, the following estimate

$$\begin{aligned} ||\Gamma_N||_{B(a,r)} &\leqslant \sum_{1\leqslant k\leqslant N} C_k\sigma_{\Theta_k}(B(a,r)) \leqslant \\ &\leqslant A_7(\varepsilon,r,n,p)\sum_k C_k\sigma_{[X_k]}(B(a,r+\varepsilon r)) < \\ &< \infty. \end{aligned}$$

Therefore, $(\Gamma_N)_N$ is an increasing sequence of positive closed currents. Further, it is locally bounded in mass independently of $N$. Let $\Theta$ denote its weak limit. Following Proposition 3.4, for any $k \in \mathbb{N}$ and any point $z \in X_k$ we have

$$\nu(\Theta_k, z) = \nu([X_k], z),$$

then according to [4], we may conclude that, for all $1 \leqslant k \leqslant N$ and any point $z \in X_k$, we have

$$\nu(\Theta, z) \ \geqslant \ \limsup_{N\to\infty} \nu(\Gamma_N, z) \ \geqslant \limsup_{N\to\infty} \sum_{1\leqslant j\leqslant N} C_j\, \nu([X_j], z) \ \geqslant \ C_k.$$

The proof is achieved. □

## 4. Proof of Theorem 0.2

There are tow mains steps. The first consists of the construction, for all $k \in \mathbb{N}$, of a plurisubhrmonic function $\tilde{v}_k$ on $\Omega$, satisfying on every open ball $B(a,r) \Subset \Omega$, the following statements as in [2].

$$\nu(dd^c\tilde{v}_k, z) = \nu([X_k], z). \tag{4.26}$$

$$\forall\, \varepsilon > 0 \text{ (small enough)}, \quad \tilde{v}_k(z) \leqslant A(\varepsilon)\nu([X_k],(1+\varepsilon)r)\log^2 r, \tag{4.27}$$

where $A(\varepsilon)$ is a positive constant. To do this, we consider a locally finite open covering $(\omega_j)$ of $\Omega$ by relatively compact open balls contained in a coordinate patches of $\Omega$. We choose concentric balls $\omega_j'' \subset \omega_j' \subset \omega_j$ of respective radii $\frac{r}{3}$, $\frac{2r}{3}$, $r$ such that $\omega_j''$ still cover $\Omega$. According to [2], for all $\varepsilon > 0$, there exists a plurisubharmonic function $v_{k,j}$ on $\Omega$ such that, analogous to the statements (4.26) and (4.27), hold for any point $z \in \omega_j''$. By a slight modification on $\omega_j \setminus \omega_j'$, we may replace the function $v_{k,j}$ by the function

$$\tilde{v}_{k,j} = \begin{cases} \max\Big(v_{k,j}, A(\varepsilon)\nu\Big([X_k],(1+\varepsilon)\dfrac{r}{3}\Big)\log^2\dfrac{r}{3}\Big), & \text{on} \quad \omega_j \setminus \omega_j' \\ v_{k,j}, & \text{on} \quad \omega_j'. \end{cases}$$

This modification ensure that statement (4.26) holds for $\tilde{v}_{k,j}$ on $\omega_j''$, and up to a positive constant, statement (4.27) holds for $\tilde{v}_{k,j}$ on $\omega_j$. Let consider now a partition of unity on $\Omega$, $\{(\rho_j,\omega_j), j \in \mathbb{N}\}$, such that $\mathrm{Supp}\rho_j \subset \omega_j'$ and $\rho_j = 1$ on $\overline{\omega}_j''$. Define $\tilde{v}_k = \sum_j \rho_j\tilde{v}_{k,j}$, then the function $\tilde{v}_k$ is plurisubharmonic on $\Omega$ since $\rho_j = 1$ on $\overline{\omega}_j''$ and $\omega_j''$ still cover $\Omega$. In addition, by definition of the function $\tilde{v}_{k,j}$ on $\omega_j \setminus \omega_j'$, we have $\tilde{v}_k \not\equiv -\infty$. It also satisfies statements (4.26) and (4.27) on any small open ball $B(a,r_0)$ contained in some open ball of the form $\omega_{j_0}''$. Moreover, for all $k \in \mathbb{N}$, the mass of $dd^c\tilde{v}_k$ over $B(a,r_0)$, satisfies the following estimate

$$\begin{aligned}||dd^c\tilde{v}_k||_{B(a,r_0)} &\leqslant A_1\int_{\omega''_{j_0}} dd^c\tilde{v}_{k,j_0}\wedge\beta^{n-1} \leqslant\\ &\leqslant A_2\int_{\omega'_{j_0}} dd^c\rho_{j_0}\wedge\tilde{v}_{k,j_0}\wedge\beta^{n-1} \leqslant\\ &\leqslant A_3\int_{\omega_{j_0}} dd^c\rho_{j_0}\wedge\tilde{v}_{k,j_0}\wedge\beta^{n-1} \leqslant\\ &\leqslant A_4\nu([X_k],(1+\varepsilon)r_0)\log^2 r_0 \leqslant\\ &\leqslant A_5 mes(X_k\cap B(a,2r_0)),\end{aligned} \tag{4.28}$$

where $A_s = A_s(\epsilon, r)$, $s = 1,2,3,4,5$ are positive constants.

The second step consists of considering the sequence of plurisubharmonic functions on $\Omega$ defined by $\tilde{V}_N = \sum\limits_{k=1}^{N} C_k\tilde{v}_k$. It is clear by construction that for all $N\in\mathbb{N}$, $\tilde{V}_N \not\equiv -\infty$. For such a sequence, we may find a contant $M$ such that $(\tilde{V}_N - M)_N$ decreases pointwise. Further, following (4.28), for all $N\in\mathbb{N}$, the mass of $dd^c\tilde{V}_N$, over any open ball $B(a,r_0)\subset\Omega$, satisfies the following estimate

$$||dd^c\tilde{V}_N||_{B(a,r_0)} \leqslant A_5(\varepsilon,r)\sum_k C_k mes(X_k\cap B(a,2r_0)) < \infty, \tag{4.29}$$

where $A_5(\varepsilon,r)$ is a positive constant. The estimate (4.29) implies that $(dd^c\tilde{V}_N)_N$ is an increasing sequence of $(1,1)$ closed positive current having a locally finite mass. Then it has a weak limit as $N\to\infty$. Put

$$dd^c\tilde{V} = \lim_{N\to\infty} dd^c\tilde{V}_N,$$

we have in the weak sens of currents $dd^c\tilde{V} = \sum\limits_{k=1}^{\infty} C_k dd^c\tilde{v}_k$. In addition, following (4.26), for any $k\in\mathbb{N}$ and for any point $z\in X_k$, we have $\nu(dd^c\tilde{V}, z)\geqslant C_k$ which means that for all $k\in\mathbb{N}$, we have $E_{C_k}(dd^c\tilde{V})\supset X_k$.

The proof is achieved.

In the following example we apply our main results to polynomial functions.

**Example.** Take $n=2$, $p=1$, $\Omega=\mathbb{C}^2$ and denote $\pi$ the projection map $(z_1,z_2)\mapsto z_1$. Consider the map $F=(f_1,f_2)$ such that $F(0)=0$ and $f_1, f_2$ are polynomial functions of degree $d$. For any natural number $k\geqslant 2$, we denote respectively $F_k$ and $G_k$ the composite functions given by

$$F_k = F\circ\cdots\circ F \quad (k\text{-times}) \quad \text{and} \quad G_k = \pi\circ F_k.$$

Then $G_k$ is a polynomial function of degree $d^k$. If $X_k = G_k^{-1}(0)$, then $(X_k)_{k\in\mathbb{N}^*}$ is a sequence of analytic subsets in $\Omega$, such that for all $k$, $\mathrm{Codim}X_k\geqslant 1$. If we choose $C_k = d^{-k}\varepsilon_k$, where $(\varepsilon_k)_k$ is any sequence of strictly positive numbers such that $\sum\limits_k \varepsilon_k < \infty$, then by Theorem 0.1, there exists a positive and closed current $\Theta$ of bidimension $(1,1)$ on $\Omega$, such that for all $k\in\mathbb{N}^*$, the sublevel sets $E_{C_k}(\Theta)$ are such that $E_{C_k}(\Theta)\supset X_k$. Moreover, by Theorem 0.2 there exists a plurisubharmonic function $v$ on $\Omega$, such that for all $k\in\mathbb{N}$, the sublevel sets $E_{C_k}(dd^c v)$, are such that $E_{C_k}(dd^c v)\supset X_k$.

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# О построении положительных замкнутых потоков с заданными числами Лелона


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**Аннотация.** Мы устанавливаем, что последовательность $(X_k)_{k\in\mathbb{N}}$ аналитических подмножеств области $\Omega$ в $\mathbb{C}^n$, рассчитанная по размеру, может быть выпущена как семейство наборов верхнего уровня для чисел Лелона некоторого положительного замкнутого тока. Это верно тогда, когда последовательность $(X_k)_{k\in\mathbb{N}}$ удовлетворяет для любого компактного подмножества $L$ в $\Omega$, условие роста $\sum\limits_{k\in\mathbb{N}} C_k \operatorname{mes}(X_k \cap L) < \infty$. Точнее, мы построили положительный замкнутый ток $\Theta$ двумерности $(p,p)$ на $\Omega$ так, чтобы общее число Лелона $m_{X_k}$ из $\Theta$ вдоль каждого $X_k$ удовлетворяло $m_{X_k} = C_k$. В частности, мы доказываем существование плюрисубгармонической функции $v$ на $\Omega$ такой, что каждый $X_k$ содержится во множестве верхнего уровня $E_{C_k}(dd^c v)$.



**Ключевые слова:** замкнутый положительный ток, плюрисубгармоническая функция, потенциал, аналитическое множество, число Лелона.

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# Magnetic Susceptibility and EPR Study of $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$

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**Abstract.** Magnetic susceptibility, microstructure and EPR of cobalt-containing solid solutions with layered perovskite-like structure $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ have been studied. Solid solutions of $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ ($x \leqslant 0.005$) can be crystallized in tetragonal syngony (sp. gr. P4/mmm), as cobalt content increases, monoclinic distortion of the unit cell emerges at $0.005 < x \leqslant 0.04$ (sp. gr. P2/m). The formation of exchange-bound aggregates of Co(III) and Co(II) atoms predominantly with antiferromagnetic exchange types has been found in the solid solutions. EPR indirectly confirms that cobalt ions are in octahedral positions of substitution of Nb(V) ions.

**Keywords:** ceramics, magnetic properties, EPR spectroscopy.

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The majority of bismuth-containing compounds with layered perovskite-like structure, analogues of the so called Aurivillius phases, are of practical and theoretical interest owing to their ferroelectric properties [1]. The composition of such compounds is described by the general formula $(Bi_2O_2)(A_{n-1}B_nO_{3n+1})$, where the bismuth-oxygen layers $Bi_2O_2$ consist of $BiO_4$ pyramids bound to each other by base edges and $A_{n-1}B_nO_{3n+1}$ are perovskite-like fragments consisting of $BO_6$ octahedra bound by vertices. The large cations A are located in the cubic octahedral sites between them [2, 3]. The coefficient n in the formula corresponds to the number of $BO_6$ octahedra forming the thickness of the perovskite-like fragment. Alongside with the layered compounds which contain the uniform perovskite-like fragments, there are the so-called mixed

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or hybrid layered compounds $(Bi_2O_2)(A_{n-1}B_nO_{3n+1})\dots(Bi_2O_2)(A_{m-1}B_mO_{3m+1})$ [4–6]. Their structure consists of alternating perovskite-like fragments of various widths. Bismuth niobate $Bi_5Nb_3O_{15}$ belongs to the group of mixed layered compounds. Its structure is characterized by the ordered alternating of fragments formed by one and two niobium-oxygen octahedra. Therefore, its structure can be described as $(Bi_2O_2)(NbO_4)(Bi_2O_2)(BiNb_2O_7)$, with $n = 1$ and $m = 3$. Oxygen-niobium octahedra are bound by side vertices and are arranged in the a-b plane of the crystal, the Nb-O-Nb bond angle is 180°. The layers of the octahedra are separated by the bismuth-oxygen layers $Bi_2O_2$ formed by the $BiO_4$ pyramids and joined by the base edges (Fig. 1). The present work discusses the results of the EPR-spectroscopy and static magnetic susceptibility measurements of electron state and the nature of exchange interactions between cobalt atoms in the solid solutions of bismuth niobate $Bi_5Nb_3O_{15}$ with layered perovskite-like structure, which were obtained by heterovalent substitution of niobium (V) with cobalt atoms (II).

# 1. Experimental

The synthesis of the solid solutions was carried out by the standard ceramic procedure from special-purity grade bismuth(III), niobium(V), and cobalt(II) oxides at 650, 850, 950 and 1050 °C Phase composition of the products was determined by means of electron scanning microscopy (using a Tescan MIRA 3LM electron scanning microscope and a X-ACT energy-dispersive spectrometer) and X-ray diffraction analysis using a DRON-4-13 diffractometer (CuK$\alpha$ radiation). The cell unit parameters of solid solutions were calculated using the CSD software package [7]. The quantitative measurement of the composition of the solid solution samples was performed by atom-emission spectrometry (a SPECTRO CIROS ISP spectrometer). The magnetic susceptibility of the samples of the solid solutions was measured by the Faraday method in the temperature range of $77-350$ K at 15 fixed temperatures and at the magnetic field strength of 724, 633, 523, and 364 mT. The semicommercial installation created in the laboratory of magnetochemistry of St. Petersburg State University and consisting of an electromagnet, an electronic balance, and cryostate was used for the magnetic susceptibility measurements. The accuracy of relative measurements was 5%. EPR measurements of finely crushed ceramic samples of the $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ solid solutions were carried out in the X-band RadioPAN SE/X 2547 spectrometer (IG Komi SC UB RAS) with 100 kHz field modulation at room temperature. The amplitude of the modulation was 0.25 mT, the microwave field power was 35 mW. A sample (near to 100 mg) was put into a thin-walled quartz test tube (internal diameter of 2.5 mm) together with the reference sample (anthracite, singlet line $g_0 = 2.003$, peak to peak distance $\Delta B_{PP} = 0.5$ mT) in a ampoule. For each sample, the spectrum in the magnetic field range of $0-700$ mT and the reference line $g_0 = 2.003$ in the scan range of 5 mT were separately recorded. The intensity of the reference line served as a measure of the gain of the instrument and, when processing spectra, was used to accurately remove background signals from the test tube and ampoule. The spectra were normalized to the reference line intensity and then to 100 mg of the sample.

# 2. Results and discussion

The cobalt-containing solid solutions $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ have been studied with $0.005 \leqslant x \leqslant 0.04$. The single-phase nature of the samples was proved by the methods of scanning electron microscopy and X-ray analyses. Solid solutions of $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ ($x \leqslant 0.005$) can be crystallized in tetragonal syngony (sp. gr. P4/mmm), unit cell parameters with x = 0.005 are: a = 0.5464, c = 2.093 nm ($Bi_5Nb_3O_{15}$, sp. gr. P4/mmm, a = 0.547, c = 2.097 nm [6]); as cobalt content increases, monoclinic distortion of the unit cell emerges at $0.005 < x \leqslant 0.04$ (Fig. 1).

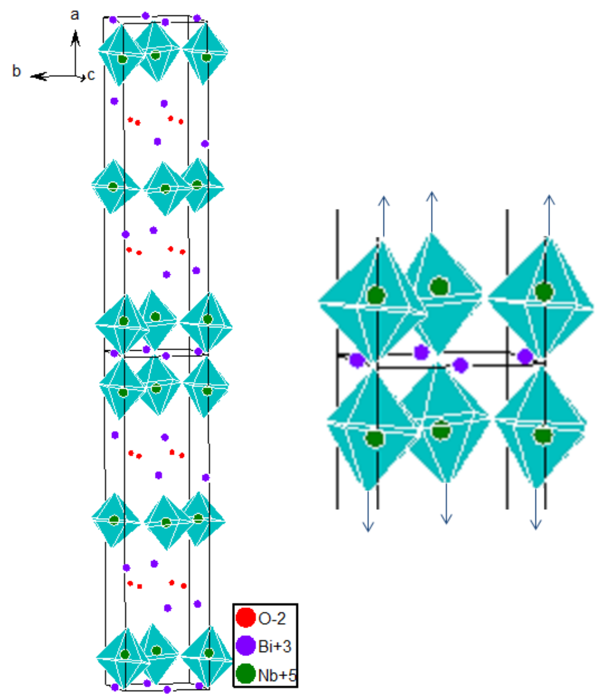


Fig. 1. The unit cell of $Bi_5Nb_3O_{15}$ and a possible type of distortion of the polyhedron of cobalt (II) atoms

Fig. 1 shows the ($Bi_5Nb_3O_{15}$ unit cell, a possible type of distortion of the polyhedron of cobalt (II) atoms is seen to the right of the crystal lattice, given the fact that it represents Jahn-Teller ions. Monoclinic distortion of the tetragonal cell of the solid solutions $Bi_5Nb_3O_{15}$ was established in previous works [6, 8] and is associated with formation of atomic defects in the structure. The X-ray patterns of the solid solutions were interpreted based on the space group P 2/m [6]. The unit cell unit cell parameters with x = 0.04 are: a = 0.5463 nm, c = 2.084 nm, b = 0.5454 nm, the $\alpha$ angle changes from 90° to 90.7°. Fig. 2 shows the surface of samples of $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ (x = 0.01, 0.02, 0.04) obtained as secondary or elastically reflected electrons.

Based on the scanning electron microscopy data, the samples are porous compacts with merged melted fine grains 1–3 $\mu$m. Using the measured magnetic susceptibility of the solid solutions, we calculated the paramagnetic components of the magnetic susceptibility and effective magnetic moments of cobalt atoms at various temperatures and concentrations of the solid solutions. The isotherms of paramagnetic component of magnetic susceptibility of cobalt atoms in $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ are typical for antiferromagnets, their comparison is shown in Fig. 3a.

The effective magnetic moments of single cobalt atoms calculated by extrapolating concentration dependencies of $[\chi^{para}(Co)]$ to infinite dilution of the solid solutions exceed pure-spin values and increase as the temperature increases from 6.18 $\mu_B$ (90 K) to 6.69 $\mu_B$ (320 K). The magnitude of the magnetic moment exceeds the pure spin values of high-spin cobalt atoms Co(II) ($\mu_{eff}$ = 3.89 $\mu_B$) and Co(III) ($\mu_{eff}$ = 4.92 $\mu_B$, therm $^5E_g$), which may indicate the formation of exchange-coupled aggregates with the antiferromagnetic type of exchange out of cobalt atoms in infinitely dilute solid solutions. The formation of aggregates out of paramagnetic atoms in highly dilute solutions did not turn out to be unexpected, it was previously observed in solid solutions of $Bi_5Nb_3O_{15}$ containing manganese or iron [8] atoms and is displayed in case of distorted coordination polyhedron caused by heterovalent substitution. Monoclinic distortion of

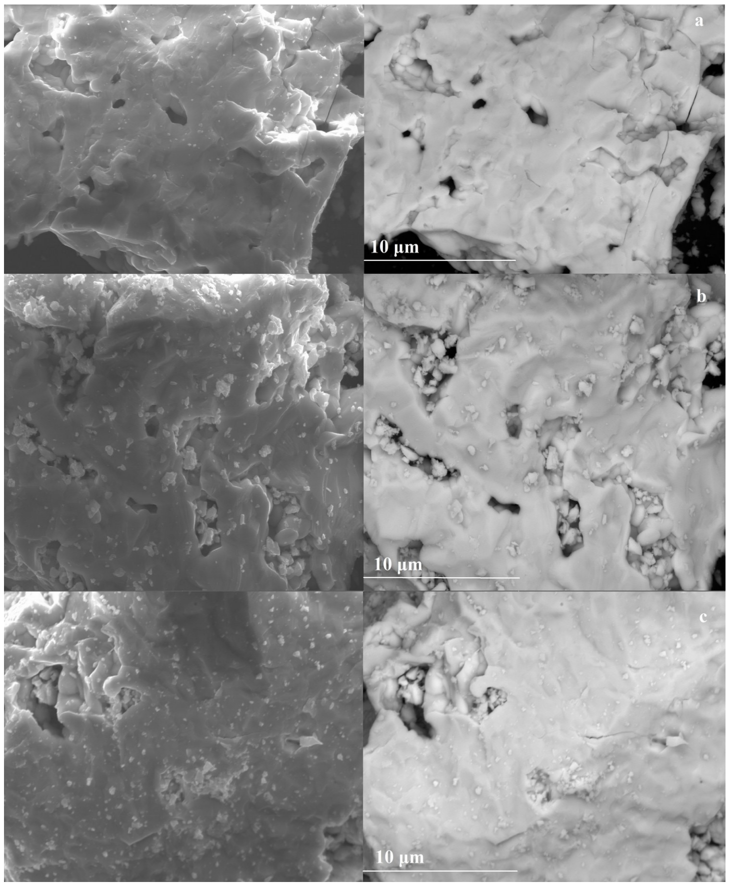


Fig. 2. Surface photomicrographs of the sample $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ at x= 0.01 (a), 0.02 (b), 0.04 (c) in the mode of secondary and elastic backscattered electrons

the tetragonal structure of the solid solutions of bismuth niobate associated with the incline of the crystallographic axis c to the plane of perovskite layers indirectly indicates such distortions. Apparently, the formation of aggregates of paramagnetic atoms near the oxygen vacancies results in stabilization of the structure of the solid solutions.

The decrease of the paramagnetic component of magnetic susceptibility of the atoms with increasing concentrations of solid solutions may be also associated with low-spin atoms of Co(III) ($\mu_{eff} = 0\ \mu_B$, $^1A_{1g}$) and increase of the portion of cobalt clusters with antiferromagnetic type of exchange between atoms [9]. Reduced magnitude of the magnetic moment in more concentrated solid solutions testifies in favour of the suggestion on the described clustering. Growing with higher temperatures dependence of the effective magnetic moment on cobalt atoms in solid solutions of various concentrations indicates the antiferromagnetic type of exchange between atoms (Fig. 3b). The antiferromagnetic type of exchange is supported by cobalt atom electrons in 3d-orbitals and layered perovskite structure of bismuth niobate ensuring indirect exchange between atoms at the angle of 180° and accessibility of exchange channels $d_{xz} \parallel p_z \parallel d_{xz}$, $d_{x^2-y^2} \parallel p_y \parallel d_{x^2-y^2}$ and between the layers through the channel – $d_{z^2} \parallel p_z \parallel d_{z^2}$. Earlier, the validity of this suggestion was shown at the example of iron-containing solid solutions with layered perovskite structure [8]. The suggestion on diamagnetic cobalt (III) atoms in concentrated solutions can be explained by a number of reasons. The oxidized state of cobalt is likely to decrease the destabilizing effect of oxygen vacancies on the structure of solid solutions; cobalt (III) atoms in the high-spin state, with their significant magnetic moment, cannot ensure such a sharp reduction of magnetic susceptibility of solid solutions, and, what is more, the accumulated distortions of oxyden polyhedra caused by heterovalent substitution of Nb(V) atoms by cobalt atoms contribute to higher tension of the crystal field of paramagnetic atoms and stabilized low-spin state of cobalt atoms.

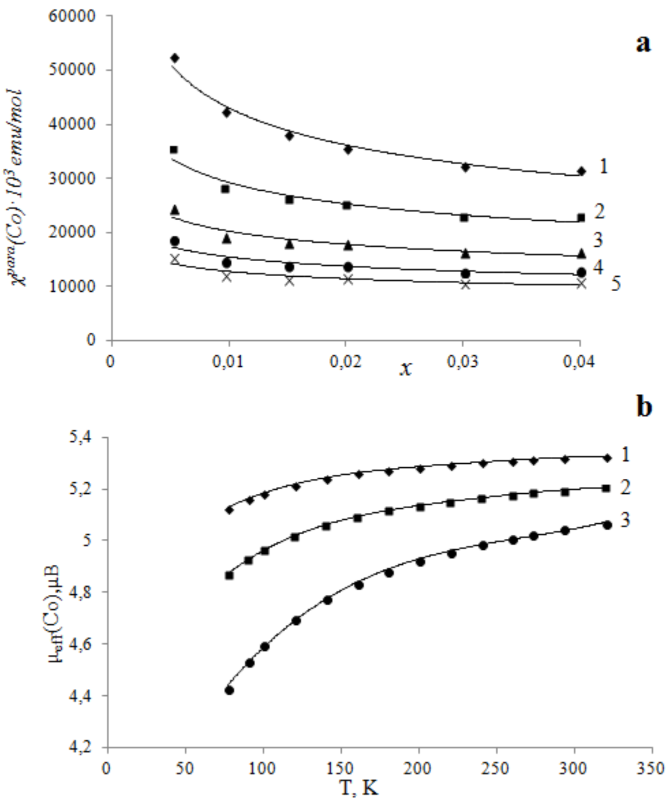


Fig. 3. **a** — Isotherms of paramagnetic component of magnetic susceptibility of the cobalt-containing solid solutions $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ at 77 K (1), 120 K (2), 180 K (3), 240 K (4) and 293 K (5); **b** — Temperature dependencies of the effective magnetic moment of cobalt in the $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ at x = 0.010 (1), 0.015 (2) and 0.04 (3)

At the top of Fig. 4 shows the EPR spectra of the $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ solid solutions for x = 0.005, 0.02, 0.04, and 0.06, reduced to standard registration options. Only one component is reliably established in all spectra. This is an asymmetric narrow line with the value of $\Delta B_{PP} =$ 17−19 mT and a g-factor of about 4.3. The integral intensity of this line monotonously decreases almost twice as $x$ increases from 0.005 to 0.06. For a nominally pure compound, this signal could be attributed to the high-spin state (S = 3/2) of $Co^{2+}$ ions in a weak, slightly distorted octahedral crystal field at $Nb^{5+}$ substitution positions. In the limit of a weak crystal field of an ideal octahedral coordination, $Co^{2+}$ ions in the EPR spectra give an isotropic line with g = 4.33 and the magnitude of hyperfine splitting on $^{59}$Co nuclei around 300 MHz, the spin-orbit interaction depending on the strength of the crystal field decreases these values [10, 11]. The asymmetry of line 4.3 can be explained by small distortions of the octahedral coordination, and its width corresponds to the unresolved hyperfine structure from $^{59}$Co. However, due to the peculiarities of the splitting of the energy levels of the octahedral $Co^{2+}$ complexes in a weak crystal field, their EPR line 4.33 can be observed in the EPR spectra only at very low temperatures. At room temperature, the spectra of $Co^{2+}$ ions are observed for low-spin states (S = 1.2) in a strong or strongly distorted crystal field, and the EPR lines are grouped in the g-factor region 2.0, rather than 4.3.

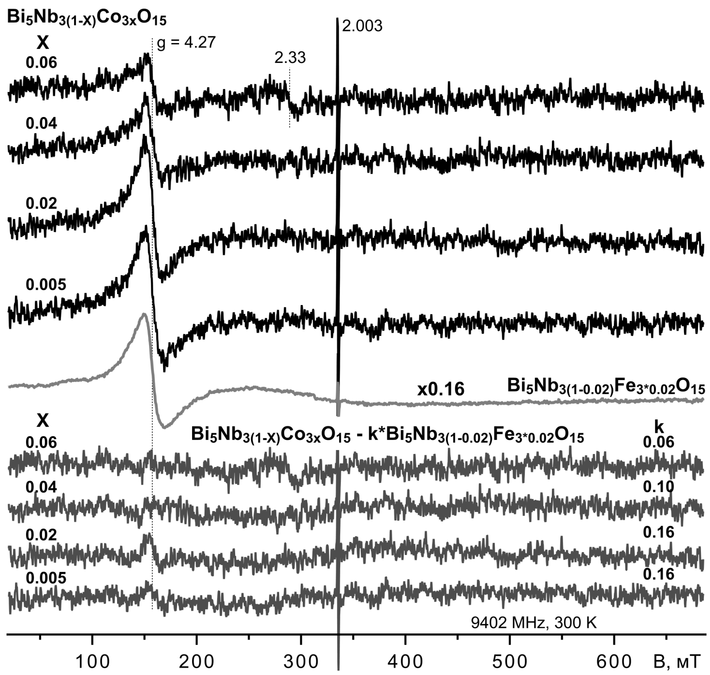


Fig. 4. EPR spectra of solid solutions $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ for $x = 0.005 - 0.06$ in comparison with the $Fe^{3+}$ spectrum of the compounds $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ from [8]. The residues of subtraction from the $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ spectra of the $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ spectrum with a weight coefficient k are shown in the bottom. The narrow line with g = 2.003 is the signal of the reference sample

A more adequate explanation of the EPR 4.3 signal in the spectra of cobalt ceramics of the $Bi_5Nb_3O_{15}$ follows from our results of the EPR studies of a similar compound doped with $Fe^{3+}$ ions [8]. In the EPR spectra of $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ at small x, the asymmetric line 4.3 with $\Delta B_{PP} = 20 - 22$ mT dominates. This is typical EPR signal for a strongly rhombic distorted octahedral oxygen environment $Fe^{3+}$ ions. We relate it to the $NbO_6$ octahedra of the compound. At high concentrations, the $Fe^{3+}$ ions in the EPR spectra increase the intensity of a wide band with g = 2.0 from the clusters of these ions. With an increase in the concentration of $Fe^{3+}$ ions, a wide band g = 2.0 of their clusters develops. A comparison of the EPR spectra of ceramics shows that the spectrum of $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ is identical to the spectrum of $Fe^{3+}$ c g = 4.3 of the $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ ceramics with a low iron content (Fig. 4, bottom). The difference in the shape of the line is reduced only to a slightly larger line width of 4.3 in the ceramics $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$. Since the line width 4.3 increases with an increase in the $Fe^{3+}$ content in the compound, we can assume that $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ ceramics contain uncontrolled traces of $Fe^{3+}$ ions causing weak 4.3 signals in the EPR spectra. The marked decrease in the integrated intensity of the line g = 4.3 of the minor content $Fe^{3+}$ in cobalt ceramics with an increase in the cobalt content is explained by the competing occurrence of both iron and cobalt ions in one structural position of the $Bi_5Nb_3O_{15}$ compound, namely $NbO_6$. Here is a Fig. 5

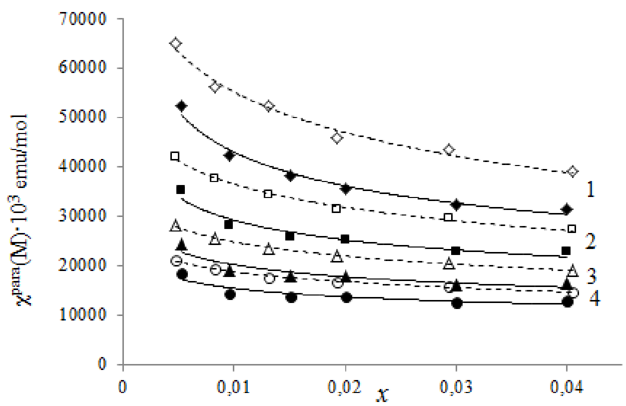


Fig. 5. Isotherms of the paramagnetic component of the magnetic susceptibility of the $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ [8] and $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ solid solutions at 77 K (1), 120 K (2), 180 K (3), 240 K (4) (the solid line is $\chi^{para}$(Co), the dashed line is $\chi^{para}$(Fe))

obtained by applying isotherms of the paramagnetic component of the magnetic susceptibility of the $Bi_5Nb_{3-3x}Fe_{3x}O_{15-\delta}$ [8] and $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ solid solutions at 77 K (1), 120 K (2), 180 K (3), 240 K (4) (the solid line is $\chi^{para}$(Co), the dashed line is $\chi^{para}$(Fe)). Comparison of magnetic susceptibilities of both series of solid solutions shows 0.815 to 1 correlation between the susceptibility values of cobalt- and iron-containing solid solutions. Assuming that the magnetic susceptibility of $Bi_5Nb_{3-3x}Co(Fe)_{3x}O_{15-\delta}$ is mainly caused by the presence of impurity iron atoms, for any $x$ the proportion of iron atoms should be approximately $4/5x$, which can hardly be seen as an impurity. It remains to admit that the absorption band in the EPR spectrum (g = 4.27) can belong to $Co(II)_{s=3/2}$ ions.

# Conclusions

Thus, it was shown that the magnetic behavior of cobalt doping bismuth niobate solid solutions with perovskite-like layered structure is generally similar and is determined mainly by the crystal structure of the solid solutions, the symmetry, and the strength of the crystal field formed by ligands. The cobalt (III), (II) atoms in solid solutions of heterovalent substitution aggregate forming strong clusters of cobalt atoms predominantly with the antiferromagnetic type of exchange, which not disintegrate even at infinite dilution. EPR indirectly confirms that cobalt ions are in octahedral positions of substitution of Nb(V) ions.

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# Исследование магнитной восприимчивости и ЭПР $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$

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**Аннотация.** Исследованы магнитная восприимчивость, микроструктура и ЭПР кобальтсодержащих твердых растворов $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ со слоистой перовскитоподобной структурой. Твердые растворы $Bi_5Nb_{3-3x}Co_{3x}O_{15-\delta}$ ($x \leqslant 0.005$) кристаллизуются в тетрагональной сингонии (пр. гр. P4/mmm), с увеличением содержания кобальта возникает моноклинное искажение элементарной ячейки при $0.005 < x \leqslant 0.04$ (пр. гр. P2/m). В твердых растворах обнаружено образование обменно-связанных кластеров из атомов Co(III) и Co(II) преимущественно с антиферромагнитным типом обмена. ЭПР косвенно подтверждает, что ионы кобальта замещают октаэдрические позиции ионов Nb (V).

**Ключевые слова:** керамика, магнитные свойства, ЭПР.

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# $L^P$-bound for the Fourier Transform of Surface-Carried Measures Supported on Hypersurfaces with $D_\infty$ Type Singularities

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**Abstract.** Estimate for Fourier transform of surface-carried measures supported on non-convex surfaces of three-dimensional Euclidean space is considered in this paper.The exact convergence exponent was found wherein the Fourier transform of measures is integrable in tree-dimensional space. This result gives an answer to the question posed by Erdösh and Salmhofer.

**Keywords:** Fourier transform, oscillatory integral, surface-carried measure.

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## 1. Introduction and preliminaries

Let $S \subset \mathbb{R}^3$ be a smooth surface and $\psi \in C_0^\infty(S)$ be a smooth function with compact support on $S$. Consider the measure $d\mu = \psi d\sigma$, where $d\sigma$ is the surface-carried measure. Fourier transform of the measure is defined by:

$$\hat{\mu}(\xi) := \int_S e^{i(\xi,x)} d\mu.$$

It is well-know that $\hat{\mu}$ is an analytic function.

In this paper the following problem is considered: find $\gamma := \inf\{p : \hat{\mu} \in L^p(\mathbb{R}^3)\}$. This problem has a long history [1,2]. Recently L. Erdös and M. Salmhofer [2] considered the problem for partial class of non-convex surfaces in $\mathbb{R}^3$. The main class of such surfaces was level set of dispersion relation of discrete Schrödinger operator on the lattice $\mathbb{Z}^3$. It should be noted that the phase function of the corresponding oscillatory integrals has singularities of type $A_1$, $A_2$, $A_3$ or $D_4$. In particular, except the case $D_4$ one of the principal curvatures does not vanish at every point. The case $D_4$ type singularities was excluded in [2]. A more general class of hypersurfaces for which the Gaussian curvature has only simple roots was considered [3]. However, it was assumed that only one of the principal curvatures can vanish. The case when both principal curvatures vanish at a point of the surface in $\mathbb{R}^3$ is still one of the open problems.

We consider the problem for hypersurfaces in $\mathbb{R}^3$. More precisely it is assumed that the phase function $(x,\omega)|_S$ (where $\omega \in S^2$ is the unite sphere centred at the origin) is small perturbation of the so-called $D_\infty$ type singularity (see [4] for definitions and basic properties of such singularities).

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It is shown that in this case $\gamma = 3$. It can be shown that for any hypersurface $S \subset \mathbb{R}^3$, $\hat{\mu} \notin L^p(\mathbb{R}^3)$ for $p \leqslant 3$, whenever $\mathrm{Supp}(\mu) \neq \emptyset$.

The main result is the following.

**Theorem 1.1.** *Let $S$ be an analytic hypersurface in $\mathbb{R}^3$. If $S$ has $D_\infty$ type singularities at the origin then there exists a neighborhood $U$ of the origin such that for any $\psi \in C_0^\infty(U)$ the inclusion $\hat{\mu} \in L^p(\mathbb{R}^3)$ holds for any $p > 3$.*

Moreover, if $S$ is any smooth surface in $\mathbb{R}^3$ and $\psi(0,0) \neq 0$ then $\hat{\mu} \notin L^3(\mathbb{R}^3)$.

The paper is organized as follows. In Section 2 the problem for the model case is considered. In this case the result is obtained with the use of simple methods. The Section 3 is devoted to special function with $D_\infty$ type singularity at the origin.

In Section 4 the general case is considered. Main theorem is proved in Section 5.

## 2. Model case $D_\infty$

Let us consider a measure supported on hypersurface $x_3 = x_1 x_2^2$. The singularity of that function is called to be $D_\infty$ type singularity at $(0,0)$. The Fourier transform of the measure can be written as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_1 x_2^2)} \psi_1(x) dx,$$

where $\psi_1(x_1, x_2) = \psi(x_1, x_2, x_1 x_2^2) / \!\!/ \sqrt{1 + x_2^4 + 4x_1^2 x_2^2}$.

Following B. Randol [3], we define the following maximal function:

$$M(\omega) = \sup_{r > o} r |\hat{\mu}(r\omega)|,$$

where $r = |\xi|$ and $\omega \in S^2$, $S^2$ is the unite sphere centred at the origin.

Let us note that $\hat{\mu}(\xi) = O(|\xi|^{-N})$ (as $|\xi| \to \infty$) provided $|\xi_3| \leqslant \max\{|\xi_1|, |\xi_2|\}$ and $\psi$ is a smooth function concentrated in a sufficiently small neighbourhood of the origin [5]. It is also assumed that $|\xi_3| \geqslant \max\{|\xi_1|, |\xi_2|\}$. Let us consider the associated oscillatory integral

$$J(\lambda, s) = \int_{\mathbb{R}^2} e^{i\lambda \Phi(x,s)} \psi_1(x) dx,$$

where $\Phi(x, s) = x_1 x_2^2 + s_1 x_1 + s_2 x_2$, $\lambda = \xi_3$, $s_j = \dfrac{\xi_j}{\lambda}$, $j = 1, 2$.

One can define the Randol type maximal function [3] associated with the oscillatory integral $J(\lambda, s)$ as

$$M(s) = \sup_{\lambda \neq 0} |\lambda| |J(\lambda, s)|.$$

Now, the following statement is proved.

**Theorem 2.1.** *The inclusion $M \in L^{3-0}_{loc}(\mathbb{R}^2)$ holds true.*

Taking into account that $\psi$ has a compact support and using integration by parts, the integral

$$J_1(\lambda, s_1, x_2) = \int_{\mathbb{R}} e^{i\lambda x_1 (x_2^2 + s_1)} \psi(x_1, x_2) dx_1$$

can be estimated by

$$|J_1(\lambda, s_1, x_2)| \leqslant \frac{c\|\psi\|_{c^2}}{1 + |\lambda|^2 |x_2^2 + s_1|^2}.$$

Consider the following integral

$$J^1(\lambda, s_1) = \int_{\mathbb{R}} \frac{dx_2}{1+|\lambda|^2|x_2^2+s_1|^2}.$$

First, we prove the auxiliary statement.

**Lemma 2.1.** *The following estimate holds true:*

$$|J^1(\lambda, s_1)| \leqslant \frac{c}{|\lambda||s_1|^{\frac{1}{2}}}.$$

*Proof.* First consider the case $\lambda|s_1| \leqslant 1$. If $s_1 = 0$ then there is nothing to prove. Let us assume that $s_1 \neq 0$. In this case we use change of variables $x_2 = |s_1|^{\frac{1}{2}} y_2$ and obtain

$$J^1(\lambda, s_1) = |s_1|^{\frac{1}{2}} \int_{\mathbb{R}} \frac{dy_2}{1+|\lambda s_1|^2|y_2^2+\mathrm{sgn(s_1)}|^2}.$$

For the sake of definiteness we assume that $\mathrm{sgn(s_1)} = -1$, e.g. $s_1 < 0$. Actually the case $\mathrm{sgn(s_1)} = 1$ or equivalently $s_1 > 0$ is much more easy to prove. Thus, we have

$$J(\lambda, s_1) = |s_1|^{\frac{1}{2}} \int_{\mathbb{R}} \frac{dy_2}{1+|\lambda s_1|^2|y_2^2-1|^2}.$$

It is easy to see that the following estimate

$$\int\limits_{|\lambda s_1||y_2^2-1|>1} \frac{dy_2}{|y_2^2-1|} \leqslant C|\lambda s_1|^{\frac{1}{2}}$$

holds. Indeed

$$\int\limits_{|y_2^2-1|>\frac{1}{\lambda s_1}} \frac{dy_2}{|y_2^2-1|} = 2 \int\limits_{y_2>\sqrt{1+\frac{1}{|\lambda s_1|}}} \frac{dy_2}{y_2^2-1} =$$

$$= \int\limits_{\sqrt{1+\frac{1}{|\lambda s_1|}}}^{\infty} \left(\frac{1}{y_2-1} - \frac{1}{y_2+1}\right) dy_2 = \ln \frac{y_2-1}{y_2+1}\Bigg|_{\sqrt{1+\frac{1}{|\lambda s_1|}}}^{\infty} =$$

$$= \ln\left(\frac{\sqrt{1+\frac{1}{|\lambda s_1|}}+1}{\sqrt{1+\frac{1}{|\lambda s_1|}}-1}\right) = \ln\left(|\lambda s_1|\left(2+\frac{1}{|\lambda s_1|}+2\sqrt{1+\frac{1}{|\lambda s_1|}}\right)\right) =$$

$$= \ln\left(1+2|\lambda s_1|+2\sqrt{|\lambda s_1|^2+|\lambda s_1|}\right) \leqslant 2|\lambda s_1| + 2\sqrt{|\lambda s_1|^2+|\lambda s_1|} =$$

$$= \sqrt{|\lambda s_1|}(2\sqrt{|\lambda s_1|}+2\sqrt{1+|\lambda s_1|}) \leqslant \sqrt{|\lambda s_1|}(2+2\sqrt{2}) = c\sqrt{|\lambda s_1|}$$

for $|\lambda s_1| \leqslant 1$. An analogical estimate holds true for $|\lambda s_1| \leqslant 2$.

Also $\mu\{y_2 : |\lambda s_1||y_2^2-1| \leqslant 1\} \leqslant \dfrac{c}{(\lambda|s_1|)^{\frac{1}{2}}}$. Hence the inequality

$$|J(\lambda, s_1)| \leqslant \frac{c}{\lambda|s_1|^{\frac{1}{2}}}$$

holds true provided $\lambda|s_1| \leqslant 2$.

Now, we consider the case $|\lambda s_1| \geqslant 2$. In this case, we have

$$\int\limits_{|y_2^2-1|\geqslant 1} \frac{dy_2}{|\lambda s_1|^2 |y_2^2-1|^2} = \frac{c}{|\lambda s_1|^2}.$$

It is easy to see that the following estimate

$$\int\limits_{1\geqslant |y_2^2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2^2-1|^2} \leqslant c|\lambda s_1|$$

holds. Indeed, using symmetry of arguments, the last integral can be estimated as

$$\int\limits_{1\geqslant |y_2^2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2^2-1|^2} \leqslant 2 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2-1|^2 |y_2+1|^2} \leqslant$$

$$\leqslant 2 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2-1|^2} \leqslant 4 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{(y_2-1)^2} = 4|\lambda s_1|.$$

On the other hand the inequality $\mu\{y_2 : |y_2^2-1| < |\lambda s_1|^{-1}\} \leqslant c|\lambda s_1|^{-1}$ holds true for the measure of the set $\{y_2 : |y_2^2-1| < |\lambda s_1|^{-1}\}$. Hence we obtain

$$|J^1(\lambda, s)| \leqslant \frac{c}{\lambda |s_1|^{\frac{1}{2}}}.$$

Lemma is proved. □

It is easy to see that the oscillatory integral $J(\lambda, s)$ can be estimated as follows:

$$|J(\lambda, s)| \leqslant \int_{-N}^{N} |J_1(\lambda, s, x_2)| dx_2,$$

where the number $N$ is

$$N = \max\{|x_2| : \text{there exist } x_1, \text{ such that } (x_1, x_2) \in \text{ Supp}\,\psi\}. \tag{1}$$

Hence

$$|J(\lambda, s)| \leqslant c\|\psi\|_{c^2} |J^1(\lambda, s)|.$$

Consequently, it follows from the Lemma that

$$|J(\lambda, s)| \leqslant \frac{c\|\psi\|_{c^1}}{|\lambda| |s_1|^{\frac{1}{2}}}$$

because $\psi$ has a compact support. If $|s| > m$, where $m$ is a big positive number depending on the support of $\psi$, then the phase function has no critical point. Hence we can use integration by parts and obtain

$$J(\lambda, s) \leqslant \frac{c}{\lambda |s|}.$$

Therefore we have

$$\chi_{\{|s|>m\}}(s) M(s) \leqslant \frac{c}{|s|} \in L^\infty(\mathbb{R}^2 \backslash B(0, m)), \tag{2}$$

where $B(0, m)$ is the ball of radius $m$ centred at the origin, and $\chi_{\{|s|>m\}}$ is the indicator function of the set $\{|s| > m\}$. Let us denote the indicator function of the set $A$ by $\chi_A$, e.g., $\chi_A(x) = 1$ for $x \in A$ otherwise $\chi_A(x) = 0$.

The relation (2) suggests that it is sufficiently to consider the oscillatory integral and the associated maximal function on the set $\{|s| \leqslant m\}$. Let us assume that $x = x^0 \in \mathrm{Supp}\,(\psi)$ is a critical point, and $s = s^0 \in \overline{B(0, m)}$ is a fixed point. If $x_0$ is not a critical point of the phase function $\Phi(x, s^0)$ then one can use integration by parts and obtain better estimate than needed. Equations for critical points are

$$(x_2^0)^2 + s_1^0 = 0, \quad 2x_1^0 x_2^0 + s_2^0 = 0.$$

Let us assume that $s_2^0 \neq 0$. Then $x_1^0 x_2^0 \neq 0$. Hence $x_1^0 \neq 0$ and also $x_2^0 \neq 0$, $s_1^0 \neq 0$. Let us consider the integral

$$J^\chi(\lambda, s) := \int_{\mathbb{R}^2} e^{i\lambda\Phi(x,s)}\psi(x)\chi(x)dx,$$

where $\chi$ is a smooth cut-off function defined in a sufficiently small neighbourhood of $x^0$ and $s$ is close to $s^0$. One can use stationary phase method in two variables because

$$Hess\Phi(x^0, s^0) = -4(x_2^0)^2 \neq 0.$$

Therefore for $|s - s^0| < \varepsilon$ we have the estimate

$$|J^\chi(\lambda, s)| \leqslant \frac{c}{\lambda}$$

provided $\chi$ is a smooth function defined in a sufficiently small neighbourhood of $x^0$. If $x^0$ is not a critical point then one can use integration by parts and obtain the same type of estimate (even better estimate than needed). Hence $M(s)$ is a bounded function in $V(s^0)$, where $V(s^0)$ is a sufficiently small neighbourhood of $s^0 \neq 0$. Let us consider the case when $s^0 = 0$, e.g., when $s$ belongs to a sufficiently small neighbourhood of the origin. This case will be considered in the next section.

## 3. Case $\{|s_1|^{\frac{1}{2}} \geqslant |s_2|\}$

Then trivial estimate for $J(\lambda, s)$ is

$$|J(\lambda, s)| \leqslant \frac{c}{|\lambda||s_1|^{\frac{1}{2}}} \leqslant \frac{c}{|\lambda||s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}}$$

and the estimate is obtained because $\dfrac{1}{|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}} \in L^{3-0}(V)$, where $V$ is a bounded neighbourhood of the origin.

Let us assume that $|s_2| \geqslant |s_1|^{\frac{1}{2}}$.

Let us consider the one-dimensional integral

$$J_2(\lambda, s_2, x_1) = \int_{\mathbb{R}} e^{i\lambda(x_1 x_2^2 + s_2 x_2)}\psi(x_1, x_2)dx_2.$$

If $|\lambda x_1| \leqslant 1$ then we have the trivial estimate

$$\int_{[0,\lambda^{-1}]} |J_2(\lambda, s_2, x_1)|dx_1 \leqslant c|\lambda|^{-1}.$$

Hence we may assume $|\lambda x_1| > 1$. If $|\lambda x_1| > 1$ and $|x_1| \leqslant |s_2|$ then the phase function has no critical point on the support of $\psi$ provided $N < \frac{1}{2}$, where $N$ is defined by relation (1). Then one can use double integration by parts and obtain

$$|J_2(\lambda, s_2, x_1)| \leqslant \frac{c\|\psi\|c_2}{|\lambda x_1|^2}.$$

Therefore

$$\int\limits_{[0,|s_2|]} |J_2(\lambda, s_2, x_1)| dx_1 \leqslant \frac{c\|\psi\|_{c_2}}{|\lambda|}.$$

Finally, let us suppose that $|x_1| > |s_2|$. Then we use stationary phase method in $x_2$ and obtain

$$J_2(\lambda, s_2, x_1) = \frac{c}{|\lambda x_1|^{\frac{1}{2}}} e^{-\frac{s_2^2}{4x_1}\lambda} \psi\Big(x_1, -\frac{s_2}{2x_1}\Big) + R(\lambda, x_1, s_2).$$

For the remainder term $R(\lambda, x_1, s)$ we have $|R(\lambda, x_1, s_2)| \leqslant \frac{c}{1 + |\lambda x_1|^{\frac{3}{2}}}$. Then $\int |R(\lambda, x_1)| dx_1 \leqslant \frac{c}{|\lambda|}$. Thus, it is sufficiently to consider the integral

$$J_1(\lambda, s) = \int_{\mathbb{R}} \frac{e^{i\lambda s_2^2(-\frac{1}{4x_1} + \frac{s_1}{s_2^2} x_1)}}{|x_1|^{\frac{1}{2}}} \psi\Big(x_1, -\frac{s_2}{2x_1}\Big) dx_1.$$

If $|\lambda s_2^2| < 1$ then we have $|J_1| \leqslant \frac{c}{|\lambda|^{\frac{1}{2}} |s_2|}$. Hence we assume $|\lambda s_2^2| > 1$. Let us estimate the integral

$$J_1^+(\lambda, s) = \int_{\mathbb{R}_+} e^{i\lambda s_2^2(-\frac{1}{4x_1} + \frac{s_1}{s_2^2} x_1)} \frac{\psi\big(x_1, -\frac{s_2}{2x_1}\big)}{x_1^{\frac{1}{2}}} dx_1.$$

Using the change of variables $x_1 = y_1^2$, we obtain

$$J_1^+(\lambda, s) = 2 \int_{\mathbb{R}_+} e^{i\lambda s_2^2(-\frac{1}{4y_1^2} + \sigma_1 y_1^2)} \psi\Big(y_1^2, -\frac{s_2}{2y_1^2}\Big) dy_1,$$

where $\sigma_1 := \frac{s_1}{s_2^2}$.

The phase function has no critical points provided $\psi$ is a smooth function defined in a sufficiently small neighbourhood of the origin so one can use integration by parts.

Thus, we obtain

$$|J_1(\lambda, s)| \leqslant \frac{c\|\psi\|_{c^1}}{|s_2||\lambda|^{\frac{1}{2}}}.$$

Let us show that

$$\frac{\chi_{\{|s_1| \leqslant s_2^2\}}}{s_2} \in L^{3-0}(V).$$

Indeed for $p < 3$ we have

$$\int_0^1 \frac{ds_2}{|s_2|^p} \int_0^{s_2} ds_1 = \int_0^1 \frac{ds_2}{|s_2|^{p-2}} < +\infty.$$

Combining the obtained estimates for the Rendol maximal function for oscillatory integral, we obtain

$$M(S) \leqslant c\bigg(\frac{\chi_{\{|s_1| \geqslant s_2^2\}}(s)}{|s_1|^{\frac{1}{2}}} + \frac{\chi_{\{s_2^2 \geqslant |s_1|\}}(s)}{|s_2|}\bigg).$$

Since $M \in L^{3-0}_{loc}(\mathbb{R}^2)$ our consideration is completed.

# 4. The general case

The following proposition holds true.

**Proposition.** Let us assume that $\Phi(x_1, x_2)$ has $D_\infty$ type singularity at the origin

$$\Phi(x_1, x_2) = x_1 x_2^2 + R(x_1, x_2),$$

where $R(x_1, x_2) = O(|x|^4)$.

Then there exist analytic functions $\varphi, \psi$ and $b$ such that function $\Phi$ can be written as

$$\Phi(x_1, x_2) = b(x_1, x_2)(x_1 - \varphi(x_2))(x_2 - \psi(x_1))^2,$$

where $\varphi(0) = \varphi'(0) = 0$, $\psi(0) = \psi'(0) = 0$, $b(0, 0) \neq 1$ (see [2] and [6]).

Let us assume that $\psi(x_1) = x_1^{m_1} \tilde{\psi}(x_1)$, $\tilde{\psi}(0) \neq 0$ and $\varphi(x_2) = x_2^{m_2} \tilde{\varphi}(x_2)$, $\tilde{\varphi}(0) \neq 0$. Then

$$\Phi(x, s) = b(x_1, x_2)(x_1 - \varphi(x_2))(x_2 - \psi(x_1))^2 + s_1 x_1 + s_2 x_2.$$

Using the change of variables

$$x_1 - \varphi(x_2) \longrightarrow x_1, \quad x_2 - \psi(x_1) \longrightarrow x_2,$$

we obtain

$$\Phi(x, s) = \tilde{b}(x_1, x_2) x_1 x_2^2 + s_1(x_1 + \varphi(x_2)) + s_2(x_2 + \psi(x_1)).$$

Let $D$ be the annulus $D = \{\frac{1}{2} \leqslant |x| \leqslant 2\}$ and $\operatorname{Supp} \chi \subset \mathrm{D}$ with $\chi \in C^\infty(D)$ satisfying

$$\sum_{\kappa=\kappa_0}^{\infty} \chi(2^{\frac{\kappa}{3}} x) = 1 \text{ for } x \neq 0, \quad |x| << 1.$$

Then we have

$$J(\lambda, s) = \int a(x_1, x_2) e^{i\lambda \Psi_1(x,s)} dx = \sum_{\kappa=\kappa_0}^{\infty} \int a(x_1, x_2) \chi(2^{\frac{\kappa}{3}} x) e^{i\lambda \Psi_1(x,s)} dx.$$

Let

$$J_\kappa = \int a(x_1, x_2) \chi(2^{\frac{\kappa}{3}} x) e^{i\lambda \Psi_1(x,s)} dx.$$

Let us use scaling $2^{\frac{\kappa}{3}} x \longrightarrow x$ and obtain

$$J_\kappa = 2^{-\frac{2\kappa}{3}} \int a(2^{-\frac{\kappa}{3}} x) \chi(x) e^{i\lambda 2^\kappa \Psi_1(x,s)} dx,$$

$$\Psi(x, s) = \tilde{b}(2^{-\frac{\kappa}{3}} x) x_1 x_2^2 + 2^{\frac{2\kappa}{3}} s_1(x_1 + x_2^{m_1} 2^{-\frac{\kappa}{3}(m_2-1)} \tilde{\varphi}(2^{-\frac{\kappa}{3}} x_2)) +$$
$$+ 2^{\frac{2\kappa}{3}} s_2(x_2 + x_1^{m_1} 2^{-\frac{\kappa}{3}(m_1-1)} \tilde{\psi}(2^{-\frac{\kappa}{3}} x_1)).$$

Note that $x \in D$. If $|2^{\frac{2\kappa}{3}} s_1| >> 1$ or $|2^{\frac{2\kappa}{3}} s_2| >> 1$ then using integration by parts, we obtain

$$|J_\kappa| \leqslant c \frac{2^{-\frac{2\kappa}{3}}}{|\lambda 2^{-\kappa}|(|2^{\frac{2\kappa}{3}} s_1| + 2^{\frac{2\kappa}{3}} |s_2|)}.$$

Let us take the integral

$$\int\limits_{2^{\frac{2\kappa}{3}} |s| > 1} \frac{2^{\frac{\kappa}{3} p} ds}{(|2^{\frac{2\kappa}{3}} s_1| + 2^{\frac{2\kappa}{3}} |s_2|)^p}.$$

After the change variable $2^{\frac{2\kappa}{3}}s=\sigma$ we have

$$\int\limits_{|\sigma|>1} 2^{\frac{\kappa p}{3}-\frac{4\kappa}{3}}\frac{d\sigma}{|\sigma|^p}=2^{\frac{\kappa}{3}(p-4)}\int\limits_{|\sigma|>1}\frac{d\sigma}{|\sigma|^p}=2^{\frac{\kappa}{3}(p-4)}c_p.$$

Thus, if $p<4$ then the series $\sum\limits_{\kappa=\kappa_0}^{\infty}\dfrac{2^{\frac{\kappa}{3}}\chi|2^{\frac{2\kappa}{3}}s|}{|2^{\frac{2\kappa}{3}}s_1|+2^{\frac{2\kappa}{3}}|s_2|}$ converges in $L^p$. Let $2^{\frac{2\kappa}{3}}s=\sigma$ and $|\sigma|\leqslant 1$.

Now, we use compactness arguments.

Let us assume that $\sigma=\sigma^0\neq 0$ and $(x_1^0,x_2^0)$ is a critical point of the phase function.

Then $\Phi_\kappa(x,\sigma)$ can be considered as a small perturbation of the function

$$\Phi=\tilde{b}(0,0)x_1x_2^2+\sigma_1^0x_1+\sigma_2^0x_2,$$

where $(x_1,x_2)\in D$. If $(\sigma_1^0,\sigma_2^0)\neq(0,0)$ then $x_2^0\neq 0$. Hence

$$Hess\Phi=\begin{vmatrix} 0 & 2x_2^0\\ 2x_2^0 & 2x_1^0\end{vmatrix}b^2(0,0)=-4(x_2^0)^2b^2(0,0)\neq 0.$$

Then we can use stationary phase method in two variables and obtain

$$|J^\chi|\leqslant\frac{c}{|\lambda|}$$

in a neighbourhood of $\sigma^0$.

Finally, let us consider the case when $(\sigma_1^0,\sigma_0^0)=(0,0)$. Since $(x_1,x_2)\in D$, then $x_2^0=0$ and $x_1^0\neq 0$. Thus $x_1^0\sim 1$.

$$\tilde{b}(2^{\frac{\kappa}{3}}x)x_1x_2^2+\sigma_1(x_2^{m_1}2^{-\frac{\kappa}{3}(m_2-1)}\tilde{\varphi}(2^{-\frac{\kappa}{3}}x_2))+\sigma_2x_2.$$

$$x_2=-\frac{\sigma_2}{2x_1}g(2^{-\frac{\kappa}{3}}x_1,2^{-\frac{\kappa}{3}(m_2-1)}\sigma_1)$$

Using stationary phase method in $x_2$, we obtain oscillatory integral with phase $g(0,0)\neq 0$.

$$\Phi_\kappa(\sigma,x_1):=\frac{\sigma_2^2}{4x_1}G\big(2^{-\frac{\kappa}{3}}x_1,2^{-\frac{\kappa}{3}(m_2-1)}\sigma_1\big)+\sigma_1x_1+\sigma_2x_1^{m_1}2^{-\frac{\kappa}{3}(m_2-1)}\tilde{\psi}(2^{-\frac{\kappa}{3}}x_1)$$

$x_1\sim 1$, $\sigma_2^2\sim\sigma_2 2^{-\frac{\kappa}{3}(m_2-1)}2^{\frac{\kappa}{3}(m_2-1)}\sigma_2\sim 1$.

Let us consider the following one-dimensional oscillatory integral

$$J_\kappa(\lambda,\sigma)=\frac{2^{\frac{\kappa}{6}}}{\lambda^{\frac12}}\int\limits_{\mathbb{R}}e^{i\lambda 2^{-\kappa}\Phi_\kappa(\sigma,x_1)}a(x_1)dx_1$$

where $|\lambda 2^{-\kappa}|>1$.

We prove the following Lemma.

**Lemma 4.1.** *Let $x_1^0\neq 0$ be a fixed point. Then there exist a cut-off function $\chi$ supported in a neighborhood of $x_1^0,k_0,c_0,c$ such that for any $\kappa>\kappa_0$ the following estimate holds true:*

$$|J_\kappa^\chi|\leqslant\frac{2^{\frac{\kappa}{3}}c}{\lambda^{\frac12}}\bigg(\frac{1}{|\sigma|^{\frac13}|\sigma_2|^{\frac13}}+\frac{\chi_{|\sigma_1|\leqslant c\sigma_2^2}(\sigma_1,\sigma_2)}{|\sigma_2|^{\frac12}|\sigma_1-c_0\sigma_2^2|^{\frac14}}\bigg).$$

Proof of the Lemma follows from the results presented in [7]. It is easy to see that for any $p < 3$ $\Psi_0 \in L^p_{loc}(\mathbb{R}^2)$, where

$$\Psi_0(\sigma_1, \sigma_2) = \frac{1}{|\sigma|^{\frac{1}{3}}|\sigma_2|^{\frac{1}{3}}} + \frac{\chi_{|\sigma_1| \leqslant c\sigma_2^2}(\sigma_1, \sigma_2)}{|\sigma_2|^{\frac{1}{2}}|\sigma_1 - c_0\sigma_2^2|^{\frac{1}{4}}}.$$

**Corollary.** There exists $\kappa_0$ such that for any $\kappa > \kappa_0$ the following estimate holds true:

$$|J_\kappa(\lambda, \sigma)| \leqslant \frac{\Psi(\sigma_1, \sigma_2)2^{\frac{\kappa}{3}}}{|\lambda|^{\frac{1}{2}}},$$

where $\Psi \in L^{3-0}_{loc}(\mathbb{R})$. The following theorem holds true.

**Theorem 4.1.** *Let $s$ be an analytic hypersurface such that it has $D_\infty$ type of singularity at the origin. Then there exists a neighbourhood $U \subset \mathbb{R}^3$ such that for any $\Psi \in C_0^\infty(U)$, $M \in L^{3-0}(S^2)$.*

# 5. Summation of the Fourier transform of measures

Let $S$ be an analytic hypersurface and

$$d\mu = \psi(x)dS.$$

We prove the following Theorem.

**Theorem 5.1.** *Let $S$ be an analytic hypersurface. If $S$ has $D_\infty$ type of singularity at the origin then there exists a neighbourhood $U$ of the origin such that for any $\Psi \in C_0^\infty(U)$ the inclusion $\hat{\mu} \in L^p(\mathbb{R}^3)$ holds for any $p > 3$.*

*Proof.* It is well known that there exists a neighbourhood $U$ of the origin such that for any $\Psi \in C_0^\infty(U)$ the following estimate holds true (see [8])

$$|\mu(\hat{\xi})| \leqslant \frac{c}{(1 + |\xi|)^{\frac{1}{2}}}. \tag{3}$$

According to Theorem 4.1, there exists a function $\Psi(\omega) \in L^{3-0}(s^2)$ such that

$$|\hat{\mu}(r\omega)| \leqslant \frac{\Psi(\omega)}{(1 + r)}. \tag{4}$$

Let $p > 3$ be a fixed number. Let us take $q < 3$. We interpolate estimates (3) and (4) and obtain

$$|\hat{d\mu}(r\omega)| \leqslant \frac{c}{(1 + r)^{\frac{\alpha}{2}+\beta}}\Psi(\omega)^\beta.$$

If $p > 3$ one can choose $\alpha$ and $\beta$ such that $p\left(\frac{\alpha}{2} + \beta\right) > 3$ and $p\beta < 3$.

For instance, we take a sufficiently small positive number $\delta > 0$ and set $\beta = \dfrac{3 - \delta}{p}$ and $\alpha = \dfrac{p - 3 + \delta}{p}$. Then it is easy to see that

$$\int_{\mathbb{R}^3} |d\hat{\mu(\xi)}|^p \xi \leqslant c \int_0^\infty \frac{r^2 dr}{(1 + z)^{(\frac{\alpha}{2}+\beta)p}} \int_{S^2} (\Psi(\omega))^{p\beta} d\omega < +\infty.$$

Theorem 5.1 is proved. □

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## $L^p$-оценки преобразования Фурье поверхностных мер, сосредоточенных на гиперповерхностях с особенностью типа $D_\infty$


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**Аннотация.** В этой статье рассматриваются оценки преобразования Фурье мер, сосредоточенных на невыпуклых поверхностях трехмерного евклидова пространства. Мы найдем точный покозатель, для которого преобразование Фурье мер с этой степенью интегрируемо по трехмерному пространству. Этот результат дает ответ на вопрос, поставленный Эрдошем и Салмхофером.



**Ключевые слова:** преобразование Фурье, осцилляторный интеграл, поверхностная мера.

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# On the Orbits of Nilpotent 7-dimensional Lie Algebras in 4-dimensional Complex Space

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**Abstract.** We study one-parameter families of 7-dimensional nilpotent indecomposable Lie algebras and the orbits of holomorphic realizations of such algebras in a 4-dimensional complex space. It is shown, in contrast to the orbits of 5-dimensional nilpotent Lie algebras in 3-dimensional case, that two such families (out of the existing nine ones) admit orbits that are Levi non-degenerate (homogeneous) real non-spherical hypersurfaces. Up to holomorphic equivalence, all obtained non-degenerate nonspherical orbits are graphs of polynomials of degree 4.

**Keywords:** Lie algebra, complex space, vector field, holomorphic function, homogeneous variety, Levi degeneracy.

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## Introduction

In connection with the problem of describing holomorphically homogeneous real hypersurfaces of 3-dimensional complex space the technique was developed in the articles [1–4] for studying holomorphic realizations of 5-dimensional Lie algebras in the space $\mathbb{C}^3$.

Under this, according to the result of [2], the orbits of nilpotent 5-dimensional Lie algebras in the space $\mathbb{C}^3$ can be either Levi degenerate hypersurfaces, or non-degenerate spherical ones, i.e. holomorphic images of non-degenerate quadrics

$$\operatorname{Im} z_3 = |z_1|^2 \pm |z_2|^2.$$

In the present work, a similar technique is used for study of holomorphically homogeneous hypersurfaces in the space $\mathbb{C}^4$. In particular, below we consider the orbits of holomorphic real-

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izations in this space of nilpotent 7-dimensional Lie algebras (representations of these algebras in the algebra of germs of holomorphic vector fields in $\mathbb{C}^4$).

Note that situations with 5-dimensional and 7-dimensional Lie algebras significantly differ from each other in the number of algebraically different objects. So, according to [5], there are only 9 nilpotent 5-dimensional Lie algebras (6 indecomposable and 3 decomposable). And the family of 7-dimensional indecomposable Lie algebras contains (see [6]) 140 isolated nilpotent algebras and 9 one-parameter families of nilpotent Lie algebras. In [6], these families are denoted by

$$147E, 1357M, 1357N, 1357S, 12457N, 123457I, 147E_1, 1357QRS_1, 12457N_2. \tag{1}$$

Below 6 such families are considered with the aim of comparing situations with holomorphic homogeneity of hypersurfaces in 3-dimensional and 4-dimensional complex spaces. The main interest of the article is related to nondegenerate orbits of Lie algebras; the main results are presented in the following two theorems.

**Theorem 1.** *Each 7-dimensional orbit in the space $\mathbb{C}^4$ of any algebra of four families $1357N$, $12457N$, $123457I$, $12457N_2$ must be degenerate in Levi sense.*

**Theorem 2.** *The two families $1357M$, $1357QRS_1$ have algebras, admitting holomorphic realizations with Levi non-degenerate non-spherical 7-dimensional orbits in the space $\mathbb{C}^4$. Up to holomorphic equivalence, all such orbits of the $1357M$ family are described by the formulas*

$$y_4 = y_1 y_3 + y_2^2 + y_1^2 y_2 + D y_1^4, \quad D \neq \frac{1}{12}; \tag{2}$$

*among Levi non-degenerate nonspherical orbits of the $1357QRS_1$ family there are holomorphic images of the surfaces*

$$y_4 = y_1 y_3 + y_2^2 + x_1 y_1 y_2 + D y_1^4, \quad D \neq \frac{1}{12}. \tag{3}$$

The families $1357S$, $147E$, $147E_1$ also contain algebras whose orbits in $\mathbb{C}^4$ are non-degenerate nonspherical hypersurfaces. However, all such surfaces are described by formula (2). Due to the limited scope of the article, we do not discuss these three families.

# 1. Families of 7-dimensional nilpotent Lie algebras

The existence of the families of nilpotent algebras depending on the real parameter, distinguishes the cases of 7-dimensional algebras and 5-dimensional ones. We denote this parameter by the common symbol $\lambda$ and discuss the families of 7-dimensional indecomposable algebras mentioned above. Each of them is described in some basis $e_1, \dots, e_7$ by the following relationships:

$$\begin{aligned} 1357M\ (\lambda \neq 0)\ &: [e_1, e_2] = e_3,\ [e_1, e_3] = e_5,\ [e_1, e_4] = e_6,\ [e_1, e_5] = e_7, \\ &[e_2, e_4] = e_5,\ [e_2, e_6] = \lambda e_7,\ [e_3, e_4] = (1-\lambda) e_7. \end{aligned} \tag{4}$$

$$\begin{aligned} 1357N\ &: [e_1, e_2] = e_3,\ [e_1, e_3] = e_5,\ [e_1, e_4] = e_6,\ [e_1, e_5] = e_7, \\ &[e_2, e_3] = \lambda e_7,\ [e_2, e_4] = e_5,\ [e_3, e_4] = e_7,\ [e_4, e_6] = e_7. \end{aligned} \tag{5}$$

$$\begin{aligned} 12457N\ &: [e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_7,\ [e_1, e_5] = e_6,\ [e_1, e_6] = e_7, \\ &[e_2, e_3] = e_5,\ [e_2, e_4] = e_6,\ [e_2, e_5] = \lambda e_7,\ [e_2, e_6] = e_7,\ [e_3, e_4] = e_7,\ [e_3, e_5] = -e_7. \end{aligned} \tag{6}$$

$$\begin{aligned} 123457I\ &: [e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_1, e_5] = e_6,\ [e_1, e_6] = e_7, \\ &[e_2, e_3] = e_5,\ [e_2, e_4] = e_6,\ [e_2, e_5] = \lambda e_7,\ [e_3, e_4] = (1-\lambda) e_7. \end{aligned} \tag{7}$$

$$\begin{aligned} 1357QRS_1\ (\lambda \neq 0)\ :\ &[e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6,\ [e_1,e_5]=e_7,\\ &[e_2,e_3]=-e_6,\ [e_2,e_4]=e_5,\ [e_2,e_6]=\lambda e_7,\ [e_3,e_4]=(1-\lambda)e_7. \end{aligned} \tag{8}$$

$$\begin{aligned} 12457N_2\ :\ &[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=-e_6,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_7,\\ &[e_2,e_3]=e_5,\ [e_2,e_4]=e_7,\ [e_2,e_5]=-e_6+\lambda e_7,\ [e_3,e_5]=-e_7. \end{aligned} \tag{9}$$

The main point of [1–4], as well as of this article, is the use of large Abelian subalgebras of studied algebras and simplification of bases of these subalgebras.

It can be noted that for nilpotent Lie algebras of arbitrary dimensions the existence of such subalgebras (and even Abelian ideals) follows from Morozov's known statement [7].

In the case of algebras from the families (4)–(9) it is easy to make sure that each of them contains a 4-dimensional Abelian ideal with the following basis:

$$12457N,\ 12457N_2,\ 123457I\ :\ I_4 =< e_4, e_5, e_6, e_7 >,$$

$$1357M,\ 1357N,\ 1357QRS_1\ :\ I'_4 =< e_3, e_5, e_6, e_7 > .$$

**Remark.** For some of the families (4)–(9), each algebra contains several different 4-dimensional Abelian ideals. For instance, in every algebra from the family of $1357M$ there are (in addition to the above ideal $I'_4 =< e_3, e_5, e_6, e_7 >$) also $I_4 =< e_4, e_5, e_6, e_7 >$ and $I''_4 =< e_2, e_3, e_5, e_7 >$ .

## 2. Degeneracy of the orbits of 7-dimensional Lie algebras

A scheme for constructing realizations of 7-dimensional Lie algebras as the algebras of holomorphic vector fields in space $\mathbb{C}^4$ essentially repeats a similar scheme implemented in [1–4] for 5-dimensional algebras. The main technical idea here is to simplify the form of basis vector fields of distinguished ideal (and then basis of the whole algebra under discussion). Algebras with simplified bases can be integrated (with overcoming certain technical difficulties).

So, write each element of the basis $e_1, \ldots, e_7$ of the discussed Lie algebra $\mathfrak{g}$ in the form of a holomorphic vector field in space $\mathbb{C}^4$:

$$e_k = a_k(z)\frac{\partial}{\partial z_1} + b_k(z)\frac{\partial}{\partial z_2} + c_k(z)\frac{\partial}{\partial z_3} + d_k(z)\frac{\partial}{\partial z_4} \quad (k = 1, \ldots, 7). \tag{10}$$

In this entry, $a_k(z),\ b_k(z),\ c_k(z),\ d_k(z)$ are holomorphic (near the discussed point of the surface) functional coefficients, $z = (z_1, z_2, z_3, z_4)$ is a vector of complex coordinates. We will also use entries of the form $e_k = (a_k, b_k, c_k, d_k)$ to shorten the formula (10).

A real hypersurface $M = \{\Phi = 0\}$ is the orbit (or integral surface) of a holomorphic realization of the algebra $\mathfrak{g}$ if for each base field $e_k$ of this algebra the condition of tangency $M$ is satisfied in the form

$$Re\,(e_k\ (\Phi)|_M) = 0. \tag{11}$$

**Lemma 1.** *Let a real hypersurface $M \subset \mathbb{C}^4$ be Levy non-degenerate near some of its point $Q$ and let it be the orbit of the 7-dimensional Lie algebra $\mathfrak{g}$ of holomorphic vector fields in this space. Let also $I_4$ be a 4-dimensional Abelian subalgebra in $\mathfrak{g}$ with a fixed basis $e_4, e_5, e_6, e_7$.*

*This basis can be reduced to one of three forms by holomorphic change of coordinates of the space $\mathbb{C}^4$ (defined near the point $Q$):*

$$1)\ \begin{array}{l} e_4 : (1,\ 0,\ 0,\ 0),\\ e_5 : (0,\ 1,\ 0,\ 0),\\ e_6 : (0,\ 0,\ 1,\ 0),\\ e_7 : (0,\ 0,\ 0,\ 1), \end{array} \quad 2)\ \begin{array}{l} (0,\ b_4(z_1),\ c_4(z_1),\ d_4(z_1)),\\ (0,\ 1,\ 0,\ 0),\\ (0,\ 0,\ 1,\ 0),\\ (0,\ 0,\ 0,\ 1), \end{array} \quad 3)\ \begin{array}{l} (0,\ 1,\ 0,\ 0),\\ (0,\ 0,\ c_5(z_1),\ d_5(z_1)),\\ (0,\ 0,\ 1,\ 0),\\ (0,\ 0,\ 0,\ 1). \end{array}$$

**Remark 1.** One can use the Abelian ideals mentioned above as Abelian subalgebras for the studied families of Lie algebras (4)–(9).

*Proof of Lemma 1.* First, we note that the assertion being proved is a simple generalization of Lemmas 3.1–3.4, formulated and proved in [4] for algebras of holomorphic vector fields in a 3-dimensional complex space. As in [1, 4], reduction the basis of the subalgebra to one of the desired types is possible by a step-by-step procedure.

For example, in the space $\mathbb{C}^n$ of any dimension $n$, a separate holomorphic vector field can be (locally) rectified, i.e. reduced by a holomorphic change of coordinates to differentiation with respect to one of the variables. In the situation discussed in Lemma 1, we bring thus the field $e_7$ to the form $\partial/\partial z_4 = (0, 0, 0, 1)$.

For another basis field $e_6$ of the subalgebra $I_4$ we consider its components $(a_6, b_6, c_6, d_6)$ in the new coordinates. Firstly, due to the commutation of the fields $e_6$ and $e_7$, these components are independent of the $z_4$ variable. And secondly, the truncated set $(a_6, b_6, c_6)$ cannot be identically zero, because a linear dependence over $\mathbb{C}$ (at each point of the surface $M$) of two fields $e_6$ and $e_7$ would mean Levy degeneracy of the discussed orbit.

Then the "truncated" vector field $(a_6(z_1, z_2, z_3), b_6(z_1, z_2, z_3),\ c_6(z_1, z_2, z_3))$ can be (locally) straightened by a holomorphic change of three variables $z_1, z_2, z_3$. Accordingly, the field $e_6$ will take the form $(0, 0, 1, d_6(z_1, z_2, z_3))$, and the field $e_7 = (0, 0, 0, 1)$ will remain rectified. The reduction of the field $e_6$ to the rectified form is completed (with preservation of the rectified field $e_7$) by another holomorphic change of coordinates $z_4^* = z_4 + \varphi(z_1, z_2, z_3)$, with an arbitrary function $\varphi(z_1, z_2, z_3)$ satisfying the condition

$$\partial\varphi/\partial z_3 = -d_6(z_1, z_2, z_3).$$

Further, when passing to the simplification of the field $e_5$, several cases arise. A common fact for all such cases is independence of $e_5$ components of two variables $z_3, z_4$ (due to the commutation of $e_5$ with the fields $e_6$, $e_7$).

In the first case, with a nonzero "truncated" field $(a_5(z_1, z_2), b_5(z_1, z_2))$ it can be straightened (just like the whole field $e_5$) due to operations similar to those described when rectifying the field $e_6$. The basis of the algebra $I_4$ takes then form

$$\begin{array}{rcllll} e_4 & : & (a_4(z_1), & b_4(z_1), & c_4(z_1), & d_4(z_1)), \\ e_5 & : & (0, & 1, & 0, & 0), \\ e_6 & : & (0, & 0, & 1, & 0), \\ e_7 & : & (0, & 0, & 0, & 1). \end{array} \tag{12}$$

Note that the components of the field $e_4$ in this situation can depend on no more than a single complex variable $z_1$ due to the commutation of all fields in $I_4$.

The second case is also possible, in which (after straightening the fields $e_6, e_7$) the basis of the subalgebra $I_4$ has the following coordinate representation

$$\begin{array}{rcllll} e_4 & : & (a_4(z_1, z_2), & b_4(z_1, z_2), & c_4(z_1, z_2), & d_4(z_1, z_2)), \\ e_5 & : & (0, & 0, & c_5(z_1, z_2), & d_5(z_1, z_2)), \\ e_6 & : & (0, & 0, & 1, & 0), \\ e_7 & : & (0, & 0, & 0, & 1). \end{array}$$

Note also that in the case of identically zero components $a_4, b_4$, the four fields $e_4, \ldots, e_7$ turn out to be linearly dependent over $\mathbb{C}$ at each point on the surface $M$, and the equation of the surface (due to the tangency conditions for these fields) get rid of the variables $z_1, z_2$. Because in a Levy non-degenerate case this is impossible, the components $a_4, b_4$ cannot be both identically zero.

Now, using the inequality $(a_4(z_1, z_2), b_4(z_1, z_2)) \neq (0, 0)$, we can, as in the discussion of the field $e_6$, straighten such a "truncated" field by a holomorphic change of two variables $z_1, z_2$. Then the field $e_4$ is rectified to the state $(0, 1, 0, 0)$. The fields $e_5, e_6$ will retain under this their rectified appearance.

In this case, the whole base tetrad of the subalgebra $I_4$ acquires the third of the possible forms indicated in the lemma that we are proving. The first two possibilities arise when completing simplifications of a basis of the form (12).

If, for example, the coefficient $a_4(z_1)$ is not identically zero, then turning it into unity and after that straightening the field $e_1$ using the procedures already described, we get the "fully" rectified subalgebra $I_4$, i.e. case 1) from the lemma being proved. If $a_4(z_1) \equiv 0$, we get the form 2) for the basis of the algebra $I_4$. Lemma 1 is completely proved. □

Further, it is proposed to consider for each algebra three possible cases from this lemma. Under this, one can significantly simplify such considerations using the following two statements.

**Proposition 1.** *Suppose that a 7-dimensional real Lie algebra $\mathfrak{g}$ has a basis $e_1, ..., e_7$ with the following properties:*

*1) $I_4 =< e_4, e_5, e_6, e_7 >$ is an Abelian ideal in $\mathfrak{g}$;*

*2) for $h =< e_1, e_2, e_3 >$ and $I_4$, the set of commutators $[h, I_4]$ is contained in the linear span $< e_5, e_6, e_7 >$;*

*3) $[e_1, e_2] = e_4$.*

*Then a holomorphic realization of the algebra $\mathfrak{g}$ in the space $\mathbb{C}^4$ with the "straightened" ideal $I_4$ is impossible.*

*Proof.* Suppose, on the contrary, that in the space $\mathbb{C}^4$ there exists a holomorphic realization of the algebra $\mathfrak{g}$ with a "straightened" basis of the ideal $I_4$, satisfying conditions 1)–3).

Consider a nonzero element $e_1$ of three-dimensional subspace $h$ of the Lie algebra $\mathfrak{g}$. Due to conditions 1)–2) of the proposition under discussion, we have

$$[e_1, e_4] = 0 \cdot e_4 + A_5 e_5 + A_6 e_6 + A_7 e_7 = (0, A_5, A_6, A_7) \tag{13}$$

with some real constants $A_5, A_6, A_7$.

But for a rectified holomorphic vector field $e_4$, the commutator $[e_1, e_4]$ is equal to $-\dfrac{\partial}{\partial z_1}(e_1)$.

This means that, by virtue of equality (13), the field $e_1$ can be represented in coordinates in the form

$$e_1 = (a_1(z_2, z_3, z_4), -A_5 z_1 + b_1(z_2, z_3, z_4), -A_6 z_1 + c_1(z_2, z_3, z_4), -A_7 z_1 + d_1(z_2, z_3, z_4)). \tag{14}$$

Similarly to (13), the commutators $[e_1, e_5], [e_1, e_6], [e_1, e_7]$, of the field $e_1$ with differentiations with respect to variables $z_2, z_3, z_4$ also do not contain $e_4$ in its expansions. This means that one can refine the form of the four functional coefficients

$$(a_1(z_2, z_3, z_4), b_1(z_2, z_3, z_4), c_1(z_2, z_3, z_4), d_1(z_2, z_3, z_4))$$

in formula (14) and write the field $e_1$ as follows:

$$e_1 = (A_1, L_{12}(z_1, z_2, z_3, z_4) + B_1, L_{13}(z_1, z_2, z_3, z_4) + C_1, L_{14}(z_1, z_2, z_3, z_4) + D_1). \tag{15}$$

Here, all linear forms $L_{1k}(z_1, z_2, z_3, z_4)$ $(k = 2, 3, 4)$ have only real coefficients, and $A_1, B_1, C_1, D_1$ are some complex constants.

It is clear that any vector field from the subspace $h$ has the form (15). Consider in such a situation a commutator of the field $e_1$ with a similar field

$$e_2 = (A_2, L_{22}(z_1, z_2, z_3, z_4) + B_2, L_{23}(z_1, z_2, z_3, z_4) + C_2, L_{24}(z_1, z_2, z_3, z_4) + D_2).$$

It is clear that the $z_1$-component of such a commutator is zero, which contradicts condition 3) of the proposition under discussion. Consequently, this proposition is proved. □

**Proposition 2.** *For each of the discussed algebras of all six families (4)–(9) there are bases satisfying conditions 1)–3) of Proposition 1.*

*Proof.* The $1357M$ and $1357QRS_1$ families contain the ideal $I_4 =< e_3, e_5, e_6, e_7 >$. Under this decomposition $[e_1, I_4]$, $[e_2, I_4]$, $[e_4, I_4]$ do not contain $e_3$ component, and $[e_1, e_2] = e_3$.

In the $1357N$ family with the same ideal $I_4 =< e_3, e_5, e_6, e_7 >$ the expansions $[e_1, I_4]$, $[e_2, I_4]$, $[e_4, I_4]$ do not contain $e_6$ components, and $[e_1, e_4] = e_6$.

Finally, the families $12457N$, $123457I$, $12457N_2$ have the same structure from the point of view interesting to us: the 4-dimensional Abelian ideal in algebras from these families is $I_4 =< e_4, e_5, e_6, e_7 >$; decompositions of $[e_1, I_4]$, $[e_2, I_4]$, $[e_3, I_4]$ do not contain $e_4$-component and at the same time $[e_1, e_3] = e_4$. Proposition 2 is proved. □

Recall that the main interest of the paper is related to Levi non-degenerate orbits of nilpotent Lie algebras. Application of Propositions 1 and 2 to each of the six families of algebras (4)–(9) (with the basis vectors ordering in 4-dimensional ideals corresponding to the Proposition 2) allows in all cases to reduce the meaningful discussions to the points 2) and 3) of Lemma 1.

## 3. An analogy with the case of 5-dimensional algebras

**Proposition 3.** *Realizations of the algebras* $1357N$, $12457N$, $123457I$, $12457N_2$ *with the fixed bases of their Abelian ideals simplified to types 2) or 3) can have only Levi-degenerate orbits in the space* $\mathbb{C}^4$.

**Remark.** We discuss Proposition 3 only for the $12457N$-family described by the maximal number of nontrivial commutation relations. The three remaining families (with the corresponding renumbering of the base fields) can be considered similarly.

*Proof.* Proposition 1 prohibits the existence of Levi-non-degenerate orbits for this family in case 1) of Lemma 1. We show that the 21 commutation relations (10 of which are trivial) in the 7-dimensional Lie algebra of the $12457N$-family also contradict cases 2) and 3) of this lemma.

In case 2) of Lemma 1 we have a triple of rectified fields

$$e_5 = (0, 1, 0, 0), \ e_6 = (0, 0, 1, 0), \ e_7 = (0, 0, 0, 1) \tag{16}$$

and the field $e_4 = (0, b_4(z_1), c_4(z_1), d_4(z_1))$.

We note, first, that the six pairwise relations for the four basis fields of the ideal have already been used (verified) for the obtaining a simplified form of the basis of the ideal $I_4$. Second, consideration of nine commutators of each of the triple rectified fields $e_5, e_6, e_7$ with each of the three fields $e_1, e_2, e_3$ from the complement to the ideal $I_4$ allows as to we get a simplified form of these additional fields:

$$\begin{aligned} e_1 &= (a_1(z_1), b_1(z_1), -z_2 + c_1(z_1), -z_3 + d_1(z_1)), \\ e_2 &= (a_2(z_1), b_2(z_1), c_2(z_1), -z_3 - \lambda z_2 + d_2(z_1)), \\ e_3 &= (a_3(z_1), b_3(z_1), c_3(z_1), z_2 + d_3(z_1)). \end{aligned} \tag{17}$$

Next, we move on to more "subtle" verification actions. For example, by coordinate-wise writing the relation $[e_2, e_4] = e_6$, we get

$$a_2(0, b_4'(z_1), c_4'(z_1), d_4'(z_1)) - (b_4(0,0,0,-\lambda) + c_4(0,0,0,-1)) = (0,0,1,0).$$

The second and the third components of this vector equality have the form

$$a_2(z_1) \cdot b_4'(z_1) = 0, \quad a_2(z_1) \cdot c_4'(z_1) = 1.$$

This means that $a_2(z_1)$ is nonzero (near the origin). In this situation, one can use the "linearization lemma" proved in [1] (see also [4], Remark 3.2). This lemma, applied to the field $e_2$, allows us to bring it after a holomorphic change of coordinates to the form

$$e_2 = (1, 0, 0, -z_3 - \lambda z_2)$$

instead of the more complicated form fixed in formulas (17). The rectified fields $e_5, e_6, e_7$ will be preserved; the fields $e_1, e_3, e_4$ will also retain their simplified structure.

Considering further the equalities $[e_1, e_4] = [e_3, e_4] = e_7$, one can obtain the next simplifications of the first four basis fields:

$$\begin{aligned} &e_1 = \big(1, b_1(z_1), -z_2 + c_1(z_1), -z_3 + d_1(z_1)\big), \quad e_2 = (1, 0, 0, -z_3 - \lambda z_2), \\ &e_3 = \big(0, b_3(z_1), c_3(z_1), z_2 + d_3(z_1)\big), \quad e_4 = \Big(0, -1, \frac{1-\lambda}{2}, \frac{1+\lambda}{2} z_1 + D_4\Big), \end{aligned} \tag{18}$$

where $D_4$ is a complex constant.

Finally, we check the remaining three commutation relations

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \qquad [e_2, e_3] = e_5. \tag{19}$$

The last of them contains restrictions on the field $e_3$, which now takes the form

$$e_3 = \Big(0, z_1 + B_3, C_3, z_2 - \frac{\lambda}{2}(z_1 + B_3)^2 + C_3(z_1 + B_3) + D_3\Big) \tag{20}$$

with arbitrary complex constants $B_3, C_3, D_3$.

Then the first of relations (19) leads to a rather complicated form of the field

$$e_1 = \Big(1, -\frac{1}{2}(z_1 + B_3)^2 + B_1, -z_2 + (-C_3 z_1 + C_1), -z_3 - \frac{\lambda}{3}(z_1 + B_3)^3 - N z_1 + D_1\Big), \tag{21}$$

where $N = (\lambda B_1 + C_2 B_3 + D_3 - C_1)$, and $B_1, C_1, D_1 \in \mathbb{C}$ are arbitrary constants.

Taking into account formulas (20), (21), the left-hand side of the second relations (19) can be written in a form

$$\begin{aligned} [e_1, e_3] = \Big( &\big(0, 1, 0, -\lambda(z_1 + B_3) + C_3\big) + \Big(-\frac{1}{2}(z_1 + B_3)^2 + B_1\Big) \cdot (0,0,0,1)\Big) - \\ &- \big((z_1 + B_3) \cdot (0,0,-1,0) + C_3(0,0,0,-1)\big). \end{aligned}$$

The second component of this vector field equals to 1, contrary to the fact that the equal field $e_4$ has $-1$ in the second component. Thus, in the second case of Lemma 1, algebras of the family $123457N$ do not admit holomorphic realizations.

Now discussing case 3) of Lemma 1, we have:

$$\begin{array}{lcl} e_4 & : & (\ 0\ ,\quad 1\ ,\qquad 0, \qquad\quad 0\ ) \\ e_5 & : & (\ 0, \quad 0\ ,\quad c_5(z_1), \quad d_5(z_1)) \\ e_6 & : & (\ 0, \quad 0\ , \qquad 1, \qquad\quad 0\ \ ) \\ e_7 & : & (\ 0, \quad 0\ , \qquad 0, \qquad\quad 1\ ) \end{array}. \tag{22}$$

Using the nine commutation relations $[e_1, e_4] = e_7,\ [e_1, e_6] = e_7,\ [e_1, e_7] = 0,$

$$[e_2, e_4] = e_6,\ [e_2, e_6] = e_7,\ [e_2, e_7] = 0, \quad [e_3, e_4] = e_7,\ [e_3, e_6] = 0,\ [e_3, e_7] = 0,$$

we obtain formulas

$$\begin{aligned} e_1 &= \big(a_1(z_1), b_1(z_1), c_1(z_1), -z_2 - z_3 + d_1(z_1)\big), \\ e_2 &= \big(a_2(z_1), b_2(z_1), -z_2 + c_2(z_1), -z_3 + d_2(z_1)\big), \\ e_3 &= \big(a_3(z_1), b_3(z_1), c_3(z_1), -z_2 + d_3(z_1)\big), \end{aligned} \tag{23}$$

similar to the collection (18) from the case 2.

We now note that in the case of identical vanishing of any two of the three coefficients $a_1(z_1), a_2(z_1), a_3(z_1)$ the six basic fields of the algebra under discussion turn out to be linearly dependent over $\mathbb{C}$. This leads to Levi degeneration of **all** orbits of the Lie algebra with a basis of the form (22)–(23).

Therefore, the search for algebras with such bases admitting non-degenerate orbits can be reduced to two subcases:

subcase 3.1. $a_3(z_1) \neq 0$;

subcase 3.2. $a_3(z_1) \equiv 0,\ a_1(z_1) \neq 0$.

Applying the linearization lemma in each of these subcases, we can, in addition to the rectified triple of fields from (22), significantly simplify one another field. In subcase 3.1, we can regard the field $e_3$ having a simplified form $e_3 = (1, 0, 0, -z_2)$, and in subcase 3.2 we have

$$e_1 = (1, 0, 0, -z_2 - z_3), \quad e_3 = \big(0, b_3(z_1), c_3(z_1), -z_2 + d_3(z_1)\big). \tag{24}$$

Given this simplification, we consider in subcase 3.1 three commutation relations $[e_1, e_3] = 0$, $[e_2, e_3] = e_5,\ [e_1, e_2] = e_3$.

The first of these relations has the expanded form

$$b_1(0, 0, 0, -1) - (a_1', b_1', c_1', d_1') = (0, 0, 0, 0).$$

It means that $e_1 = (A_1, B_1, C_1, -z_2 - z_3 - B_1 z_1 + D_1)$ with some complex constants $A_1, B_1, C_1, D_1$. Similarly, from $[e_2, e_3] = e_5$ we get

$$e_2 = \big(A_2, B_2, -z_2 + c_2(z_1), -z_3 + d_2(z_1)\big), \quad e_5 = \big(0, 0, -c_2'(z_1), -d_2'(z_1) - B_2\big)$$

with complex constants $A_2, B_2$ and holomorphic functions $c_2(z_1),\ d_2(z_1)$.

But taking into account the formulas obtained for the fields $e_1, e_2$, the first component of the commutator

$$\begin{aligned} [e_1, e_2] = \big(&A_1(0, 0, c_2'(z_1), d_2'(z_1)) + B_1(0, 0, -1, 0) + C_1(0, 0, 0, -1)\big) - \\ &- \big(A_2(0, 0, 0, -B_1) + B_2(0, 0, 0 - 1) + (-z_2 + c_2(z_1))(0, 0, 0, -1)\big) \end{aligned}$$

is zero, contrary to the fact that $e_3 = (1, 0, 0, -z_2)$.

A similar contradiction also arises in subcase 3.2. Here, from the equality $[e_1, e_3] = 0$ we obtain the formula

$$e_3 = \big(0, B_3, C_3, -z_2 - (B_3 + C_3)z_1 + D_3\big).$$

Then a refined form of the field $e_2$ is derived from the relation $[e_1, e_2] = e_3$:

$$e_2 = \big(A_2, B_3 z_1 + B_2, -z_2 + C_3 z_1 + C_2, -z_3 + (B_3 + C_3)z_1^2 + (D_3 - B_2 - C_2)z_1 + D_2\big).$$

And the field $e_5$, equal to the commutator of these two fields, will now take the form

$$e_5 = \big(0, 0, B_3, -B_3 z_1 + (-C_3 - B_2 - A_2 B_3 - A_2 C_3)\big). \tag{25}$$

In this subcase, the last three commutation relations were not considered: $[e_1, e_5] = e_6$, $[e_2, e_5] = \lambda e_7$, $[e_3, e_5] = -e_7$.

Taking into account formulas (24) and (25), the first of them leads to a contradiction, since

$$[e_1, e_5] = (0, 0, 0, -B_3) - B_3(0, 0, 0, -1) = (0, 0, 0, 0) \neq e_6.$$

Proposition 3 for the family of Lie algebras $12457N$ is proved. □

## 4. Integration of the $1357M$ and $1357QRS1$ families

The technique of the previous section allows us to obtain similar conclusions in the study of other algebras. The description of all possible holomorphic realizations of algebras from the families $1357M$ and $1357QRS1$ was received exactly in such manner. We omit here technical details of reasonings (close to fragments of Section 3) and note only the following two points.

1) The descriptions of holomorphic realizations of the two families $1357M$ and $1357QRS1$ almost coincide, because descriptions (4) and (8) of these families themselves have the only difference: the trivial commutator $[e_2, e_3] = 0$ in the family $1357M$ is replaced by a nontrivial relation $[e_2, e_3] = -e_6$ for the $1357QRS1$ family.

2) This description is connected with the ideal $I'_4 =< e_4, e_5, e_6, e_7 >$, which is most often found in the six (and even in nine) algebra families under consideration, but not with the ideal $I_4 =< e_3, e_5, e_6, e_7 >$ distinguished in Proposition 2. Thereby, all three cases of Lemma 1 were directly verified, while Proposition 1 was not used.

**Proposition 4.** *Holomorphic realizations of the families $1357M$, $1357QRS1$ connected with the ideal $I'_4 =< e_4, e_5, e_6, e_7 >$ and admitting Levi non-degenerate orbits are possible only in case 3) of Lemma 1. The bases of such realizations have the form ($Im\, B_3 \neq 0$, $Im\, C_5 \neq 0$):*

$$\begin{array}{llllll}
e_1 : ( & 1, & 0, & -z_2, & 0 & ),\\
e_2 : ( & -\lambda C_5, & B_3 z_1 + B_2, & c_2(z_1, z_2) & d_2(z_1, z_2, z_3) & ),\\
e_3 : ( & 0, & B_3, & (C_5 - B_3)z_1 + C_3, & z_1^2/2 + D_5 z_1 + (\lambda - 1)z_2 + D_3 & ),\\
e_4 : ( & 0, & 1, & 0, & 0 & ),\\
e_5 : ( & 0, & 0, & C_5, & z_1 + D_5 & ),\\
e_6 : ( & 0, & 0, & 1, & 0 & ),\\
e_7 : ( & 0, & 0, & 0, & 1 & ),
\end{array} \tag{26}$$

*where*

$$c_2(z_1, z_2) = (C_5 - 2B_3)z_1^2/2 + (C_3 - B_2)z_1 - C_5 z_2 + C_2,$$
$$d_2(z_1, z_2, z_3) = z_1^3/6 + D_5 z_1^2/2 + D_3 z_1 - (z_1 + D_5)z_2 - \lambda z_3 + D_2.$$

*Conditions for the* $1357M$ *family:*

$$-\lambda C_5 + (\lambda + 1)B_3 = 0, \quad -\lambda C_5 D_5 + B_2(\lambda - 1) + B_3 D_5 + C_3 \lambda = 0.$$

*Conditions for the* $1357QRS1$ *family:*

$$-\lambda C_5(C_5 - B_3) + B_3 C_5 = -1, \quad -\lambda C_5 D_5 + B_2(\lambda - 1) + B_3 D_5 + C_3 \lambda = 0.$$

To complete the proof of Theorem 2, it remains to integrate the algebras of vector fields, obtained in Proposition 5 and present non-degenerate nonspherical orbits of these algebras.

Recall that the defining function for the orbit of an arbitrary algebra of holomorphic vector fields with basis $e_1, \ldots, e_7$ in $\mathbb{C}^4$ is a solution to a system of seven partial differential equations of the form (11) for $k = 1, \ldots, 7$.

Given the presence in the realizations of all the algebras discussed, the triple of rectified fields and being interested only in non-degenerate orbits, one can consider that each of them is described by an equation of the form

$$y_4 = F(x_1, y_1, y_2, y_3).$$

But even with such simplification the integration of the system of (only) four equations presents, generally speaking, considerable technical difficulties.

**Proposition 5.** *For* $\lambda \neq -1$ *Levi non-degenerate orbits of algebras with bases (26) from the family of* $1357M$ *are (up to local holomorphic transformations) only algebraic tubular surfaces with affine-homogeneous bases*

$$y_4 = y_1 y_3 + A y_2^2 + B y_1^2 y_2 + C y_1^4, \tag{27}$$

*where*

$$A = \frac{1 - \lambda}{2(1 + \lambda)}, \quad B = \frac{1}{1 + \lambda}, \quad C = \frac{1}{4(1 + \lambda)}. \tag{28}$$

*Proof.* We use another Abelian ideal $I_4'' =< e_2, e_3, e_5, e_7 >$ in the algebras of the $1357M$ family. Coordinate description in $\mathbb{C}^4$ of the basis of this ideal (in the holomorphic realization from Proposition 4) is upper triangular:

$$\begin{array}{lllll} e_2 : ( & -\lambda C_5, & B_3 z_1 + B_2, & c_2(z_1, z_2) & d_2(z_1, z_2, z_3) \quad ), \\ e_3 : ( & 0, & B_3, & (C_5 - B_3)z_1 + C_3, & z_1^2/2 + D_5 z_1 + (\lambda - 1)z_2 + D_3 \,), \\ e_5 : ( & 0, & 0, & C_5, & z_1 + D_5 \quad ), \\ e_7 : ( & 0, & 0, & 0, & 1 \quad ). \end{array} \tag{29}$$

Moreover, $Im\, B_3 \neq 0, \; Im\, C_3 \neq 0$. Then after complex dilation of variables

$$z_1 = -\lambda C_5 z_1^*, \quad z_2 = B_3 z_2^*, \quad z_3 = C_5 z_3^*$$

each of the four diagonal elements of the matrix (29) will be equal to unity.

We apply to the fields (29) the sequential straightening procedure described in Section 2 above using the commutation of any pair of these fields. Then the basis of the 1357$M$ family will take much simpler affine form

$$\begin{aligned} &e_1 = i\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_3} - z_3\frac{\partial}{\partial z_4}, \quad e_2 = \frac{\partial}{\partial z_1}, \quad e_3 = \frac{\partial}{\partial z_2}, \\ &e_4 = -i(1+\lambda)\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3} + (1-\lambda)z_2\frac{\partial}{\partial z_4}, \\ &e_5 = \frac{\partial}{\partial z_3}, \quad e_6 = -i\lambda\frac{\partial}{\partial z_3} + \lambda z_1\frac{\partial}{\partial z_4}, \quad e_7 = \frac{\partial}{\partial z_4}. \end{aligned} \tag{30}$$

For $\lambda \neq -1$, integration of an algebra with such a simplified basis leads precisely to equation (27) with coefficients of the form (28).

Note also that by dilation the variables

$$z_1^* = tz_1, \quad z_2^* = rt_2, \quad z_3^* = sz_3, \quad z_4^* = qz_4$$

with real coefficients $t, r, s, q$ the three nonzero coefficients $(A, B, C)$ from equation (27) can be transformed to the form $(1, 1, AC/B^2)$.

With this in mind, for $\lambda \neq 1$, equation (27)–(28) can be reduced to

$$y_4 = y_1y_3 + y_2^2 + y_1^2y_2 + Dy_1^4, \tag{31}$$

where $D = CA/B^2 = (1-\lambda)/8$.

Concluding the proof of Proposition 5, we note that the quadratic form $y_1y_3 + y_2^2$ from the right-hand side of equation (31) turns in complex coordinates into indefinite nondegenerate Levi form

$$H(z_1, z_2, z_3) = z_1\bar{z}_3 + z_3\bar{z}_1 + |z_2|^2.$$

Therefore, for $\lambda \neq \pm 1$ all the orbits of (4) are non-degenerate. Proposition 5 is proved. □

**Remark 1.** For $\lambda = -1$, all orbits of such an algebra in the space $\mathbf{C}^4$ are affine equivalent to a Levy degenerate hypersurface $y_1 = y_2^2$.

**Remark 2.** According to [8], the surface with equation (4) is spherical, i.e. holomorphically equivalent to the corresponding quadric

$$y_4 = z_1\bar{z}_3 + z_3\bar{z}_1 + |z_2|^2,$$

only for $A = 1/12$, i.e. with $\lambda = 1/3$.

Thus, the first part of Theorem 2 is proved, and the family of surfaces (4) illustrates the difference between the situation in $\mathbf{C}^4$ and the 3-dimensional case.

**Remark 3.** For $\lambda = 1$, the surface (27)–(28), i.e.

$$y_4 = y_1\Big(y_3 + \frac{1}{2}y_1y_2 + \frac{1}{8}y_1^3\Big). \tag{32}$$

is Levi degenerate.

**Remark 4.** Any surface from family (31) admits a consistent expansion of variables that preserves both the surface and the origin of $\mathbf{C}^4$ lying on them. This means that an algebra with basis (30) is a subalgebra of an 8-dimensional algebra, the additional basis field of which is

$$e_8 = z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + 3z_3\frac{\partial}{\partial z_3} + 4z_4\frac{\partial}{\partial z_4}.$$

The following statement completes the proof of Theorem 2.

**Proposition 6.** *For arbitrary $\lambda > 1$ and for the following parameter values*

$$B_3 = C_5 = i, \ D_5 = \frac{\lambda}{\lambda - 1}, \ B_2 = C_3 = D_3 = 0$$

*the orbits of any algebra from the family $1357QRS_1$ with a basis of the form (26) are affinely equivalent to the surfaces*

$$y_4 = y_1 y_3 + y_2^2 + x_1 y_1 y_2 + \frac{5}{48(\lambda - 1)} y_1^4. \tag{33}$$

*Proof of Proposition 6.* Let us consider a system of partial differential equations corresponding to an algebra from family $1357QRS_1$. Due to the specified restrictions on the parameters four meaningful equations for the determining function of the surface $y_4 = F(x_1, y_1, y_2, y_3)$ acquires a relatively simple form:

$$\begin{aligned}
&\frac{\partial F}{\partial x_1} - y_2 \frac{\partial F}{\partial y_3} = 0, \qquad \frac{\partial F}{\partial y_2} = x_1 y_1 + \frac{\lambda}{\lambda - 1} y_1 + (\lambda - 1) y_2, \qquad \frac{\partial F}{\partial y_3} = y_1, \\
&-\lambda \frac{\partial F}{\partial y_1} + x_1 \frac{\partial F}{\partial y_2} + \left( -\frac{1}{2}(x_1^2 - y_1^2) - x_2 \right) \frac{\partial F}{\partial y_3} = \\
&\qquad = \left( \frac{1}{2} x_1^2 y_1 + \frac{\lambda}{\lambda - 1} x_1 y_1 - \frac{1}{6} y_1^3 \right) - \left( (x_1 + \frac{\lambda}{\lambda - 1}) y_2 + y_1 x_2 \right) - \lambda y_3.
\end{aligned} \tag{34}$$

A step-by-step solution of the individual equations of this system leads to its final solution of the form

$$F = y_1 y_3 + \frac{1}{2}(\lambda - 1) y_2^2 + y_1 y_2 (x_1 + \mu) - \frac{5}{24} y_1^4 + C, \quad C = const, \ \mu = \frac{\lambda}{\lambda - 1}.$$

Due to affine transformations and, in particular, consistent coordinate expansion of the complex space $\mathbb{C}^4$, the equations of the desired orbits in this case can be written in the form (3), which is very close in form to (2). Proposition 6 and Theorem 2 are proved. □

**Remark 1.** It is also possible to write out and integrate the equations corresponding to algebras from the family $1357QRS_1$ with arbitrary parameter values (the authors did this using the Maple package). The resulting equations of the orbits are very cumbersome, but in all non-degenerate cases they are reduced by holomorphic transformations to equations (3).

**Remark 2.** Currently, the authors are not aware of the answer to the question of holomorphic equivalence (or nonequivalence) for the surfaces (2) and (3). In multidimensional complex analysis, the task of practical verification of holomorphic equivalence of specific varieties is often hard enough to solve. Therefore, the study of the (possibly simple) question about the surfaces (2) and (3) can be be considered as going beyond the scope of this article.

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# Об орбитах нильпотентных 7-мерных алгебр Ли в 4-мерном комплексном пространстве


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**Аннотация.** В работе изучены однопараметрические семейства 7-мерных нильпотентных неразложимых алгебр Ли и орбиты голоморфных реализаций таких алгебр в 4-мерном комплексном пространстве. Показано, что в отличие от орбит 5-мерных нильпотентных алгебр Ли в 3-мерном пространстве два таких семейства (из имеющихся девяти) допускают орбиты, являющиеся невырожденными по Леви (однородными) вещественными несферическими гиперповерхностями. С точностью до голоморфной эквивалентности все полученные невырожденные несферические орбиты являются графиками многочленов 4-й степени.



**Ключевые слова:** алгебра Ли, комплексное пространство, векторное поле, голоморфная функция, однородное многообразие, вырождение по Леви.

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# Commutative Hypercomplex Numbers and the Geometry of Two Sets

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**Abstract.** The main task of the theory of phenomenologically symmetric geometries of two sets is the classification of such geometries. In this paper, by complexing with associative hypercomplex numbers, functions of a pair of points of new geometries are found by the functions of a pair of points of some well-known phenomenologically symmetric geometries of two sets (FS GDM). The equations of the groups of motions of these geometries are also found. The phenomenological symmetry of these geometries is established, that is, functional relationships are found between the functions of a pair of points for a certain finite number of arbitrary points. In particular, the $s+1$-component functions of a pair of points of the same ranks are determined by single-component functions of a pair of points of the FS of GDM ranks (n,n) and (n + 1,n). Finite equations of motion group and equation expressing their phenomenological symmetry are found.

**Keywords:** geometry of two sets, phenomenological symmetry, group symmetry, hyper-complex numbers.

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## Introduction

**0.1.** In the works [1–3] the definition of one-dimensional phenomenologically symmetric geometry of two sets (PS of GTS) of rank $(n+1, m+1)$ is given, which is given by a differentiable non-degenerate function of a pair of points with open and dense in $R^m \times R^n$ domain:

$$f : R^m \times R^n \to R.$$

The axiom of phenomenological symmetry is fulfilled: the functional relation

$$\Phi(f(\mu_1, \nu_1), f(\mu_1, \nu_2), \ldots, f(\mu_{n+1}, \nu_{m+1})) = 0,$$

for an open and dense subset of the sequences $\langle \mu_1, \mu_2, \ldots, \mu_n, \mu_{n+1}; \nu_1, \nu_2, \ldots, \nu_m, \nu_{m+1} \rangle$ of length $n+m+2$ from neighborhood $V(\langle \mu_1, \mu_2, \ldots, \mu_n, \mu_{n+1}; \nu_1, \nu_2, \ldots, \nu_m, \nu_{m+1} \rangle) \subset R^{m(n+1)} \times R^{n(m+1)}$. The function $\Phi$ is differentiable and rang$\Phi = 1$. Points from the first set are denoted $\mu$, $\mu_1$, $\mu_2 \ldots$, and points from the second set are $\nu$, $\nu_1$, $\nu_2 \ldots$

In the coordinates, the function of a pair of points of the PS of GTS of rank $(n+1, m+1)$ is given as

$$f(\mu, \nu) = f(x^1(\mu), \ldots, x^m(\mu), \xi^1(\nu), \ldots, \xi^n(\nu)),$$

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where $(x^1(\mu), \dots, x^m(\mu))$ are the coordinates of the point $\mu \in R^m$, and $(\xi^1(\nu), \dots, \xi^n(\nu))$ are the coordinates of the point $\nu \in R^n$.

There is a complete classification of one-dimensional PS of GTS [4], according to which there are PS of GTS only of the ranks $(n+1, n+1)$, $(n+2, n+1)$ and $(4, 2)$, where $n \geqslant 1$.

PS of GTS rank $(n+1, n+1)$:

$$f(\mu, \nu) = x^1(\mu)\xi^1(\nu) + \dots + x^n(\mu)\xi^n(\nu); \tag{1}$$

$$f(\mu, \nu) = x^1(\mu)\xi^1(\nu) + \dots + x^{n-1}(\mu)\xi^{n-1}(\nu) + x^n(\mu) + \xi^n(\nu), \tag{2}$$

where $n \geqslant 1$;

PS of GTS rank $(n+2, n+1)$:

$$f = f(\mu, \nu) = x^1(\mu)\xi^1(\nu) + \dots + x^n(\mu)\xi^n(\nu) + \xi^{n+1}(\nu), \tag{3}$$

where $n \geqslant 1$;

PS of GTS $(4, 2)$:

$$f = f(\mu, \nu) = \frac{x^1(\mu)\xi^1(\nu) + \xi^2(\nu)}{x^1(\mu) + \xi^3(\nu)}. \tag{4}$$

**0.2.** As above, one can define a $s$-metric PS of GTS of rank $(n+1, m+1)$, which is given by a differentiable non-degenerate function of a pair of points with open and dense in $R^{sm} \times R^{sn}$ domain:

$$f' : R^{sm} \times R^{sn} \to R^s.$$

The axiom of phenomenological symmetry is fulfilled: the functional relation

$$\Phi'(f'(\mu_1, \nu_1), f'(\mu_1, \nu_2), \dots, f'(\mu_{n+1}, \nu_{m+1})) = 0,$$

for an open and dense subset of the sequences $\langle \mu_1, \mu_2, \dots, \mu_n, \mu_{n+1}; \nu_1, \nu_2, \dots, \nu_m, \nu_{m+1} \rangle$ of length $n+m+2$ from $V(\langle \mu_1, \mu_2, \dots, \mu_n, \mu_{n+1}; \nu_1, \nu_2, \dots, \nu_m, \nu_{m+1} \rangle) \subset R^{sm(n+1)} \times R^{sn(m+1)}$ [3]. The function $\Phi'$ is differentiable and rang$\Phi' = s$. There is no complete classification of $s$-metric PS of GTS.

**0.3.** This work is a continuation of the research published in the article [3]. Here, complexifications of one-dimensional PS of GTS of ranks $(n+1, n+1)$ with $n \geqslant 3$ and PS of GTS of ranks $(n+2, n+1)$ with $n \geqslant 2$ by associative commutative hypercomplex numbers of rank are constructed $s$. For example, hypercomplex numbers of rank 2 are: ordinary complex numbers ($i^2 = -1$), double complex numbers ($i^2 = 1$) and dual complex numbers ($i^2 = 0$) [3, 5, 6], associative but noncommutative hypercomplex numbers of rank 4 are quaternions [5]. As a result of complexification, functions of a pair of points of $s$-metric PS of GTS are obtained. This method was tested in [3] and [5].

Note that the cases $n = 1$ and $n = 2$ for PS of GTS rank $(n+1, n+1)$, as well as the case $n = 1$ for PS of GTS rank $(n+2, n+1)$ was previously considered in [3] over the algebra of associative hypercomplex numbers.

# 1. Algebra of hypercomplex numbers and matrix algebra over hypercomplex numbers

**1.1.** The results of this item are given by article [3]. Consider the real associative commutative algebra $L$ of hypercomplex numbers of order $s$ ( [7], p. 462). An arbitrary hypercomplex number has the form: $x = x_0 + x_1 i_1 + \cdots + x_{s-1} i_{s-1}$, where $x_0, x_1, \ldots, x_{s-1} \in R$, $i_0 = 1$, $i_1, \ldots, i_{s-1}$ are imaginary units. Addition and multiplication by a real number are component determined, and the product is written so: for arbitrary $x, y \in L$,

$$xy = \sum_{k,l=0}^{n} x_k y_l i_k i_l.$$

The product of imaginary units $i_k i_l \in L$ is defined by a special table. Denote by $U(L) \subset L$ the set of invertible elements. The set $U(L)$ is open and dense in $L$ and is a group by multiplication.

**1.2.** Let $M_m$ be the set of matrices of size $m \times m$ over the algebra of associative commutative hypercomplex numbers $L$, $m > 1$. The addition of such matrices and multiplication by a hypercomplex number is determined in the usual way. The product of matrices is also determined by the rule "row by column". One can prove that $M_m$ is a linear associative algebra ( [7], p. 184).

Consider the matrix $A = (a_{ij}) \in M_m$, where $i, j = 1, \ldots, m$. Denote by $|A|$ the determinant of this matrix, and by $A_{ik}$ — the algebraic complement of the element $a_{ik}$. Further we use the well-known statement from linear algebra.

**Proposition 1.1** ( [8], p. 50)**.** *A square matrix $A$ with elements from a commutative ring $K$ with unity has an inverse matrix with elements from $K$ when and only if the determinant of the matrix $A$ is invertible into $K$.*

From this statement it follows that over an algebra of associative commutative hypercomplex numbers $L$, the matrix $A$ is invertible if and only if $|A| \in U(L)$. The set of invertible matrices in $M_m$ is denoted by $UM_m$). This set is open and dense in $M_m$. The inverse matrix to the matrix $A \in U(M_m)$ is denoted $A^{-1} = (\alpha_{ik})$ and its element $\alpha_{ik}$ is calculated by the formula ( [9], p. 26):

$$\alpha_{ik} = A_{ki} |A|^{-1}.$$

# 2. Classification of one-metric phenomenologically symmetric geometries of two sets (PS of GTS)

**2.1.** In the introduction, the definition of a one-metric PS of GTS of rank $(n+1, m+1)$ is given, and the functions of a pair of points for a PS of GTS of rank $(n+1, n+1)$, of a PS of GTS of rank $(n+1, n)$ and PS of GTS rank $(4, 2)$. For them, functional connections are known [1] and [4]:

For PS of GTS rank $(n+1, n+1)$:

first solution (function of a pair of points (1)):

$$\begin{vmatrix} f(\mu_1,\nu_1) & f(\mu_1,\nu_2) & \cdots & f(\mu_1,\nu_n) \\ f(\mu_2,\nu_1) & f(\mu_2,\nu_2) & \cdots & f(\mu_2,\nu_n) \\ \cdots & \cdots & \cdots & \cdots \\ f(\mu_n,\nu_1) & f(\mu_n,\nu_2) & \cdots & f(\mu_n,\nu_n) \end{vmatrix} = 0, \tag{5}$$

second solution (function of a pair of points (2)):

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & f(\mu_1,\nu_1) & f(\mu_1,\nu_2) & \cdots & f(\mu_1,\nu_n) \\ 1 & f(\mu_2,\nu_1) & f(\mu_2,\nu_2) & \cdots & f(\mu_2,\nu_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & f(\mu_n,\nu_1) & f(\mu_n,\nu_2) & \cdots & f(\mu_n,\nu_n) \end{vmatrix} = 0. \tag{6}$$

For PS of GTS rank $(n+1, n)$:

$$\begin{vmatrix} f(\mu_1,\nu_1) & f(\mu_1,\nu_2) & \cdots & f(\mu_1,\nu_{n-1}) & 1 \\ f(\mu_2,\nu_1) & f(\mu_2,\nu_2) & \cdots & f(\mu_2,\nu_{n-1}) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f(\mu_n,\nu_1) & f(\mu_n,\nu_2) & \cdots & f(\mu_n,\nu_{n-1}) & 1 \end{vmatrix} = 0. \tag{7}$$

For PS of GTS rank $(4, 2)$:

$$\begin{vmatrix} f(\mu_1,\nu_1) & f(\mu_1,\nu_2) & f(\mu_1,\nu_1)f(\mu_1,\nu_2) & 1 \\ f(\mu_2,\nu_1) & f(\mu_2,\nu_2) & f(\mu_2,\nu_1)f(\mu_2,\nu_2) & 1 \\ f(\mu_3,\nu_1) & f(\mu_3,\nu_2) & f(\mu_3,\nu_1)f(\mu_3,\nu_2) & 1 \\ f(\mu_4,\nu_1) & f(\mu_4,\nu_2) & f(\mu_4,\nu_1)f(\mu_4,\nu_2) & 1 \end{vmatrix} = 0. \tag{8}$$

**2.2.** The concept of motion in GTS is introduced as a set of locally diffeomorphic transformations

$$x' = \lambda(x), \quad \xi' = \sigma(\xi)$$

of $R^m$ and $R^n$ varieties preserving the function pairs of points:

$$f(x', \xi') = f(\lambda(x), \quad \sigma(\xi)) = f(x, \xi). \tag{9}$$

Relation (9) is a functional equation for a group of motions, solving which are the equations of this group:

For PS of GTS rank $(n+1, n+1)$:

first solution:

$$X' = AX, \quad \Xi' = A^{-1}\Xi, \tag{10}$$

where $X' = \begin{pmatrix} x'^1 \\ \cdots \\ x'^n \end{pmatrix}$, $X = \begin{pmatrix} x^1 \\ \cdots \\ x^n \end{pmatrix}$, $\Xi' = \begin{pmatrix} \xi'^1 \\ \cdots \\ \xi'^n \end{pmatrix}$, $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^n \end{pmatrix}$, $A = \begin{pmatrix} a^{11} & \cdots & a^{1n} \\ \cdots & \cdots & \cdots \\ a^{n1} & \cdots & a^{nn} \end{pmatrix}$ is nondegenerate matrix;

second solution:

$$X' = AX + B, \quad x'^n = x^n + C^T X + b^n,$$
$$\Xi' = A^{-1}(\Xi - C), \quad \xi'^n = \xi^n - B^T A^{-1}(\Xi - C) - b^n, \tag{11}$$

where $X' = \begin{pmatrix} x'^1 \\ \cdots \\ x'^{n-1} \end{pmatrix}$, $X = \begin{pmatrix} x^1 \\ \cdots \\ x^{n-1} \end{pmatrix}$, $\Xi' = \begin{pmatrix} \xi'^1 \\ \cdots \\ \xi'^{n-1} \end{pmatrix}$, $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^{n-1} \end{pmatrix}$, $B = \begin{pmatrix} b^1 \\ \cdots \\ b^{n-1} \end{pmatrix}$, $C = \begin{pmatrix} c^1 \\ \cdots \\ c^{n-1} \end{pmatrix}$, $A = \begin{pmatrix} a^{11} & \cdots & a^{1(n-1)} \\ \cdots & \cdots & \cdots \\ a^{(n-1)1} & \cdots & a^{(n-1)(n-1)} \end{pmatrix}$ nondegenerate matrix.

For PS of GTS rank $(n+1, n)$:

$$X' = AX + B, \quad \Xi' = A^{-1}\Xi, \quad \xi'^n = \xi^n - B^T A^{-1}\Xi, \tag{12}$$

where $X' = \begin{pmatrix} x'^1 \\ \cdots \\ x'^{n-1} \end{pmatrix}$, $X = \begin{pmatrix} x^1 \\ \cdots \\ x^{n-1} \end{pmatrix}$, $\Xi' = \begin{pmatrix} \xi'^1 \\ \cdots \\ \xi'^{n-1} \end{pmatrix}$, $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^{n-1} \end{pmatrix}$, $B = \begin{pmatrix} b^1 \\ \cdots \\ b^{n-1} \end{pmatrix}$, $A = \begin{pmatrix} a^{11} & \cdots & a^{1(n-1)} \\ \cdots & \cdots & \cdots \\ a^{(n-1)1} & \cdots & a^{(n-1)(n-1)} \end{pmatrix}$ nondegenerate matrix.

For PS of GTS rank $(4, 2)$:

$$\begin{aligned} &x^{1\prime} = (ax^1 + b)/(cx^1 + d), \quad \xi^{1\prime} = (d\xi^1 - c\xi^2)/(d - c\xi^3), \\ &\xi^{2\prime} = (a\xi^2 - b\xi^1)/(d - c\xi^3), \quad \xi^{3\prime} = (a\xi^3 - b)/(d - c\xi^3), \end{aligned} \tag{13}$$

where $ad - bc = 1$.

It should be noted that group and phenomenological symmetries for PS of GTS are equivalent in the following sense: by the function of a pair of points, you can find a group of motions, and by a group of motions — a function of a pair of points [1, 2, 4].

# 3. Complexification of one-metric PS of GTS rank $(n+1, n+1)$ hypercomplex numbers

**3.1.** Consider a one-metric FS of GDM rank $(n + 1, n + 1)$, which exists in two variants ( [1], p. 63), defined by the functions of a pair of points (1) and (2) in $R^n \times R^n$. For $n \geqslant 2$ these options are not equivalent. For the first solution, the functional relationship is expressed by equation (5).

The group symmetry of degree $n^2$ is determined by the $n^2$-parametric group of motions with equations (10) for the function of the pair of points (1), which satisfies the identity (9).

**3.2.** We carry out the complexification of the function of the pair of points (1), passing to the corresponding hypercomplex functions and coordinates, assuming

$$f = \sum_{k=1}^{s} f_k i_k, \quad x = \sum_{k=1}^{s} x_k i_k, \quad \xi = \sum_{k=1}^{s} \xi_k i_k.$$

As a result, we obtain the $s$-component function of a pair of points

$$f_{\text{к}} = x^1\xi^1 + \cdots + x^n\xi^n, \tag{14}$$

where $x^1, \ldots, x^n, \xi^1, \ldots, \xi^n, f_k \in L$. Phenomenological symmetry, as is easily seen, if we use the formula (14), is given by the identity:

$$\begin{vmatrix} f_{\text{к}}(\mu_1, \nu_1) & f_{\text{к}}(\mu_1, \nu_2) & \cdots & f_{\text{к}}(\mu_1, \nu_n) \\ f_{\text{к}}(\mu_2, \nu_1) & f_{\text{к}}(\mu_2, \nu_2) & \cdots & f_{\text{к}}(\mu_2, \nu_n) \\ \cdots & \cdots & \cdots & \cdots \\ f_{\text{к}}(\mu_n, \nu_1) & f_{\text{к}}(\mu_n, \nu_2) & \cdots & f_{\text{к}}(\mu_n, \nu_n) \end{vmatrix} = 0.$$

**3.3.** Let us find the group of motions for the complexification PS of GTS rank $(n+1, n+1)$. To do this, we solve the functional equation

$$x'^1\xi'^1 + \cdots + x'^n\xi'^n = x^1\xi^1 + \cdots + x^n\xi^n \tag{15}$$

on a set of motions.

**Theorem 1.** *The group of motions of the complexification PS of GTS of rank* $(n+1, n+1)$ *with the function of a pair of points* (14) *is given by the equations*

$$X' = XA, \quad \Xi' = A^{-1}\Xi, \tag{16}$$

*in which* $X = \begin{pmatrix} x^1 & \cdots & x^n \end{pmatrix}$, $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^n \end{pmatrix}$, $A = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \cdots & \cdots & \cdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix} = \text{const} \in U(M_n)$.

*Proof.* We write the identity (15) for $n$ pairs of points $\langle \mu\nu_1\rangle, \dots, \langle \mu\nu_n\rangle$:

$$x'^1\xi'^1(\nu_1) + \dots + x'^n\xi'^n(\nu_1) = x^1\xi^1(\nu_1) + \dots + x^n\xi^n(\nu_1),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x'^1\xi'^1(\nu_n) + \dots + x'^n\xi'^n(\nu_n) = x^1\xi^1(\nu_n) + \dots + x^n\xi^n(\nu_n).$$

For convenience, the last equalities are rewritten in a matrix form:

$$X'D' = XD,$$

where $D = \begin{pmatrix} \xi^1(\nu_1) & \cdots & \xi^1(\nu_n) \\ \cdots & \cdots & \cdots \\ \xi^n(\nu_1) & \cdots & \xi^n(\nu_n) \end{pmatrix} \in U(M_n)$. Resolving, we have

$$X' = XDD'^{-1}.$$

It can be seen that the variables are divided. Therefore $A = DD'^{-1} = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \cdots & \cdots & \cdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix} =$ $=$ const $\in U(M_n)$. In this way,

$$X' = XA.$$

Similarly, identity (15) is written for the sequences $\langle \mu_1\nu\rangle, \dots, \langle \mu_n\nu\rangle$, and then rewritten in a matrix form:

$$U'\Xi' = U\Xi,$$

where $U = \begin{pmatrix} x^1(\mu_1) & \cdots & x^n(\mu_1) \\ \cdots & \cdots & \cdots \\ x^1(\mu_n) & \cdots & x^n(\mu_n) \end{pmatrix} \in U(M_n)$, $U' = UA$. Then

$$UA\Xi' = U\Xi, \quad \Xi' = (UA)^{-1}U\Xi = A^{-1}U^{-1}U\Xi = A^{-1}\Xi.$$

Thus, we obtain (16). □

Obviously, the group of motions (16) is a $sn^2$-parametric group, which includes $sn^2$ real parameters.

**3.4.** For the second solution, the functional relationship is expressed by equation (6). The group symmetry of degree $n^2$ is determined by the $n^2$-parameter group of motions with equations (11) for the function of the pair of points (2), which satisfies the identity (9).

**3.5.** When passing to hypercomplex coordinates in expression (2), we obtain the $s$-component function of a pair of points

$$f_{\kappa} = x^1\xi^1 + \dots + x^{n-1}\xi^{n-1} + x^n + \xi^n, \tag{17}$$

where $x^1, \dots, x^n, \xi^1, \dots, \xi^n, f_k \in L$.

Phenomenological symmetry, as is easily seen, if we use the formula (17), is given by the identity:

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & f_{\kappa}(\mu_1,\nu_1) & f_{\kappa}(\mu_1,\nu_2) & \cdots & f_{\kappa}(\mu_1,\nu_n) \\ 1 & f_{\kappa}(\mu_2,\nu_1) & f_{\kappa}(\mu_2,\nu_2) & \cdots & f_{\kappa}(\mu_2,\nu_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & f_{\kappa}(\mu_n,\nu_1) & f_{\kappa}(\mu_n,\nu_2) & \cdots & f_{\kappa}(\mu_n,\nu_n) \end{vmatrix} = 0.$$

**3.6.** We find the group of motions for the second complexification PS of GTS rank $(n+1, n+1)$. For this, we solve the functional equation

$$x'^1\xi'^1 + \cdots + x'^{n-1}\xi'^{n-1} + x'^n + \xi'^n = x^1\xi^1 + \cdots + x^{n-1}\xi^{n-1} + x^n + \xi^n$$

on a set of movements.

**Theorem 2.** *The group of motions of the complexification second PS of GTS of rank* $(n+1, n+1)$ *with the function of a pair of points* (17) *is given by the equations*

$$X' = XA + B, \quad x'^n = x^n + XC^T + b^n;$$

$$\Xi' = A^{-1}(\Xi - C^T), \quad \xi'^n = \xi^n - BA^{-1}(\Xi - C^T) - b^n, \tag{18}$$

*in which* $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^{n-1} \end{pmatrix}$, $X = \begin{pmatrix} x^1 & \cdots & x^{n-1} \end{pmatrix}$, $B = \begin{pmatrix} b^1 \\ \cdots \\ b^{n-1} \end{pmatrix} = const$, $C = \begin{pmatrix} c^1 \\ \cdots \\ c^{n-1} \end{pmatrix} = const$, $b^n = const$, $A = \begin{pmatrix} a^1_1 & \cdots & a^{n-1}_1 \\ \cdots & \cdots & \cdots \\ a^1_{n-1} & \cdots & a^{n-1}_{n-1} \end{pmatrix} = const \in U(M_{n-1})$.

The proof of this theorem is similar to the proof of Theorem 1; therefore, it is omitted.

It is obvious that the group of motions (18) is a $sn^2$-parametric group, which includes $sn^2$ real parameters.

# 4. Complexification of one-metric PS of GTS rank $(n+1, n)$ hypercomplex numbers

**4.1.** We now turn to a one-metric FS of a GDM of rank $(n+1, n)$, ( [1], p. 63) defined by the function of the pair of points (3) in $R^{n-1} \times R^n$. The functional relationship is given by formula (7). The group symmetry of degree $n(n-1)2$ is determined by the $n(n-1)$-parametric group of motions with equations (12) for the metric function (3), which satisfies the identity (9).

**4.2.** In the transition to the hypercomplex coordinates in expression (3), we get the $s$-component function of a pair of points

$$f_{\kappa} = x^1\xi^1 + \cdots + x^{n-1}\xi^{n-1} + \xi^n, \tag{19}$$

where $x^1, \ldots, x^{n-1}$, $\xi^1, \ldots, \xi^n$, $f_{\kappa} \in L$. Phenomenological symmetry, as is easily seen, if we use the formula (19), is given by the identity:

$$\begin{vmatrix} f_{\kappa}(\mu_1, \nu_1) & f_{\kappa}(\mu_1, \nu_2) & \cdots & f_{\kappa}(\mu_1, \nu_{n-1}) & 1 \\ f_{\kappa}(\mu_2, \nu_1) & f_{\kappa}(\mu_2, \nu_2) & \cdots & f_{\kappa}(\mu_2, \nu_{n-1}) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{\kappa}(\mu_n, \nu_1) & f_{\kappa}(\mu_n, \nu_2) & \cdots & f_{\kappa}(\mu_n, \nu_{n-1}) & 1 \end{vmatrix} = 0.$$

**4.3.** Let us find the group of motions for the complexification PS of GTS rank $(n+1, n)$. For this, we solve the functional equation

$$x'^1\xi'^1 + \cdots + x'^{n-1}\xi'^{n-1} + \xi'^n = x^1\xi^1 + \cdots + x^{n-1}\xi^{n-1} + \xi^n$$

on a set of movements.

**Theorem 3.** *The group of motions of the complexification second PS of GTS of rank* $(n+1,n)$ *with the function of a pair of points* (19) *is given by the equations*

$$X' = XA + B, \quad \Xi' = A^{-1}\Xi, \quad \xi'^n = \xi^n - BA^{-1}\Xi, \tag{20}$$

*in which the notation is entered* $\Xi = \begin{pmatrix} \xi^1 \\ \cdots \\ \xi^{n-1} \end{pmatrix}$, $X = \begin{pmatrix} x^1 & \cdots & x^{n-1} \end{pmatrix}$, $B = \begin{pmatrix} b^1 \\ \cdots \\ b^{n-1} \end{pmatrix} = const,$

$A = \begin{pmatrix} a_1^1 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots \\ a_{n-1}^1 & \cdots & a_{n-1}^{n-1} \end{pmatrix} = const \in U(M_{n-1}).$

The proof of this theorem, as well as the previous one, is similar to the proof of Theorem 1; therefore, it is omitted.

It is obvious that the group of motions (20) is a $sn(n-1)$-parametric group, which includes $sn(n-1)$ real parameters.

# 5. Complexification of one-metric PS of GTS rank $(4,2)$ hypercomplex numbers

**5.1.** Finally, we turn to a one-metric PS of GTS of rank $(4,2)$, ( [1], p. 63) defined by the function of a pair of points (4) in $R \times R^3$. The functional relationship is given by formula (8). The group symmetry of degree $n(n-1)2$ is determined by the $n(n-1)$-parametric group of motions with equations (13) for the metric function (3), which satisfies the identity (9).

**5.2.** In the transition in expression (4) to hypercomplex coordinates, we obtain the $s$-component function of a pair of points

$$f_\kappa = (x^1\xi^1 + \xi^2)(x^1 + \xi^3)^{-1}, \tag{21}$$

where $x^1$, $\xi^1, \xi^2, \xi^3$, $f_\kappa \in L$, $x^1 + \xi^3 \in U(L)$. Phenomenological symmetry, as is easily seen, if we use the formula (21), is given by the identity:

$$\begin{vmatrix} f_\kappa(\mu_1,\nu_1) & f_\kappa(\mu_1,\nu_2) & f_\kappa(\mu_1,\nu_1)f_\kappa(\mu_1,\nu_2) & 1 \\ f_\kappa(\mu_2,\nu_1) & f_\kappa(\mu_2,\nu_2) & f_\kappa(\mu_2,\nu_1)f_\kappa(\mu_2,\nu_2) & 1 \\ f_\kappa(\mu_3,\nu_1) & f_\kappa(\mu_3,\nu_2) & f_\kappa(\mu_3,\nu_1)f_\kappa(\mu_3,\nu_2) & 1 \\ f_\kappa(\mu_4,\nu_1) & f_\kappa(\mu_4,\nu_2) & f_\kappa(\mu_4,\nu_1)f_\kappa(\mu_4,\nu_2) & 1 \end{vmatrix} = 0.$$

**5.3.** Let us find the group of motions for the complexification PS of GTS rank $(4,2)$. For this, we solve the functional equation

$$(x^{1\prime}\xi^{1\prime} + \xi^{2\prime})(x^{1\prime} + \xi^{3\prime})^{-1} = (x^1\xi^1 + \xi^2)(x^1 + \xi^3)^{-1} \tag{22}$$

on a set of movements.

**Theorem 4.** *The group of motions of the complexification second PS of GTS of rank* $(4,2)$ *with the function of a pair of points* (21) *is given by the equations*

$$\begin{cases} x^{1\prime} = (ax^1 + b)(cx^1 + d)^{-1}, & \xi^{1\prime} = (d\xi^1 - c\xi^2)(d - c\xi^3)^{-1}, \\ \xi^{2\prime} = (a\xi^2 - b\xi^1)(d - c\xi^3)^{-1}, & \xi^{3\prime} = (a\xi^3 - b)(d - c\xi^3)^{-1}, \end{cases} \tag{23}$$

*where* $ad - bc = 1$, $a,b,c,d \in L$, $cx^1 + d, d - c\xi^3 \in U(L)$.

*Proof.* Identity (22) is solvable with respect to $x^{1\prime}$, after which we fix the coordinates of the points of the second set. After redefinition, we obtain the first equation of system (23).

Further, the identity (18) we write for sequences $\langle\mu_1\nu\rangle$, $\langle\mu_2\nu\rangle$, $\langle\mu_3\nu\rangle$:

$$(x^{1\prime}(\mu_1)\xi^{1\prime}+\xi^{2\prime})(x^{1\prime}(\mu_1)+\xi^{3\prime})^{-1}=(x^1(\mu_1)\xi^1+\xi^2)(x^1(\mu_1)+\xi^3)^{-1},$$

$$(x^{1\prime}(\mu_2)\xi^{1\prime}+\xi^{2\prime})(x^{1\prime}(\mu_2)+\xi^{3\prime})^{-1}=(x^1(\mu_2)\xi^1+\xi^2)(x^1(\mu_2)+\xi^3)^{-1},$$

$$(x^{1\prime}(\mu_3)\xi^{1\prime}+\xi^{2\prime})(x^{1\prime}(\mu_3)+\xi^{3\prime})^{-1}=(x^1(\mu_3)\xi^1+\xi^2)(x^1(\mu_3)+\xi^3)^{-1}.$$

Then the resulting system is resolved with respect to $\xi^{1\prime}$, $\xi^{2\prime}$ and $\xi^{3\prime}$, whereupon fix the coordinates of the points of the first set and go to the identity (22). As a result, after redefinitions, we get equalities (23). □

Obviously, the group of motions (23) is a $3s$-parametric group, which includes $3s$ real parameters.

## Conclusion

Complexification by ordinary complex numbers PS of GTS rank (2,2), (3,3), (4,4) and (5,5) are interpreted by Yu.S. Vladimirov in the theory of physical interactions [10,11]. They are given the definition of spinors through the PS of GTS rank (2,2), which are used to describe elementary particles. Complex PS of GTS high ranks are used to describe the fundamental interactions of elementary particles.

In works [12–14], respectively, the complexification affine group and the complexification projective group are investigated as the PS of GTS.

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# Коммутативные гиперкомплексные числа и геометрия двух множеств

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**Аннотация.** Главной задачей теории феноменологически симметричных геометрий двух множеств является классификация таких геометрий. В данной работе по функциям пары точек некоторых известных феноменологически симметричных геометрий двух множеств (ФС ГДМ) с помощью комплексификации ассоциативными гиперкомплексными числами находим функции пары точек новых геометрий. Находим также уравнения групп движений этих геометрий. Устанавливаем феноменологическую симметрию этих геометрий, то есть находим функциональные связи между функциями пары точек для определенного конечного числа произвольных точек. В частности, по однокомпонентным функциям пары точек ФС ГДМ рангов (n,n) и (n+1,n) определяем $s+1$-компонентные функции пары точек тех же рангов. Для них находим конечные уравнения групп движений и уравнения, выражающие их феноменологическую симметрию.

**Ключевые слова:** геометрия двух множеств, феноменологическая симметрия, групповая симметрия, гиперкомплексные числа.