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On Some Decompositions of Matrices over Algebraically Closed and Finite Fields

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Abstract. Decomposition of every square matrix over an algebraically closed field or over a finite field into a sum of a potent matrix and a nilpotent matrix of order 2 is considered. This can be related to our recent paper, published in Linear & Multilinear Algebra (2022).

The question of when each square matrix over an infinite field can be decomposed into a periodic matrix and a nilpotent matrix of order 2 is also completely considered.

Keywords: nilpotent matrix, potent matrix, Jordan normal form, rational form, field.

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1. Introduction and conventions

Nilpotent and potent elements in matrix rings is mainly considered in this paper. Let us recall that an element q of an arbitrary ring R is said to be a *nilpotent* if there is an integer $n \geqslant 1$ that depends on q such that $q^n = 0$ (the minimal n with this property is called an *exponent* for q; in particular, if $n = 2$ the non-zero nilpotent is shortly called *square-zero*). Element $p \in R$ is said to be *potent* if there is a natural number $m \geqslant 2$ that depends on p and $p^m = p$ (p is called m-potent). If $m = 2$, this element is called *idempotent*. Common generalization of potent element is *periodic* element. An element t is said to be *periodic* if there are two different natural numbers m, n that depend on t and $t^m = t^n$.

Representation of an arbitrary matrix over a field as the sum of a nilpotent matrix and an idempotent matrix was considered in pioneering [3]. It was proved that this presentation is possible precisely when the field contains only two elements. This was further extended by showing in some cases the exact exponent of the nilpotent matrix [13], [12]. The valuable discussion on the decomposition of a matrix as the sum of an idempotent and a square-zero matrix was given [9]. On the other hand, as generalization of the aforementioned main fact from [3] it was proved that every matrix over any finite field of cardinality d is representable as the sum of a nilpotent matrix and a d-potent matrix [1]. Furthermore, this representation was refined by proving that if d is odd then the exponent of a nilpotent matrix is not more than 3 [2]. Moreover, it was constructed a 3×3 matrix over the field of three elements which is *not* presentable as the sum of a 3-potent matrix and a square-zero matrix [2, Example 6].

Hence the following intriguing problem is considered.

<u>Question 1</u>: *When every square matrix over a field K can be expressed as*

$$P + Q,$$

where P is a potent matrix and Q is a square-zero matrix?

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Let us consider below two situations, namely, algebraically closed fields (see Corollary 2.4) and finite fields (see Corollary 3.2). The results can be viewed and treated as the development of method and ideas presented in [8] and [6], respectively. Some closely related studies can also be found in [5].

It is well known that any element in finite rings is periodic, so any matrix over a finite ring is also periodic itself. This immediately rises the question on matrices over infinite rings. Attention will be concentrated only on infinite fields, so the following interesting problem will also be examined.

<u>Question 2</u>: *When each square matrix over an infinite field F can be expressed as*

$$T+Q,$$

where T is a periodic matrix and Q is a square-zero matrix?

It should be noted that a part of the established here results can be found in [7].

2. Decomposition into potent matrix and zero-square matrix over algebraically closed fields

As a first approach to the problem, it will be shown that all square matrices over an algebraically closed field admit decomposition into a diagonalizable matrix and a nilpotent matrix of order two. This decomposition will be significantly improved in the next section, where the same result is proved for non necessarily algebraically closed fields by using the rational canonical form. Nevertheless, in this section a simple argument is provided in terms of Jordan blocks. It is included here for the sake of completeness. Moreover, it provides a decomposition for nilpotent matrices over non necessarily algebraically closed fields.

The construction is based on Jordan blocks and roots of unity. Let us consider the explicit decomposition for a Jordan block (see Remarks 2.8 and 2.9 from [8]).

Lemma 2.1. *Let K be a field and let J be a Jordan block in $\mathbb{M}_n(K)$, $n \geqslant 3$ associated with $a \in K$*

$$J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

(i) Suppose that $\operatorname{char}(K)$ *does not divide n. If K contains n (different) roots of the polynomial $x^n - 1 \in K[x]$ then J has the following decomposition*

$$J = \underbrace{(J + e_{1n})}_{D} + \underbrace{(-e_{1n})}_{Q}$$

where e_{1n} denotes the nilpotent matrix with 1 in the $(1n)$-entry and zero in the rest of entries, and matrix D is diagonalizable. Moreover, if $a = 0$ then $D^n = I$.

(ii) Suppose that $\operatorname{char}(K)$ *divides n. If K contains $n-1$ (different) roots of the polynomial $x^{n-1} - 1 \in K[X]$ then J has the following decomposition*

$$J = \underbrace{(J + e_{2n})}_{D} + \underbrace{(-e_{2n})}_{Q}$$

where e_{2n} denotes the nilpotent matrix with 1 in the $(2n)$-entry and zero in the rest of entries, and matrix D is diagonalizable. Moreover, if $a = 0$ then $D^n = D$ and D^{n-1} is similar to the diagonal matrix $\mathrm{diag}(1, \ldots, 1, 0)$.

Proof. (i) If $q = \mathrm{char}(K)$ does not divide n, then J can be written as

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 1 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}$$

The minimal polynomial of D is $p(x) = (x - a)^n - 1$ and it has n different roots in K (by hypothesis K contains all roots of $p(x) = (x - a)^n - 1$ and they are all different because $p'(x) = n(x - a)^{n-1} \neq 0$ since $q \nmid n$). In particular, D is diagonalizable. Moreover, one can see that $Q^2 = 0$.

(ii) If $q = \mathrm{char}(K)$ divides n then

$$J = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 1 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}$$

The minimal polynomial of D is $p(x) = (x - a)^n - (x - a)$ and its n roots belong to K by hypothesis and they are all different because $p'(x) = -1 \neq 0$ (recall $q|n$). In particular, it follows that D is diagonalizable. Moreover, $Q^2 = 0$ as required. □

Remark 2.2. The decomposition of each Jordan block into $D + Q$ given in Lemma 2.1 has the following properties:

- Each D is diagonalizable with no multiple eigenvalues.
- $Q^2 = 0$ and $\mathrm{rank}(Q) \leqslant 1$.

Proposition 2.3. *Let K be an algebraically closed field. Then any matrix $A \in \mathbb{M}_n(K)$ can be written as $D + Q$, where D is a diagonalizable matrix and Q is a nilpotent matrix for which $Q^2 = 0$.*

Proof. Since K is algebraically closed and A is similar to a direct sum of Jordan blocks then it is sufficient to decompose each Jordan block. Let J be a Jordan block of size $m \times m$ for some $m \leqslant n$ and q is the characteristic of K. If $m \leqslant 2$ the decomposition is straightforward (see Section 1 of [8]). When $m \geqslant 3$ and if q does no divide m then decomposition of J is presented in Lemma 2.1(i). If q does divide m then decomposition of J is presented in 2.1(ii). □

The assumption of algebraic closeness of the field can be removed when dealing with nilpotent matrices over a field for which the decomposition into Jordan blocks always holds. Let us notice that this can be related to [10, Sec. 2] where minimal conditions for a nilpotent element in a ring are given to admit decomposition into Jordan blocks. As a consequence, any nilpotent matrix can be expressed as the sum of a potent matrix and a nilpotent matrix of zero square. This result can be related to [4, Corollary 8] where any nilpotent matrix is decomposed into an idempotent matrix and a nilpotent matrix.

Corollary 2.4. *Every nilpotent matrix over a field can be written as $D+Q$, where D is a potent matrix (i.e., $D^q = D$ for a certain $q \in \mathbb{N}$) and Q is a nilpotent matrix with $Q^2 = 0$.*

Proof. Let $A \in \mathbb{M}_n(K)$ be a nilpotent matrix over the field K. Then A is similar to the direct sum of Jordan blocks $J_1, \ldots, J_s$, each of them is associated with the eigenvalue 0. Any of these Jordan blocks $J_i \in \mathbb{M}_{m_i}(K)$ is decomposed as in Lemma 2.1: $J_i = D_i + Q_i$. Let us define

$$k_i := \begin{cases} m_i, & \text{if } \mathrm{char}(K) \text{ does not divide } m_i, \\ m_i - 1 & \text{if } \mathrm{char}(K) \text{ divides } m_i, \end{cases}$$

and let $q = lcm\{k_i \mid i = 1, \ldots, s\} + 1$. Then

$$(\bigoplus_{i=1}^{s} D_i)^q = \bigoplus_{i=1}^{s} D_i^q = \bigoplus_{i=1}^{s} D_i,$$

i.e., $\bigoplus_{i=1}^{s} D_i$ is q-potent. Finally, A is decomposed into $D+Q$ as in Proposition 2.3 and D is similar to $\bigoplus_{i=1}^{s} D_i$. □

It is also worth to notice that the last statement can be proved by using the rational (Frobenius) canonical form [4], [10].

In what follows, Corollary 2.4 and Remark 2.9 from [8] will be substantially generalized with the use of another approach.

Proposition 2.5. *Every nilpotent matrix over a von Neumann regular ring is decomposable as the sum of a potent matrix and a nilpotent matrix of order two.*

Proof. Let R be a von Neumann regular ring. Then A^s is a von Neumann regular matrix for all $s \in \mathbb{N}$ for any nilpotent matrix A over R. Hence A is decomposed into a direct sum of Jordan blocks (see, e.g., [10]). Each of these Jordan blocks can be represented as the sum of a potent matrix and a zero-square matrix. In particular, it is not too hard to verify that A itself can be represented as the sum of a potent matrix and a zero-square matrix, as asserted in Proposition 2.5. □

3. Decomposition into potent matrix and zero-square matrix over finite fields

In what follows the following approach is used [8].

Lemma 3.1. *Let K be a field, $n \geqslant 3$ and $A \in \mathbb{M}_n(K)$ is the companion matrix of a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Then*

- *If $c_{n-1} = 0$ and $|K| \geqslant n$ then A admits decomposition into $D+Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\mathrm{rank}(Q) \leqslant 1$.*
- *If $c_{n-1} \neq 0$ and $|K| \geqslant n+1$ then A admits decomposition into $D+Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\mathrm{rank}(Q) \leqslant 1$.*

In this section the following assertion which is devoted to a non-trivial property of matrices over finite fields is considered (see Proposition 3.1 and Corollary 3.2 from [8]).

Corollary 3.2. *Let K be a finite field and $n \in \mathbb{N}$. Then every matrix in $\mathbb{M}_n(K)$ admits decomposition into the sum of an r-potent matrix, for certain $1 < r \in \mathbb{N}$, and a square-zero matrix.*

Proof. Let $\mathbb{F}_q$ be the finite field of q elements and assume that $A \in \mathbb{M}_n(\mathbb{F}_q)$. Let us consider the decomposition of A with respect to its invariant factors. Then A is similar to the direct sum of s companion matrices. Each companion matrix is of size $m_i \leqslant n$, $i = 1, \dots, s$. Let us take for each companion matrix an irreducible polynomial $q_i(x)$ of degree m_i with the same trace, and write this block as the sum of the companion matrix $C(q_i(x))$ and a nilpotent matrix of zero square (see Lemma 3.1 and its proof). The decomposition field F of $q_i(x)$ is an extension of degree m_i of $\mathbb{F}_q$, i.e., $F = \mathbb{F}_{q^{m_i}}$. Since matrix $C(q_i(x))$ is diagonalizable with different eigenvalues in F (finite fields are perfect), $C(q_i(x))$ is similar to a diagonal matrix $D_i \in \mathbb{M}_{m_i}(F)$. Therefore,

$$D_i^{q^{m_i}-1} = \begin{cases} I, & \text{if } q_i(0) \neq 0; \\ \operatorname{diag}(1, \dots, 1, 0), & \text{if } q_i(0) = 0. \end{cases}$$

Let us define $r = lcm\{q^{m_i} - 1 \mid i = 1, \dots, s\} + 1$. Since each $D_i^r = D_i$ one can express A as the sum of an r-potent matrix and a square-zero matrix. □

Actually, the more general question is whether or not every square matrix over an arbitrary (possibly infinite) field is presentable as the sum of a nilpotent matrix and a potent matrix. Let us notice that for finite fields this was settled independently in Corollary 3.2 and [1]. However, the answer seems to be definitely "not" as the next example illustrates but such matrix is rather the sum of a non-singular matrix and a nilpotent matrix (see, e.g., [11]). In particular, this fact surely implies that the matrix over the field $\mathbb{F}_4$ is the sum of a potent matrix and a nilpotent matrix (compare also with Corollary 3.2 and [1]).

Example 3.3. Let us consider matrix $A = 2\,\mathrm{Id} \in \mathbb{M}_n(\mathbb{R})$, and show that A cannot be expressed as the sum of a k-potent matrix and an r-nilpotent matrix. Otherwise, $A = Q + N$ with $Q^k = Q$ and $N^r = 0$ for some natural numbers k and r. On the one hand this surely implies that $Q = A - N$ satisfies the polynomial $X^k - X$ (because $Q^k = Q$). On the other hand, since $0 = N^r = (A - Q)^r = (2\,\mathrm{Id} - Q)^r$, matrix Q also satisfies the polynomial $(2 - X)^r$. This means that minimal polynomial of Q must divide both $X^k - X$ and $(X - 2)^r$ but these two polynomials have no common roots in $\mathbb{R}$. Then the minimal polynomial of Q is 1. This is a contradiction. The proof is completed.

4. Decomposition into periodic matrix and zero-square matrix over infinite fields

To begin with, it will be shown that Question 2 is *not* true even over algebraically closed fields. Let us consider the following example.

Example 4.1. Let $\mathbb{C}$ be the field of complex numbers, and consider the matrix $A = 2\mathrm{Id} \in \mathbb{M}_n(\mathbb{C})$. Let us assume that $A = T + N$, where $N^2 = 0$. Then $N = A - T$ and, therefore, $0 = N^2 = (A - T)^2 = 4\mathrm{Id} + T^2 - 4T$. This means that matrix T satisfies the polynomial $x^2 - 4x + 4 = (x - 2)^2$.

Furthermore, it is easy to verify that characteristic polynomial of T is of the form $(x - 2)^n$ for some $n \in \mathbb{N}$. Then the determinant of T is equal to $(-1)^n 2^n$. Consequently, T cannot be periodic because either $\det(P) = 0$, $\det(P) = 1$ or $\det(P) = -1$.

One can propose a complete answer to Question 2 as follows.

Proposition 4.2. *Let F be a field. Then the next three statements are valid:*

(1) If $\mathrm{char}(F) = 0$ then the answer is NO.

(2) If $\mathrm{char}(F) = p$ and the extension of F over its prime field F_p is transcendental then the answer is NO.

(3) If $\mathrm{char}(F) = p$ *and the extension of* F *over its prime field* F_p *is algebraic then the answer is YES.*

Proof. (1) The unique prime field of zero characteristic is precisely the field of rationals $\mathbb{Q}$. Hence the matrix 2Id cannot be decomposed into the sum of periodic matrix and zero-square matrix (compare with the stated above example).

(2) There exists an element $a \in F$ that is not algebraic over F_p. Let us consider then the matrix aId. Using the same argument as in (1), one can obtain that this matrix cannot be decomposed into the sum of periodic matrix and zero-square matrix.

(3) Indeed, square matrix can be decomposed even into the sum of potent matrix and zero-square matrix. Let $A \in \mathbb{M}_n(F)$. Let us consider the finite field L generated by F_p and by the entries of A. If L has more elements than the matrix size n then applying the main result from [8], the matrix A is decomposed into the sum of a diagonalizable matrix over L and a zero-square matrix. Otherwise, one can extend L by adding elements from F until some finite field L' with more elements than n is obtained. Since still $A \in \mathbb{M}_n(L')$ matrix A is decomposed into the sum of a diagonalizable matrix over L' and a zero-square matrix. Taking into account that a diagonalizable matrix over a finite field is always potent and hence periodic, the proof is complete. □

At the end of the paper let us consider the following interesting question.

Problem. Can any square matrix over the indecomposable ring $\mathbb{Z}_4$ be decomposed into the sum of a square-zero matrix and a potent matrix?

Let us note that it was established in [13] that every such matrix can be decomposed into the sum of a nilpotent matrix of order at most 8 and an idempotent matrix. So, it is rather realistic to replace the idempotent matrix by a potent matrix and thereby to expect that the order of the nilpotent matrix could be decreased to order 2 or, in a worse variant, to order 4.

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О некоторых разложениях матриц над алгебраически замкнутыми и конечными полями

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Аннотация. Мы доказываем, что каждая квадратная матрица над алгебраически замкнутым полем или над конечным полем разложима в сумму потентной матрицы и нильпотентной матрицы порядка 2. Это отчасти продолжает исследование из нашей недавней статьи, опубликованной в Linear & Multilinear Algebra (2022 г.).

Мы также полностью решаем вопрос, когда каждую квадратную матрицу над бесконечным полем можно разложить на периодическую матрицу и нильпотентную матрицу порядка 2.

Ключевые слова: нильпотентная матрица, потентная матрица, жорданова нормальная форма, рациональная форма, поле.

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Energy Method for the Elliptic Boundary Value Problems with Asymmetric Operators in a Spherical Layer

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Abstract. Three-dimensional elliptic boundary value problems arising in the mathematical modeling of quasi-stationary electric fields and currents in conductors with gyrotropic conductivity tensor in domains homeomorphic to the spherical layer are considered. The same problems are mathematical models of thermal conductivity or diffusion in moving or gyrotropic media. The operators of the problems in the traditional formulation are non-symmetric. New statements of the problems with symmetric positive definite operators are proposed. For the four boundary value problems the quadratic energy functionals, to the minimization of which the solutions of these problems are reduced, are constructed. Estimates of the obtained quadratic forms are made in comparison with the form appearing in the Dirichlet principle for the Poisson equation.

Keywords: mathematical modeling, energy method, elliptic equation, asymmetric operator.

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Introduction

The tree-dimensional elliptic boundary value problems with asymmetric operators arise, for example, in mathematical modeling of quasi-stationary electric fields and currents in the global conductor, consisting of the Earth's ionosphere and atmosphere, using the domain decomposition method [1]. In such models as [2], a significant simplification of the description of the D-layer of the Earth's ionosphere, lying at heights of $50 - 90$ km, is used. It is in this layer that the two-dimensional model, which is adequate for higher layers, is inapplicable, and the conductivity ceases to be a scalar, as in the underlying atmosphere. Therefore, in this layer, bounded by two surfaces close to concentric spheres, it is necessary to solve a three-dimensional problem of current continuity with a gyrotropic conductivity tensor. Within the framework of the decomposition method [1], the boundary conditions arise at the selected boundaries of subdomains. In the case of interest to us, they are mixed: the condition of ideal conductivity at the upper boundary of the D-layer and the condition on the ideal insulator on the bottom. Both conditions are heterogeneous. This boundary value problem is studied in detail in this work. For three problems which differ from the main problem by the boundary conditions, the necessary changes in the formulations

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and proofs are described. We use the approach previously used for the problem in a simply connected domain with a given normal component of the current density on the boundary [3].

Note that mathematical models of thermal conductivity or diffusion in moving media can be reduced to the same boundary value problems [4, 5], and the equations of heat conduction or diffusion in stationary gyrotropic media differ from the electric current continuity problem only in notation.

In the present work, the quadratic energy functionals are constructed, which makes it possible to reduce the solution of the boundary value problems for a three-dimensional elliptic equation with asymmetric coefficients to minimisation of functionals. It is shown that the energy norm is equivalent to the norm of the space $W_2^{(1)}(\Omega)$. The corresponding estimates are obtained with specific values of the constants that will allow us to estimate the condition number of the matrix of a system of linear algebraic equations, which will arise in the numerical solution of the problem.

1. The electric current continuity problem

In the three-dimensional domain Ω occupied by a conductor, the electric field strength $\mathbf{E}$ and the current density $\mathbf{J}$ in the quasistationary approximation satisfy Faraday's law, charge conservation law and Ohm's law:

$$\operatorname{rot}\mathbf{E} = \mathbf{G}, \quad \operatorname{div}\mathbf{J} = Q, \quad \mathbf{J} = \hat{\sigma}\mathbf{E}, \tag{1}$$

where $\mathbf{G} \neq 0$, if there is a given magnetic field that varies with time, $Q \neq 0$, if there are external currents, $\hat{\sigma}$ is the conductivity tensor. We assume that the norms of the given functions, Q and the Cartesian components of $\mathbf{G}$, are bounded in the space $L_2(\Omega)$. All vectors in this paper are considered as column vectors, which are transformed into row vectors by transposition, denoted by the symbol $*$. The system (1) with proper boundary conditions is referred to as the electric current continuity problem [6].

The conductivity of some substances in a magnetic field, for example, plasma in the Earth's ionosphere, is a gyrotropic tensor. In Cartesian coordinates x, y, z with the z axis directed along the magnetic induction vector, the conductivity tensor has the form:

$$\hat{\sigma} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}. \tag{2}$$

Its components are called field-aligned ($\sigma_\parallel$), Pedersen (σ_P) and Hall (σ_H) conductivities [6]. We also use the Cowling conductivity

$$\sigma_C = (\sigma_P^2 + \sigma_H^2)/\sigma_P.$$

In the article [3] a more general form of $\hat{\sigma}$ is considered. Here we use the form (2), to obtain more accurate estimates which are important for the numerical solution of the problems. Since the passage of electric current is accompanied by dissipation of the electric energy with density $\mathbf{J}^*\mathbf{E}$, the symmetric part of $\hat{\sigma}$ is positive definite. For a tensor of the form (2), this means the positiveness of the diagonal elements. Excluding ideal conductors and insulators from consideration, we assume uniform in the domain Ω boundedness of all coefficients of $\hat{\sigma}$ and uniform positive definiteness. It is convenient to write these conditions in the form:

$$\sigma_1 \leqslant \sigma_C \leqslant \sigma_2, \qquad \sigma_1 \leqslant \sigma_\parallel \leqslant \sigma_2, \tag{3}$$

where σ_1, σ_2 are positive constants.

We assume that the domain Ω is bounded and homeomorphic to the spherical layer. Its outer Γ and inner γ boundaries are twice continuously differentiable surfaces homeomorphic to the sphere. Normal (positive outward direction) and tangent to the boundary components of vectors are marked with indices n and τ.

To carry out all the proofs by simple means, we assume the convexity of the surface Γ and the boundedness of the curvature of γ. We also consider the differences of both surfaces to be limited from two spheres with a common center and not too different radii. We will give a concrete form to all these restrictions in Section 2.

As the main problem, consider a problem with boundary conditions arising in mathematical modeling of the D-layer of the ionosphere [1]. If the domain under consideration borders on an ideal insulator, then at the boundary, the normal component of $\mathbf{J}$ equals to zero. On the border with the ideal conductor the tangent components of $\mathbf{E}$ equal to zero. The conditions can be heterogeneous:

$$J_n|_\gamma = q, \qquad \mathbf{E}_\tau|_\Gamma = \mathbf{g}, \tag{4}$$

where q, $\mathbf{g}$ are given functions. Assume that the norms of these functions are bounded in the spaces $L_2(\Gamma)$ and $L_2(\gamma)$, respectively, that is, the boundedness of the integrals of the squares of the moduli of these functions over the surfaces on which they are given.

For the solvability of the problem (1), (4) the right-hand sides must satisfy some constraints. First, it is necessary

$$\operatorname{div} \mathbf{G} = 0, \tag{5}$$

since div of rot is identically equal to zero, and without (5) the first equation (1) cannot hold.

Let us calculate the components of rot normal to the boundary from the left and right sides in the second boundary condition (4):

$$\operatorname{rot}_n \mathbf{E}_\tau \,|_\Gamma = \operatorname{rot}_n \mathbf{g} \,|_\Gamma \ .$$

The resulting left side also satisfies the first equation (1). This imposes on the given functions the second condition necessary for the solvability of the problem

$$G_n \,|_\Gamma = \operatorname{rot}_n \mathbf{g} \,|_\Gamma \ . \tag{6}$$

In Section 4, a new problem will be formulated with an conjugately factorized operator by definition [7], the solution of which is the solution to the original problem (1), (4). The existence of the solution to the new problem implies the existence of a solution to the original one, but the uniqueness must be proven independently. It does not differ from the proof given in [3] for the problem in a domain homeomorphic to a ball. Heuristic considerations are also given there, allowing us to propose a new formulation and to construct a quadratic energy functional.

2. The energy scalar product

We consider the set of pairs of smooth functions F, $\mathbf{P}$, satisfying the conditions:

$$F \,|_\Gamma = 0, \quad P_n \,|_\Gamma = 0, \quad \mathbf{P}_\tau \,|_\gamma = 0. \tag{7}$$

We denote with square brackets the symmetric bilinear form:

$$\left[\begin{pmatrix} u \\ \mathbf{v} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix} \right] = \int \left(\begin{pmatrix} \operatorname{grad} u \\ \operatorname{rot} \mathbf{v} \end{pmatrix}^* \begin{pmatrix} \frac{1}{\sigma_0} \hat\sigma \hat S \hat\sigma^* & -\hat\sigma \hat S \\ -\hat S \hat\sigma^* & \sigma_0 \hat S \end{pmatrix} \begin{pmatrix} \operatorname{grad} F \\ \operatorname{rot} \mathbf{P} \end{pmatrix} + \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{P} \right) d\Omega, \tag{8}$$

where $u, \mathbf{v}$ and $F, \mathbf{P}$ are the pairs of smooth functions satisfying conditions (7), $\hat{S}$ is a symmetric positive definite matrix, which we will choose later, as well as the value of the positive constant σ_0. This bilinear form will be used as the energy scalar product. Let us check that it has the necessary properties for this.

Consider the corresponding quadratic form. We start with the auxiliary integral

$$\int (\operatorname{grad} F)^* \operatorname{rot} \mathbf{P} \, d\Omega. \tag{9}$$

We transform the integrand:

$$\int (\operatorname{div}(F \operatorname{rot} \mathbf{P}) - F \operatorname{div} \operatorname{rot} \mathbf{P}) \, d\Omega.$$

The second term is identically zero. The remaining integral is transformed using the Gauss–Ostrogradsky theorem:

$$\oint_{\Gamma} F \operatorname{rot}_n \mathbf{P} \, d\Gamma + \oint_{\gamma} F \operatorname{rot}_n \mathbf{P} \, d\gamma.$$

Both integrands are equal to zero due to the first and third conditions (7), respectively. Therefore, the integral (9) is equal to zero, and therefore it can be added with any coefficient to the corresponding (8) quadratic form without changing its value.

The matrix appearing in (8) is degenerate, since its upper blocks are obtained from the lower ones through multiplication by $-\hat{\sigma}/\sigma_0$. By adding the doubled integral (9) the matrix of the integrand quadratic form becomes equal to

$$K = \begin{pmatrix} \frac{1}{\sigma_0} \hat{\sigma} \hat{S} \hat{\sigma}^* & -\hat{\sigma} \hat{S} + \hat{I} \\ -\hat{S} \hat{\sigma}^* + \hat{I} & \sigma_0 \hat{S}, \end{pmatrix},$$

where $\hat{I}$ is the identity matrix.

To get more accurate estimates than in [3], we use a special form $\hat{\sigma}$ (2) and constraints (3), taking

$$\sigma_0 = \sqrt{\sigma_1 \sigma_2}, \quad \hat{S}^{-1} = (\hat{\sigma} + \hat{\sigma}^*)/2. \tag{10}$$

With this choice, $\hat{S}$ is a diagonal matrix, and the symmetric matrix K in the same local Cartesian coordinates as (2) takes the form:

$$K = \begin{pmatrix} \frac{\sigma_C}{\sigma_0} & 0 & 0 & 0 & \frac{\sigma_H}{\sigma_P} & 0 \\ 0 & \frac{\sigma_C}{\sigma_0} & 0 & -\frac{\sigma_H}{\sigma_P} & 0 & 0 \\ 0 & 0 & \frac{\sigma_\parallel}{\sigma_0} & 0 & 0 & 0 \\ 0 & -\frac{\sigma_H}{\sigma_P} & 0 & \frac{\sigma_0}{\sigma_P} & 0 & 0 \\ \frac{\sigma_H}{\sigma_P} & 0 & 0 & 0 & \frac{\sigma_0}{\sigma_P} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sigma_0}{\sigma_\parallel} \end{pmatrix}.$$

The eigenvalues of the matrix do not change with the simultaneous permutation of rows and corresponding columns. This allows us to reduce the matrix K to a block-diagonal form with blocks

$$\begin{pmatrix} \dfrac{\sigma_C}{\sigma_0} & \dfrac{\sigma_H}{\sigma_P} \\ \dfrac{\sigma_H}{\sigma_P} & \dfrac{\sigma_0}{\sigma_P} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\sigma_C}{\sigma_0} & -\dfrac{\sigma_H}{\sigma_P} \\ -\dfrac{\sigma_H}{\sigma_P} & \dfrac{\sigma_0}{\sigma_P} \end{pmatrix}, \quad \frac{\sigma_\parallel}{\sigma_0}, \quad \frac{\sigma_0}{\sigma_\parallel}. \tag{11}$$

The eigenvalues of the two matrices are equal. The bigger of them λ_{max} does not exceed the matrix trace that is equal to

$$\frac{\sigma_C}{\sigma_0} + \frac{\sigma_0}{\sigma_P}.$$

Due to (3), the eigenvalues of blocks and the last two numbers (11) do not exceed

$$\lambda_{max} \leqslant 2\sqrt{\sigma_2/\sigma_1}. \tag{12}$$

The determinants of the blocks (11) are equal to one, which means that smaller eigenvalues

$$\lambda_{min} = 1/\lambda_{max} \geqslant \sqrt{\sigma_1/\sigma_2}/2. \tag{13}$$

Since the eigenvalues are not changed when the coordinate system is rotated, the parameter (12) for all points of the domain Ω estimates the eigenvalues of the matrix K from above, and the inverse value from below. Therefore, the condition number of the matrix K does not exceed $4\sigma_2/\sigma_1$.

By adding an auxiliary integral (9), we have not changed the value of the quadratic form (8). Therefore, from (12), (13), the upper and the lower estimations follow:

$$\left[\begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}\right] \leqslant \sqrt{\frac{4\sigma_2}{\sigma_1}} \int ((\operatorname{grad} F)^2 + (\operatorname{rot} \mathbf{P})^2 + (\operatorname{div} \mathbf{P})^2)\, d\Omega, \tag{14}$$

$$\left[\begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}\right] \geqslant \sqrt{\frac{\sigma_1}{4\sigma_2}} \int ((\operatorname{grad} F)^2 + (\operatorname{rot} \mathbf{P})^2 + (\operatorname{div} \mathbf{P})^2)\, d\Omega. \tag{15}$$

Now consider the functions F and $\mathbf{P}$ separately in order to estimate the right-hand side (15) from below. Since the function F is equal to zero on the surface Γ (7), it satisfies the Friedrichs inequality

$$\int F^2\, d\Omega \leqslant c_0 \int (\operatorname{grad} F)^2\, d\Omega, \tag{16}$$

where the constant c_0 is determined only by the shape of the domain and does not depend on the specific function F. Usually this inequality is formulated for functions equal to zero on the entire boundary, however, it suffices equality to zero on a segment of finite area (the theorem on equivalent norms [8]). For the spherical layer under consideration, it is possible to obtain a specific value of the constant c_0 by the same method, which is used for the more complex case of vector functions in [9].

In [9] for the functions $\mathbf{P}$ satisfying the boundary conditions (7), the following inequalities are proved

$$\int |\mathbf{P}|^2\, d\Omega \leqslant c_1 \int |\operatorname{grad} \mathbf{P}|^2\, d\Omega, \tag{17}$$

$$c_2 \int |\operatorname{grad} \mathbf{P}|^2\, d\Omega \leqslant \int ((\operatorname{rot} \mathbf{P})^2 + (\operatorname{div} \mathbf{P})^2)\, d\Omega, \tag{18}$$

where $|\operatorname{grad} \mathbf{P}|^2$ is the sum of the squared moduli of the gradients of all Cartesian components of $\mathbf{P}$, and the constants c_1, c_2 are determined only by the shape of the domain Ω and do not depend on the specific function $\mathbf{P}$.

The first inequality is similar to the Friedrichs inequality for the scalar functions which are equal to zero at the boundary. If the entire vector $\mathbf{P}$ is equal to zero at the boundary, for each of its Cartesian components one can use Friedrichs inequality to obtain the required inequality.

However, of interest are the functions with only normal or only tangent components equal to zero at the boundary. In [10], both inequalities were proved for the functions with one of these conditions posed at the entire boundary of an arbitrary multiply connected domain. For mixed boundary conditions (7) new proofs are required.

To restrict ourselves to simple means, in [9] the additional restrictions are imposed on the domain shape: convexity of the surface Γ and bounded curvature of γ, and also the limited difference of both surfaces from two spheres with a common center and not too different radii. Let R denote the minimum radius of curvature of γ. Let the surfaces Γ and γ be defined in spherical coordinates using the functions $R_\Gamma(\theta,\varphi)$ and $R_\gamma(\theta,\varphi)$. We assume that

$$0 < R_1 \leqslant R_\gamma(\theta,\varphi) \leqslant R_2, \quad 0 < R_\Gamma(\theta,\varphi) - R_\gamma(\theta,\varphi) \leqslant \delta R,$$

and the constants $R, R_1, R_2, \delta R$ will be subject to one more general restriction.

Another condition limits the angle between the boundary normal and the radial direction at each point. Let us write it down in a convenient form as a constraint on the radial component of the unit vector of the outer normal:

$$n_{\Gamma,r} \geqslant \xi_1, \quad -n_{\gamma,r} \geqslant \xi_1.$$

We also require that the scalar product of the normals $\mathbf{n}_\Gamma$ and $\mathbf{n}_\gamma$, calculated on one ray is positive:

$$\mathbf{n}_\Gamma(\theta,\varphi)\cdot\mathbf{n}_\gamma(\theta,\varphi) \geqslant \xi_2 > 0.$$

Then the constant obtained in [9],

$$c_2 = 1 - \frac{2R_2^2\delta R}{\xi_1\xi_2^2R_1^2R}. \tag{19}$$

The inequality (18) makes sense only for positive c_2, which gives one more general constraint on the values of geometric parameters. In mathematical modeling of the D-layer of the ionosphere, this condition is satisfied by a large margin: the fraction in (19) is less than 0.02. If the boundaries γ and Γ are the spheres with radii R_1 and $R_1+\delta R$, then $R_2 = R = R_1$, $\xi_1 = \xi_2 = 1$, and the only condition $2\delta R < R_1$ is enough.

Denoting by c_4 the lesser of the constants $1/c_0$, c_2/c_1 in the inequalities (15)–(18) we obtain an inequality that means positive definiteness of the bilinear form (8)

$$\left[\begin{pmatrix} F \\ \mathbf{P}\end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P}\end{pmatrix}\right] \geqslant c_4\sqrt{\frac{4\sigma_2}{\sigma_1}}\int (F^2 + |\mathbf{P}|^2)\,d\Omega.$$

The same set of inequalities gives a lower estimate for (8) in terms of the sum of the squares of the norms of F and the Cartesian components of $\mathbf{P}$ as elements of the space $W_2^{(1)}(\Omega)$.

It is easy to prove the inequality:

$$(\operatorname{rot}\mathbf{P})^2 + (\operatorname{div}\mathbf{P})^2 \leqslant 3\sum_{i,j=1}^{3}|\operatorname{grad}\mathbf{P}|^2. \tag{20}$$

We expand the expressions on the left, immediately replacing the products with the sums of squares, without decreasing the value of the entire expression. Bringing similar terms, we get the sum, in which the squares of all derivatives of all components of the vector $\mathbf{P}$ enter with the coefficients 2 or 3, while on the right-hand side (20) they all appear with coefficient 3.

With (14) and (20) the energy norm is estimated in terms of the same norms of F, $\mathbf{P}$ from above.

The upper and lower estimates mean the equivalence of the introduced energy norm to the norm of the space $W_2^{(1)}(\Omega)$. In particular, this allows for the numerical solution of the problem to use the same approximating functions, as for the Poisson equation.

3. The energy functional

In accordance with the energy method [11], we define the energy functional:

$$W(F,\mathbf{P}) = \frac{1}{2}\left[\begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}\right] - \frac{1}{\sigma_0}\int FQ\,d\Omega - \int \mathbf{P}^*\mathbf{G}\,d\Omega + \int_\Gamma \mathbf{P}^*\mathbf{g}\,d\Gamma + \frac{1}{\sigma_0}\int_\gamma Fq\,d\gamma. \tag{21}$$

We use the Cauchy-Bunyakovsky inequality to estimate the linear functionals:

$$\left|\int FQ\,d\Omega\right|^2 \leqslant \int F^2\,d\Omega \int Q^2\,d\Omega, \quad \left|\int \mathbf{P}^*\mathbf{G}\,d\Omega\right|^2 \leqslant \int |\mathbf{P}|^2\,d\Omega \int |\mathbf{G}|^2\,d\Omega, \tag{22}$$

$$\left|\int_\gamma Fq\,d\gamma\right|^2 \leqslant \int_\gamma F^2\,d\gamma \int_\gamma q^2\,d\gamma, \quad \left|\int_\Gamma \mathbf{P}^*\mathbf{g}\,d\Gamma\right|^2 \leqslant \int_\Gamma |\mathbf{P}|^2\,d\Gamma \int_\Gamma |\mathbf{g}|^2\,d\Gamma. \tag{23}$$

Due to the inequalities (16), (17), the right-hand sides (22) are estimated from above by the energy norm with some factor independent of $F, \mathbf{P}$. The second factors are bounded due to belonging of Q and Cartesian components of $\mathbf{G}$ to the space $L_2(\Omega)$, specified in the formulation of equations (1). Therefore, the first two linear functionals are bounded.

The integral $|\mathbf{P}|^2$ over the boundary Γ is estimated from above in terms of the integral $|\operatorname{grad}\mathbf{P}|^2$ over the domain Ω in [9]. It is not difficult to estimate in a similar way the integral of F^2 over the boundary γ in terms of the integral of $|\operatorname{grad} F|^2$. Since the functions F, $\mathbf{P}$ from the energy space are elements of $W_2^{(1)}(\Omega)$, the right-hand sides of the inequalities (23) are estimated from above by the energy norm of the pair of $F, \mathbf{P}$ with some factor independent of $F, \mathbf{P}$. The second factors in (23) are bounded due to q and Cartesian components of $\mathbf{g}$ belonging to the spaces $L_2(\Gamma)$ and $L_2(\gamma)$, respectively, which was as agreed in the formulation of the boundary conditions (4). This means that the last linear functionals in (21) are also bounded.

By F. Rees's theorem, any bounded linear functional can be represented in the form of a scalar product by some uniquely defined element of Hilbert space. Therefore, the energy functional (21) can be written as

$$W(F,\mathbf{P}) = \frac{1}{2}\left[\begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}\right] - \left[\begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P} \end{pmatrix}\right],$$

where $a, \mathbf{b}$ is some element of the energy space.

This expression can be transformed by highlighting the square of the difference:

$$W(F,\mathbf{P}) = \frac{1}{2}\left[\begin{pmatrix} F-a \\ \mathbf{P}-\mathbf{b} \end{pmatrix}, \begin{pmatrix} F-a \\ \mathbf{P}-\mathbf{b} \end{pmatrix}\right] - \frac{1}{2}\left[\begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}\right]. \tag{24}$$

The second term does not depend on $F, \mathbf{P}$, and the first is positive definite. Therefore the minimum of $W(F,\mathbf{P})$ is attained at $F = a$, $\mathbf{P} = \mathbf{b}$.

Since the element $a, \mathbf{b}$ belongs to the energy space and is defined uniquely, it is proved that in the energy space there exists, and the only one, an element that supplies the energy functional with a minimum value.

4. The generalized solution

Let us introduce the notation

$$\mathbf{E} = -\frac{1}{\sigma_0}\hat{S}\hat{\sigma}^* \mathrm{grad}\, F + \hat{S}\mathrm{rot}\,\mathbf{P}, \quad \mathbf{J} = \hat{\sigma}\mathbf{E}, \tag{25}$$

where $F, \mathbf{P}$ are the functions which provide the minimum value to the energy functional. Let us prove that these $\mathbf{J}, \mathbf{E}$ are the solution to the original boundary value problem (1), (4).

The condition for the minimality of the energy functional, taking into account the notation (25), can be written as an identity valid for arbitrary smooth functions $u, \mathbf{v}$ satisfying the conditions (7):

$$\int \big(-(\mathrm{grad}\, u)^* \mathbf{J}/\sigma_0 + (\mathrm{rot}\,\mathbf{v})^* \mathbf{E} + \mathrm{div}\,\mathbf{v}\,\mathrm{div}\,\mathbf{P} - uQ/\sigma_0 - \mathbf{v}^*\mathbf{G}\big)\, d\Omega + \\ + \frac{1}{\sigma_0}\int_\gamma uq\, d\gamma + \int_\Gamma \mathbf{v}^*\mathbf{g}\, d\Gamma. \tag{26}$$

If we additionally assume the smoothness of all functions, we can use integration by parts, for transformation of this identity to the form:

$$\frac{1}{\sigma_0}\int u(\mathrm{div}\,\mathbf{J} - Q)\, d\Omega + \int \mathbf{v}^*(\mathrm{rot}\,\mathbf{E} - \mathbf{G})\, d\Omega + \int \mathrm{div}\,\mathbf{v}\,\mathrm{div}\,\mathbf{P}\, d\Omega + \\ + \frac{1}{\sigma_0}\int_\gamma u(-J_n + q)\, d\gamma + \int_\Gamma \mathbf{v}^*(-\mathbf{E}_\tau + \mathbf{g})\, d\Gamma = 0. \tag{27}$$

Take $u = 0$ and the function $\mathbf{v}$ of the form

$$\mathbf{v} = \mathrm{grad}\, V, \quad V|_\Gamma = 0, \quad V|_\gamma = 0. \tag{28}$$

Then the identity (27) takes the form:

$$\int (\mathrm{grad}\, V)^*(\mathrm{rot}\,\mathbf{E} - \mathbf{G})\, d\Omega + \int \mathrm{div}\,\mathrm{grad}\, V\,\mathrm{div}\,\mathbf{P}\, d\Omega = 0. \tag{29}$$

The first integral by the Gauss–Ostrogradskii theorem is

$$\oint_\Gamma V(\mathrm{rot}\,\mathbf{E} - \mathbf{G})_n\, d\Gamma + \oint_\gamma V(\mathrm{rot}\,\mathbf{E} - \mathbf{G})_n\, d\gamma - \int V(\mathrm{div}\,\mathrm{rot}\,\mathbf{E} + \mathrm{div}\,\mathbf{G})\, d\Omega.$$

The integrands are equal to zero, since V is equal to zero at the boundaries Γ, γ due to (28), div rot is identically zero, and the function $\mathbf{G}$ satisfies the condition (5).

Therefore, from (29) we obtain the identity

$$\int \mathrm{div}\,\mathrm{grad}\, V\,\mathrm{div}\,\mathbf{P}\, d\Omega = 0,$$

which is valid for any function V equal to zero on the boundary.

This identity allows us to prove the equality to zero $\mathrm{div}\,\mathbf{P}$ in the usual way. Assuming the opposite, take a point in Ω, where $\mathrm{div}\,\mathbf{P} \neq 0$, and its neighborhood, where $\mathrm{div}\,\mathbf{P}$ is sign-preserving. We construct V as a solution to the Dirichlet problem for the Poisson equation with the right-hand side, equal to 1 in the selected neighborhood, and zero for the rest of Ω. Substituting such a function V, we obtain the nonzero value of the last integral, which contradicts the identity.

Proving that

$$\operatorname{div}\mathbf{P} = 0, \tag{30}$$

the third integral in (27) can be eliminated. After that, using the arbitrariness of the functions $u, \mathbf{v}$, it is easy to prove that all factors for $u, \mathbf{v}$ are equal to zero. Since $\mathbf{E}$ and $\mathbf{J}$ are used in the present section only for the abbreviated notation of the expression (25), the following equations are fulfilled:

$$\begin{aligned} \operatorname{div}\left(-\frac{1}{\sigma_0^2}\hat{\sigma}\hat{S}\hat{\sigma}^*\operatorname{grad}F + \frac{1}{\sigma_0}\hat{\sigma}S\operatorname{rot}\mathbf{P}\right) &= Q/\sigma_0, \\ \operatorname{rot}\left(-\frac{1}{\sigma_0}\hat{S}\hat{\sigma}^*\operatorname{grad}F + \hat{S}\operatorname{rot}\mathbf{P}\right) &= \mathbf{G}, \end{aligned} \tag{31}$$

$$\left(-\frac{1}{\sigma_0^2}\hat{\sigma}\hat{S}\hat{\sigma}^*\operatorname{grad}F + \frac{1}{\sigma_0}\hat{\sigma}\hat{S}\operatorname{rot}\mathbf{P}\right)_n\Bigg|_\gamma = q, \tag{32}$$

$$\left(-\frac{1}{\sigma_0}\hat{S}\hat{\sigma}^*\operatorname{grad}F + \hat{S}\operatorname{rot}\mathbf{P}\right)_\tau\Bigg|_\Gamma = \mathbf{g}. \tag{33}$$

The converse statement is also easy to prove: the solution of the problem (30)–(33), (7) gives the minimum value in the energy space to the energy functional. Indeed, let $F, \mathbf{P}$ be a solution to this boundary value problem. Let us write down the energy functional for the sum of the functions $F + tu$, $\mathbf{P} + t\mathbf{v}$, where t — an arbitrary number, $u, \mathbf{v}$ are smooth functions satisfying the conditions (7):

$$\begin{aligned} W(F+tu, \mathbf{P}+t\mathbf{v}) = W(F,\mathbf{P}) + \frac{t^2}{2}\left[\begin{pmatrix} u \\ \mathbf{v}\end{pmatrix}, \begin{pmatrix} u \\ \mathbf{v}\end{pmatrix}\right] + t\left(\left[\begin{pmatrix} u \\ \mathbf{v}\end{pmatrix}, \begin{pmatrix} F \\ \mathbf{P}\end{pmatrix}\right] - \right. \\ \left. - \int (uQ/\sigma_0 + \mathbf{v}^*\mathbf{G})\, d\Omega + \int_\gamma uq/\sigma_0\, d\gamma + \int_\Gamma \mathbf{v}^*\mathbf{g}\, d\Gamma\right). \end{aligned} \tag{34}$$

Since $F, \mathbf{P}$ is a solution to the boundary value problem, the identity (27) is satisfied for arbitrary $u, \mathbf{v}$. By integrating by parts from (27) we obtain the identity (26), which means equality to zero of the factor of t in the square trinomial (34).

The coefficient of t^2 is positive, since the positive definiteness of the energy quadratic form has been proved, and we are interested in $u, \mathbf{v}$, which are not identically zero. Therefore

$$W(F+tu, \mathbf{P}+t\mathbf{v}) \geqslant W(F, \mathbf{P}),$$

and equality is obtained only for $t = 0$, that is, the minimum of W is attained at the element $F, \mathbf{P}$.

As can be seen from (27), the constructed functions $\mathbf{E}$ and $\mathbf{J}$ satisfy equations and boundary conditions of the original boundary value problem (1), (4). As already noted, the uniqueness of the solution is proved in almost the same way as in [3].

Thus, we have proposed a new formulation of the problem (30)–(33), (7), which, unlike the original problem, has a symmetric positive definite operator.

At the beginning of the section, an additional smoothness assumption was made. If this condition is not satisfied, the identity (26) cannot be transformed into (27). In this case, the pair of functions $F, \mathbf{P}$, providing the energy functional the minimum value, we consider as the generalized solution to the problem (30)–(33), (7), and the pair $\mathbf{E}$, $\mathbf{J}$ constructed from them by formulae (25) is the generalized solution of the original problem (1), (4).

The existence and uniqueness of the generalized solution of the problem (30)–(33), (7) have been proved, therefore the existence of the generalized solution of the original boundary value problem (1), (4) is proved, and the principle of the minimum of the energy functional is substantiated for it.

5. Other boundary value problems

Consider three problems which differ from the main problem by the boundary conditions. In the first problem, we swap the parts of the boundary γ and Γ in the boundary conditions (4). Then, naturally, the condition of the form (6), that is necessary for the solvability of this first problem, is moved to γ.

As described in [3] and demonstrated above when constructing the generalized solution (26), the conditions for the set of pairs of smooth functions F, $\mathbf{P}$, are the conjugate ones to the boundary conditions of the original problem. Therefore, in the conditions (7) and in the energy functional, all changes are limited to the permutation of γ and Γ. Note that the value of the constant in the inequalities (17), (18) [9] varies.

As the second boundary value problem, consider the domain Ω surrounded by an insulator. Then the first of the conditions (4) is set on both sections of the boundary γ and Γ. From the charge conservation law, integrated over the entire domain, a necessary condition for the solvability of this problem arises:

$$\int Q\, d\Omega - \int_{\gamma} q\, d\gamma - \int_{\Gamma} q\, d\Gamma = 0. \tag{35}$$

The conditions for the pairs of functions F, $\mathbf{P}$ (7) take the form

$$\mathbf{P}_{\tau}\,|_{\Gamma} = 0, \quad \mathbf{P}_{\tau}\,|_{\gamma} = 0, \quad \int F\, d\Omega = 0. \tag{36}$$

The last condition is the conjugate one to the condition (35). It has to be added, since without the equality $F = 0$ on the boundary that was in (7), the Friedrichs inequality cannot be used for the function F. The inequality (16) with a different constant is now fulfilled as Poincare's inequality. The condition in question can be replaced by fixing some other average value of the function F, for example, the average over γ or Γ. Then the inequality (16) will be true as a consequence of the equivalent normalization theorem [8]. In fact, such a condition should eliminate the ambiguity of the solution corresponding to the addition of an arbitrary constant to F. Inequalities (17), (18) for the vector functions $\mathbf{P}$ whose tangent components are equal to zero on the whole boundary (36), were proved in [10] for bounded multiply connected domains of general form, however, without specifying the values of the constants.

The third boundary value problem describes the domain Ω surrounded by an ideal conductor. Then, on both sections of the boundary γ and Γ, the second of the conditions (4) is set. In contrast to the second boundary value problem, the additional solvability condition does not arise. It would arise if the domain Ω is, for example, a torus, in which this problem has a nonzero solution for zero right-hand sides. For isotropic constant conductivity in an axisymmetric torus, this would be an azimuthal electric field, not changing with displacement in the direction of the axis of symmetry and decreasing in inverse proportion to the distance to the axis, and the current density proportional to it. Of course, the nonzero Joule dissipation of such a current system must be compensated by some external sources. In this example, the energy source is a time-varying magnetic field in the domain encompassed by this torus. For multiply connected domains of general form, the ambiguity of solutions to similar problems is analyzed in [10].

The conditions for the set of pairs of functions F, $\mathbf{P}$ in the third boundary value problem have the form

$$F\mid_{\gamma,\Gamma}=0, \quad P_n\mid_{\gamma,\Gamma}=0.$$

The inequality (16) for the function F remains the Friedrichs inequality, and for the vector functions $\mathbf{P}$ with zero normal component on the entire boundary the inequalities (17), (18) are also proved in [10].

Thus, for three boundary value problems, new formulations of problems are proposed, which, in contrast to the original problems, have symmetric positive definite operators, and the principles of the minimum of energy functionals are justified for them.

Taking into account the notation (25), with the tensor $\hat{S}$ chosen in accordance with (10), the quadratic form, corresponding to the energy scalar product (8) can be written in terms of the original electric current continuity problem (1), (4):

$$\frac{1}{\sigma_0}\int \left(\mathbf{E}^*\mathbf{J} + (\operatorname{div}\mathbf{P})^2 \right) \, d\Omega. \tag{37}$$

The product $\mathbf{E}^*\mathbf{J}$ is the Joule dissipation density, that is, heat release accompanying the passage of the electric current. Since the minimization of the energy functional means satisfaction of the equality (30), the quadratic form (37) is equal to the total Joule dissipation in the domain Ω, which justifies the name "energy". Thus, the introduced energy norm makes sense from the point of view of nonequilibrium thermodynamics.

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Энергетический метод для эллиптических краевых задач с несимметричными операторами в шаровом слое

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Аннотация. Рассмотрены трехмерные эллиптические краевые задачи, возникающие при математическом моделировании квазистационарных электрических полей и токов в проводниках с гиротропным тензором проводимости в областях, гомеоморфных шаровому слою. Аналогичные задачи формулируются при моделировании теплопроводности или диффузии в движущихся или гиротропных средах. Операторы задач в традиционной формулировке являются несимметричными. Предложены новые формулировки задач с симметричными положительно определенными операторами. Для четырех краевых задач построены квадратичные функционалы энергии, к минимизации которых сведено решение этих задач. Выполнены оценки полученных квадратичных форм в сравнении с формой, фигурирующей в принципе Дирихле для уравнения Пуассона.

Ключевые слова: математическое моделирование, энергетический метод, эллиптическое уравнение, несимметричный оператор.

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On Reductants of Two Groups

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Abstract. In this paper we consider the reductant of the dihedral group D_n, consisting of a set of axial symmetries, and the sphere S^2 as a reductant of the group $\mathrm{SU}(2,\mathbb{C}) \cong S^3$ (the group of unit quaternions). By introducing the Sabinin's multiplication on the reductant of D_n, we get a quasigroup with unit.

Keywords: groups reductants, quasigroups

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Introduction

Nonassociative structures in modern algebra is not only mathematical curiosity but actively developed direction. The subtraction on a set of integers is not associative. Indeed, $3-(2-1) = 2$, and $(3-2)-1 = 0$. On the set of real numbers we introduce the averaging operation $a*b = \dfrac{a+b}{2}$, then

$$(a*b)*c = \frac{\frac{a+b}{2}+c}{2},$$

and

$$a*(b*c) = \frac{a+\frac{b+c}{2}}{2}.$$

The vector product in three-dimensional space is nonassociative. Recall that

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1),$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are three-dimensional vectors.

Multiplication of octonionic imaginary units, see Tab. 1, does not have the property of associativity.

Therein

$$\imath^2 = J^2 = \mathcal{J}^2 = -1;$$
$$Jz = z^*J;$$
$$h(q\mathcal{J}) = (qh)\mathcal{J},\ (h\mathcal{J})q = (hq^*)\mathcal{J},\ (h\mathcal{J})(q\mathcal{J}) = -q^*h,$$

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Table 1.

	1	$\imath$	J	$\imath J$	$\mathcal{J}$	$\imath\mathcal{J}$	$J\mathcal{J}$	$\imath J\mathcal{J}$
1	1	$\imath$	J	$\imath J$	$\mathcal{J}$	$\imath\mathcal{J}$	$J\mathcal{J}$	$\imath J\mathcal{J}$
$\imath$	$\imath$	-1	$\imath J$	$-J$	$\imath J$	$-\mathcal{J}$	$-\imath J\mathcal{J}$	$J\mathcal{J}$
J	J	$-\imath J$	-1	$\imath$	$J\mathcal{J}$	$\imath J\mathcal{J}$	$-\mathcal{J}$	$-\imath\mathcal{J}$
$\imath J$	$\imath J$	J	$-\imath$	-1	$\imath J\mathcal{J}$	$-J\mathcal{J}$	$\imath\mathcal{J}$	$-\mathcal{J}$
$\mathcal{J}$	$\mathcal{J}$	$-\imath\mathcal{J}$	$-J\mathcal{J}$	$-\imath J\mathcal{J}$	-1	$\imath$	J	$\imath J$
$\imath\mathcal{J}$	$\imath\mathcal{J}$	$\mathcal{J}$	$-\imath J\mathcal{J}$	$J\mathcal{J}$	$-\imath$	-1	$-\imath J$	J
$J\mathcal{J}$	$J\mathcal{J}$	$\imath J\mathcal{J}$	$\mathcal{J}$	$-\imath\mathcal{J}$	$-J$	$\imath J$	-1	$-\imath$
$\imath J\mathcal{J}$	$\imath J\mathcal{J}$	$-J\mathcal{J}$	$\imath\mathcal{J}$	$\mathcal{J}$	$-\imath J$	$-J$	$\imath$	-1

where z is an arbitrary complex number, and h, q are quaternions. Also z^* and q^* are conjugate quantities, see, for example, [1]. If we add a fourth imaginary unit W with the property $W^2 = -1$ and expand the system of relations

$$\begin{aligned} &\imath^2 = J^2 = \mathcal{J}^2 = \Omega^2 = -1; \\ &Jz = z^* J; \\ &h(q\mathcal{J}) = (qh)\mathcal{J},\ (h\mathcal{J})q = (hq^*)\mathcal{J},\ (h\mathcal{J})(q\mathcal{J}) = -q^*h; \\ &o(pW) = (po)W,\ (oW)p = (op^*)W,\ (oW)(pW) = -p^*o; \end{aligned}$$

then we get a set of sedenions, therein $o, p \in \mathbb{O}$. These are hexadecimal numbers. Thus we have a sequence of embeddings of hypercomplex number systems

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{S} \subset \dots,$$

obtained one from the other using the Cayley–Dickson doubling procedure. Here $\mathbb{R}$ and $\mathbb{C}$ are fields of real and complex numbers, respectively, $\mathbb{H}$ is algebra (body in Russian) of quaternions, $\mathbb{O}$ is analytic Moufang loop of octonions, see for example [2], and $\mathbb{S}$ is loop of sedenions.

In Sections 1 and 2 we recall the basic concepts from the theory of quasigroups, and also point out the close relationship between quasigroups and finite automata. The application of FA in the theory of periodic groups was discussed in the article [3].

In Section 3 we discuss the reductant of the dihedral group D_n consisting of axial symmetries. Thus, we illustrate the main ideas proposed in [4]. Following Sabinin we call by a reductant an arbitrary subset of a group. Very close questions are discussed in the paper [5], where twisted subsets of the dihedral group are considered. The authors of the paper call a subset K of the group G twisted if $e \in K$ and $xy^{-1}x \in K$ for all $x, y \in K$. An example of a twisted subset in D_n is given by the involutions together with the unit.

In section 4 we show that the sphere S^2 is a reductant of the group $\mathrm{SU}(2, \mathbb{C})$.

1. Quasigroups

The set Q, considered together with some binary operation $*$, will be called a groupoid (or magma) $\langle Q, * \rangle$.

Definition 1. *The groupoid $\langle Q, * \rangle$ is called a quasigroup if for any elements a and b of the Q equations*

$$a * x = b,\ y * a = b$$

are always uniquely solvable.

The Definition 1 is equivalent to the invertibility of the $*$ operation on the right and left, see, for example, [6, Chapter I]. In the finite-dimensional case this means that each row and column of the Cayley table of the groupoid $\langle Q, *\rangle$ are permutations of elements from Q. The quasigroup $\langle Q, *\rangle$ with a unit will be called a unital quasigroup or a loop. Note that in the definition of a quasigroup, the binary operation, in general, does not require associativity. In other words, a finite quasigroup is a nonassociative Latin square.

By a reductant $\mathcal{R}$ of the group $\langle \mathcal{G}, \circ\rangle$ we will call any subset of it. We introduce, quite naturally, the law of composition on $\mathcal{R}$, see, for example, [4].

$$m_1 * m_2 = \operatorname{proj}(m_1 \circ m_2),$$

where m_1 and m_2 are arbitrary elements of $\mathcal{R}$, and $\operatorname{proj} : \mathcal{G} \to \mathcal{R}$ is a projector on a reductant. We know this concept from the works of Sabinin.

In this paper we show that if we consider the set of axial symmetries in the dihedral group D_n as $\mathcal{R}$, and arrange proj as

$$\operatorname{proj} : r_{2\pi k/n} \mapsto m_{\pi k/n},$$

$k = 0, \dots, n-1$, then $\mathcal{R}$ will be endowed with the unital quasigroup structure. Here r_φ is a counterclockwise rotation of φ, and m_ψ is an axial symmetry relative to a straight line with an angle of inclination ψ.

2. Quasigroup on a finite automaton

Let K_d be a complete graph, whose numbered vertices are represented by beads moving along the edges. The symbol 0 denotes an empty position where any of the beads can be moved. Fig. 1 shows the complete graph K_5.

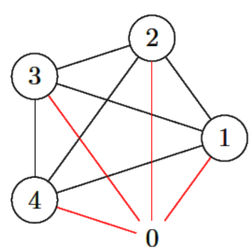

Fig. 1

Let $A = (Q, \mathcal{A}, \delta, q_0, F)$ be a deterministic finite automaton with the transition function $\delta : Q \times \mathcal{A} \to Q$, and $S = (Q, \mathcal{A}, \delta)$ be a semiatomaton see, for example, [7]. Consider a semiautomaton S whose state set Q coincides with the permutation group S_d of vertices $\{1, 2, \dots, d-1, 0\}$ of the graph K_d, and the input alphabet $\mathcal{A}$ is equal to the set $\{1, 2, \dots, d-1\}$ with $\delta : (s, j) \mapsto \sigma$, where the permutation $\sigma \in S_d$ is equal to s up to the permutation of the elements j and 0, $j = 1, 2, \dots, d-1$.

For $d = 3$ we get the graph, which can be depicted in the way shown in Fig. 2.

Here a, b, c, d, e and f are elements of the symmetric group S_3, where $a = (1, 2, 0)$, $b = (1, 0, 2)$, $c = (0, 1, 2)$, $d = (2, 1, 0)$, $e = (2, 0, 1)$, $f = (0, 2, 1)$. It is easy to see that the set $S = \{a, c, e\}$ is a normal subgroup in S_3. On the set of involutions $\mathcal{R} = \{b, d, f\} \subset S_3$ we can introduce a quasigroup multiplication by the rule

$$m_i * m_j = \operatorname{proj}(m_i m_j),$$

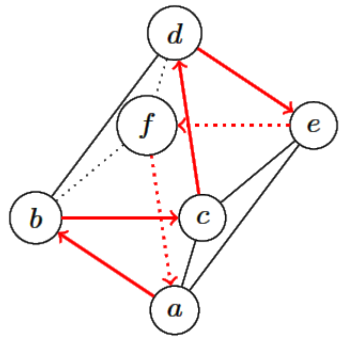

Fig. 2

where proj is the projection operator on $\mathcal{R}$ along the Hamiltonian cycle highlighted in red in Fig. 2, and $m_i m_j$ is the product in the group S_3. We get a nonassociative multiplication table (Tab. 2).

Table 2.

$*$	b	d	f
b	b	d	f
d	f	b	d
f	d	f	b

Indeed, different ways of placing parentheses lead to different results $(fd)b = d$, and $f(db) = b$. We have obtained a third-order quasigroup.

3. Reductant of a dihedral group

Consider a regular n-gon. The group of its symmetries (dihedral group) consists of a subgroup of rotations $S = \{r_{2\pi k/n}\}$ and axial symmetries $\mathcal{R} = \{m_{\pi k/n}\}$, $k = 0, 1, \ldots, n-1$ which are represented by matrices

$$\begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix},$$

and

$$\begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ \sin 2\pi k/n & -\cos 2\pi k/n \end{pmatrix}$$

respectively. It is easy to see that the projector $\mathrm{proj} : S \to \mathcal{R}$ just throws the minus sign in the second column from the first row to the second. More strictly, the projection is carried out by multiplying the rotation matrix by the Pauli matrix on the right:

$$\begin{aligned} \mathrm{proj}\,(r_{2\pi k/n}) &= \begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ \sin 2\pi k/n & -\cos 2\pi k/n \end{pmatrix}. \end{aligned}$$

Obviously, $\mathrm{proj}^2 = \mathrm{id}$, where id is an identical transformation. Multiplying the matrices of two axial symmetries, we get

$$\begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ \sin 2\pi k/n & -\cos 2\pi k/n \end{pmatrix} \begin{pmatrix} \cos 2\pi l/n & \sin 2\pi l/n \\ \sin 2\pi l/n & -\cos 2\pi l/n \end{pmatrix}$$
$$= \begin{pmatrix} \cos 2\pi(k-l)/n & -\sin 2\pi(k-l)/n \\ \sin 2\pi(k-l)/n & \cos 2\pi(k-l)/n \end{pmatrix}.$$

Let $0 < \alpha < \pi$, then $m_\alpha = m_{\pi+\alpha}$ and $m_{-\alpha} = m_{\pi-\alpha}$. If $k - l \geqslant 0$, then

$$m_{\pi k/n} * m_{\pi l/n} = m_{\pi(k-l)/n},$$

if $k - l < 0$, then

$$m_{\pi k/n} * m_{\pi l/n} = m_{\pi(n+k-l)/n}.$$

Let's write out the multiplication table of the axial symmetries of the dihedral group D_n (Tab. 3). For more elegance, instead of $m_{\pi k/n}$, $k \in \mathbb{Z} \cap [-n+1, n-1]$ we will simply write the k.

Table 3.

$*$	0	1	2	3	4	5	...	$n-1$
0	0	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	...	1
1	1	0	$n-1$	$n-2$	$n-3$	$n-4$	...	2
2	2	1	0	$n-1$	$n-2$	$n-3$	...	3
3	3	2	1	0	$n-1$	$n-2$	...	4
4	4	3	2	1	0	$n-1$	...	5
5	5	4	3	2	1	0	...	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$n-1$	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	...	0

4. S^2 sphere

Calculate the product of the points $w = (w_1, w_2, w_3)$ and $v = (v_1, v_2, v_3)$ from sphere $S^2 \subset \mathrm{SU}(2, \mathbb{C})$ represented by complex matrices

$$\begin{pmatrix} \imath w_1 & -w_2 - \imath w_3 \\ w_2 - \imath w_3 & -\imath w_1 \end{pmatrix} \begin{pmatrix} \imath v_1 & -v_2 - \imath v_3 \\ v_2 - \imath v_3 & -\imath v_1 \end{pmatrix}$$
$$= \begin{pmatrix} -\langle w, v\rangle + \imath(w_2 v_3 - w_3 v_2) & -(w_3 v_1 - w_1 v_3 + \imath(w_1 v_2 - w_2 v_1)) \\ w_3 v_1 - w_1 v_3 - \imath(w_1 v_2 - w_2 v_1) & -\langle w, v\rangle - \imath(w_2 v_3 - w_3 v_2) \end{pmatrix}$$
$$= \begin{pmatrix} \xi_0 + \imath\xi_1 & -(\xi_2 + \imath\xi_3) \\ \xi_2 - \imath\xi_3 & \xi_0 - \imath\xi_1 \end{pmatrix}$$
$$= \xi,$$

where the norm of a vector $\|(\xi_0, \xi_1, \xi_2, \xi_3))\|_{\mathbb{R}^4}$ coincides with the determinant of the matrix ξ and is equal to one ($\|\xi\| = \det \xi = 1$).

Now we project $\xi \in \mathrm{SU}(2,\mathbb{C})$ back onto the reductant. To do this, we will twist the ξ as follows:

$$\begin{aligned}
&\begin{pmatrix} a+\imath b & 0 \\ 0 & a-\imath b \end{pmatrix} \begin{pmatrix} \xi_0+\imath\xi_1 & -(\xi_2+\imath\xi_3) \\ \xi_2-\imath\xi_3 & \xi_0-\imath\xi_1 \end{pmatrix} \\
&= \begin{pmatrix} a\xi_0-b\xi_1+\imath(a\xi_1+b\xi_0) & -(a\xi_2-b\xi_3+\imath(a\xi_3+b\xi_2)) \\ a\xi_2-b\xi_3-\imath(a\xi_3+b\xi_2) & a\xi_0-b\xi_1+\imath(a\xi_1+b\xi_0) \end{pmatrix} \\
&= \begin{pmatrix} \eta_0+\imath\eta_1 & -(\eta_2+\imath\eta_3) \\ \eta_2-\imath\eta_3 & \eta_0-\imath\eta_1 \end{pmatrix},
\end{aligned}$$

where $a^2+b^2=1$. We are looking for a and b such that $\eta_0=0$ $(a\xi_0=b\xi_1)$ and

$$\begin{aligned}
1 &= \eta_1^2+\eta_2^2+\eta_3^2 \\
&= (a\xi_1+b\xi_0)^2+(a\xi_2-b\xi_3)^2+(a\xi_3+b\xi_2)^2 \\
&= a^2(1-\xi_0^2)+b^2(1+\xi_1^2).
\end{aligned}$$

Some location configurations of an arbitrary ellipse and circle $a^2+b^2=1$ are shown in Fig. 3.

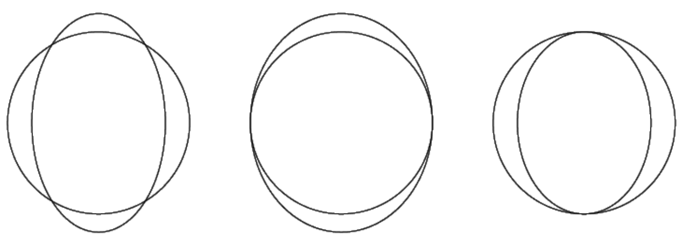

Fig. 3

The solution of the system

$$\begin{cases} a^2(1-\xi_0^2)+b^2(1+\xi_1^2)=1, \\ a^2+b^2=1 \end{cases}$$

have the form

$$\begin{aligned}
&b=-\sqrt{1-a^2}, \quad a^2-1\neq 0, \quad \xi_1=-\frac{a\xi_0}{\sqrt{1-a^2}}; \\
&b=-\sqrt{1-a^2}, \quad a^2-1\neq 0, \quad \xi_1=\frac{a\xi_0}{\sqrt{1-a^2}}; \\
&b=\sqrt{1-a^2}, \quad a^2-1\neq 0, \quad \xi_1=-\frac{a\xi_0}{\sqrt{1-a^2}}; \\
&b=\sqrt{1-a^2}, \quad a^2-1\neq 0, \quad \xi_1=\frac{a\xi_0}{\sqrt{1-a^2}}; \\
&a=-1, \quad b=0, \quad \xi_0=0; \\
&a=1, \quad b=0, \quad \xi_0=0.
\end{aligned}$$

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О редуктантах двух групп

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Аннотация. В работе рассматриваются редуктант диэдральной группы D_n, состоящий из множества осевых симметрий, и сфера S^2 как редуктант группы $\mathrm{SU}(2,\mathbb{C})$ (группы единичных кватернионов). Введя сабининское умножение на редуктанте из D_n, мы получим квазигруппу с единицей.

Ключевые слова: редуктанты групп, квазигруппы.

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On Estimation of the Convergence Rate to Invariant Measures in Markov Branching Processes with Possibly Infinite Variance and Immigration

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Abstract. The continuous-time Markov Branching Process with Immigration is discussed in the paper. A critical case wherein the second moment of offspring law and the first moment of immigration law are possibly infinite is considered. Assuming that the non-linear parts of the appropriate generating functions are regularly varying in the sense of Karamata, theorems on convergence of transition functions of the process to invariant measures are proved. The rate of convergence is determined provided that slowly varying factors are with remainder.

Keywords: Markov branching process, generating functions, immigration, transition functions, slowly varying function, invariant measures, convergence rate.

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1. Introduction and preliminaries

The discussion of the population growth model called the continuous-time Markov Branching Process with Immigration (MBPI) which was considered in [5] is continued in this paper. Recall that this process has simple physical interpretation: the population size changes not only as a result of reproduction and disappearance of existing individuals but also as a result of the random influx of "extraneous" individuals of the same type from the outside. Namely, the process develops according to the following scheme. Each individual existing at time $t \in \mathcal{T} := [0, +\infty)$ independently of his history and of each other for a small time interval $(t, t + \varepsilon)$ is transformed into $j \in \mathbb{N}_0 \backslash \{1\}$ individuals with probability $a_j\varepsilon + o(\varepsilon)$, and with probability $1 + a_1\varepsilon + o(\varepsilon)$ stays to live or makes evenly one descendant (as $\varepsilon \downarrow 0$). Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N}$ is the set of natural numbers, and $\{a_j\}$ are intensities of individual transformation, $a_j \geqslant 0$ for $j \in \mathbb{N}_0 \backslash \{1\}$ and $0 < a_0 < -a_1 = \sum_{j \in \mathbb{N}_0 \backslash \{1\}} a_j < \infty$. Independently of these for each time interval $j \in \mathbb{N}$ new individuals enter the population with probability $b_j\varepsilon + o(\varepsilon)$, and immigration does not occur with probability $1 + b_0\varepsilon + o(\varepsilon)$. Immigration intensities $b_j \geqslant 0$ for $j \in \mathbb{N}$ and $0 < -b_0 = \sum_{j \in \mathbb{N}} b_j < \infty$. Newly arrived individuals undergo transformation in accordance with the reproduction law generated by intensities $\{a_j\}$; see [11, p. 217]. Thus, the process under consideration is completely determined by infinitesimal generating functions(GFs)

$$f(s) := \sum_{j \in \mathbb{N}_0} a_j s^j \quad \text{and} \quad g(s) := \sum_{j \in \mathbb{N}_0} b_j s^j \quad \text{for} \quad s \in [0, 1).$$

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Let us denote the population size at the time $t \in \mathcal{T}$ in MBPI by $X(t)$. This is homogeneous continuous-time Markov chain with state space $\mathcal{S} \subset \mathbb{N}_0$ and transition functions

$$p_{ij}(t) := \mathbb{P}_i\bigl\{X(t) = j\bigr\} = \mathbb{P}\left\{X(t+\tau) = j \,\middle|\, X(\tau) = i\right\}$$

for all $i, j \in \mathcal{S}$ and $\tau, t \in \mathcal{T}$.

Only critical case is considered in the paper, i.e., $f'(1-) = \sum_{j\in\mathbb{N}} j a_j = 0$, and limit behaviours of transition functions $p_{ij}(t)$ as $t \to \infty$ is observed. Pakes [9] was one of the first who studied invariant measures for MBPI with finite variance and found an integral form of GF of invariant measures. He has proved that limits $\pi_j := \lim_{t\to\infty} t^{\lambda} p_{ij}(t)$ exist independently on j, iff $\sum_{j\in\mathbb{N}} a_j j^2 \ln j < \infty$ and $\sum_{j\in\mathbb{N}} b_j j \ln j < \infty$, where $\lambda = 2g'(1-)\big/ f''(1-)$, besides the set $\{\pi_j, j \in \mathcal{S}\}$ presents an invariant measure for MBPI. The invariant measure of MBPI can also be constructed by the strong ratio limit property of transition functions but slightly different [7]. Namely, the set of positive numbers $\bigl\{\upsilon_j := \lim_{t\to\infty} p_{0j}(t)/p_{00}(t)\bigr\}$ is an invariant measure. Moreover one can see a close relation between the sets $\{\pi_j, j \in \mathcal{S}\}$ and $\{\upsilon_j, j \in \mathcal{S}\}$, and their GFs $\pi(s) = \sum_{j\in\mathcal{S}} \pi_j s^j$ and $\mathcal{U}(s) = \sum_{j\in\mathcal{S}} \upsilon_j s^j$. In fact, they are really only different versions of the same limit law. So, it is easy to see that $\mathcal{U}(s) = \pi(s)/\pi(0)$, and this is consistent with uniqueness, up to a multiplicative constant, of the invariant measure of MBPI.

An estimation of the rate of convergence to invariant measures is of exceptional interest. The rate of convergence of $t^{\lambda} p_{ij}(t)$ to π_j for all $i, j \in \mathcal{S}$ was studied under the condition $\max\bigl\{f'''(1-), g''(1-)\bigr\} < \infty$ [5]. It was found that the convergence rate is $\mathcal{O}\bigl(\ln t/t\bigr)$ as $t \to \infty$.

Throughout the paper, the following *Basic assumptions* for $f(s)$ and $g(s)$ are used

$$f(s) = (1-s)^{1+\nu} \mathcal{L}\left(\frac{1}{1-s}\right), \qquad [f_\nu]$$

and

$$g(s) = -(1-s)^{\delta} \ell\left(\frac{1}{1-s}\right) \qquad [g_\delta]$$

for all $s \in [0,1)$, where $0 < \nu, \delta < 1$ and $\mathcal{L}(\cdot)$, $\ell(\cdot)$ are *slowly varying at infinity* ($\mathbf{SV}_\infty$) in the sense of Karamata (see, for instance, [2] and [10]). Basic assumptions imply that the offspring distribution belongs to the domain of attraction of the $(1+\nu)$-stable law, and the immigration distribution belongs to the domain of attraction of the δ-stable law. In the critical case assumption $[f_\nu]$ implies that $2b := f''(1-) = \infty$. If $b < \infty$ then representation $[f_\nu]$ holds with $\nu = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$. Similarly, GF $g(s)$ of the form $[g_\delta]$ generates the immigration law with the δ-order moment. However, if $g'(1-) < \infty$ then assumption $[g_\delta]$ is fulfilled with $\delta = 1$ and $\ell(t) \to g'(1-)$ as $t \to \infty$.

An additional requirement for $\mathcal{L}(x)$ and $\ell(x)$ is introduced:

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = 1 + \mathcal{O}\bigl(\alpha(x)\bigr) \quad \text{as} \quad x \to \infty \qquad [\mathcal{L}_\nu]$$

for each $\lambda > 0$, where $\alpha(x)$ is known positive decreasing function so that $\alpha(x) \to 0$ as $x \to \infty$. In this case $\mathcal{L}(x)$ is called SV_∞ with remainder $\mathcal{O}\bigl(\alpha(x)\bigr)$ (see [2, p. 185, condition SR1]). When employing condition $[\mathcal{L}_\nu]$ it is assumed that

$$\alpha(x) = \mathcal{O}\left(\frac{\mathcal{L}(x)}{x^{\nu}}\right) \quad \text{as} \quad x \to \infty.$$

Similarly, the condition

$$\frac{\ell(\lambda x)}{\ell(x)} = 1 + \mathcal{O}\big(\beta(x)\big) \quad \text{as} \quad x \to \infty \qquad [\ell_\delta]$$

is also allowed for each $\lambda > 0$, where

$$\beta(x) = \mathcal{O}\left(\frac{\ell(x)}{x^\delta}\right) \quad \text{as} \quad x \to \infty.$$

It was shown that the asymptotes of the transition functions depend on the sign of the parameter $\gamma := \delta - \nu$ [3]. In addition, the limit functions $U(s) := \lim_{t\to\infty} \mathcal{P}(t;s)$ for $\gamma > 0$ and $\pi(s) := \lim_{t\to\infty} e^{T(t)}\mathcal{P}(t;s)$ for $\gamma < 0$ and for some $T(t)$ were found.

In this paper the rate of convergence is determined provided that conditions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold.

The rest of this paper is organized as follows. Section 2. contains main results. Auxiliary statements that are used in the proof of theorems are considered in Section 3.. Proof of main results is presented in Section 4..

2. Main results

Let us consider GF $\mathcal{P}_i(t;s) := \sum_{j\in\mathcal{S}} p_{ij}(t)s^j$. It is not difficult to see that (see [9])

$$\mathcal{P}_i(t;s) = \Big(F(t;s)\Big)^i \exp\left\{\int_0^t g\big(F(u;s)\big)du\right\}, \tag{1}$$

where $F(t;s)$ is GF of Markov Branching Process initiated by single individual without immigration. Since $F(t;s) \to 1$ as $t \to \infty$ uniformly in $s \in [0,d]$, $d < 1$ (see Lemma 1 below), it is sufficient to consider $\mathcal{P}(t;s) := \mathcal{P}_0(t;s)$. Then taking into account Basic assumptions and the Kolmogorov backward equation $\partial F/\partial t = f(F)$, it follows from (1) that

$$\mathcal{P}(t;s) = \exp\left\{\int_s^{F(t;s)} \frac{g(u)}{f(u)}\,du\right\}. \tag{2}$$

Taking into account Basic assumptions, the integrand is

$$\frac{g(u)}{f(u)} = -(1-u)^{\gamma-1}\, \mathsf{L}\left(\frac{1}{1-u}\right), \tag{3}$$

where $\gamma := \delta - \nu$ and

$$\mathsf{L}(t) := \frac{\ell(t)}{\mathcal{L}(t)}.$$

State space $\mathcal{S}$ can be classified in accordance with the sign of γ. By virtue of (3), integral $\int_s^1 [g(u)/f(u)]\,du$ converges if $\gamma > 0$, and diverges if $\gamma < 0$. It was shown that $\mathcal{S}$ is *positive-recurrent* if $\gamma > 0$, and it is *transient* if $\gamma < 0$. The special case $\gamma = 0$ implies that $g(s) = f'(s)$ and $\mathsf{L}(t) \to 1+\nu$ as $t \to \infty$. It is another population process called *Markov Q-process* (see [4], [6], [1, pp. 56–58] and [8] for the discrete-time case).

Main results are formulated only for the case $\gamma \neq 0$ in the following two theorems. Let

$$\tau(t) := \frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)} \qquad \text{and} \qquad T(t) := \big(\tau(t)\big)^{|\gamma|},$$

where $\mathcal{N}(x)$ is SV_∞ defined in Lemma 1 below.

Theorem 2.1. *Let $\gamma > 0$. Then $\mathcal{P}(t;s)$ converges to the function $U(s) = \exp\left\{\int\limits_s^1 \left[g(u)/f(u)\right] du\right\}$ for $s \in [0,1)$, and its power series expansion $U(s) = \sum\limits_{j\in\mathcal{S}} u_j s^j$ generates an invariant distribution $\{u_j, j \in \mathcal{S}\}$ for MBPI. The convergence is uniform over compact subsets of $[0,1)$. In addition, if assumptions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold then*

$$\mathcal{P}(t;s) = U(s)\left(1 + \Delta(t;s)\mathcal{K}\big(\tau(t)\big)\right), \tag{4}$$

where $\mathcal{K}(x) = \mathcal{L}^{-\delta/\nu}(x)\ell(x)$, function $\mathcal{N}(x)$ is SV_∞ defined in (13) below and

$$\Delta(t;s) = \frac{1}{\gamma}\frac{1}{\big(\lambda(t;s)\big)^{\gamma/\nu}} + \mathcal{O}\left(\frac{\ln\big[\Lambda(1-s)\lambda(t;s)\big]}{\big(\lambda(t;s)\big)^{\delta/\nu}}\right) \quad as \quad t \to \infty,$$

where $\lambda(t;s) = \nu t + \Lambda^{-1}(1-s)$ and $\Lambda(y) = y^\nu \mathcal{L}(1/y)$. The transition functions are

$$p_{ij}(t) = u_j\left(1 + \mathcal{O}\left(\frac{K(t)}{t^{\gamma/\nu}}\right)\right) \quad as \quad t \to \infty, \tag{5}$$

where $K(t)$ is SV_∞.

Another asymptotic property comes out for $\mathcal{P}(t;s)$ when $\gamma < 0$. Taking into account Basic assumptions, one can easily verify that

$$-\frac{\ln p_{00}(t)}{T(t)} \sim \frac{1}{|\gamma|}\mathsf{L}\big(\tau(t)\big) \quad \text{as} \quad t \to \infty.$$

This asymptotic relation shows that $\big(T(t)\big)^{-1}\ln p_{00}(t)$ is asymptotically SV_∞. Then one should consider the limit of the function $e^{T(t)}\mathcal{P}(t;s)$ as $t \to \infty$. First one needs to consider SV_∞ property of $\mathsf{L}(t)$. In accordance with the slowly varying theory, functions $\ell(\cdot)$ and $\mathcal{L}(\cdot)$ are positive. Then by virtue of [2, p. 185, Theorem 3.12.2 (SR1)], one can obtain the following propositions:

- $[\mathcal{L}_\nu] \iff \mathcal{L}(x) = C_{\mathcal{L}} + \mathcal{O}\big(\alpha(x)\big)$ as $t \to \infty$, $[C_{\mathcal{L}}]$
- $[\ell_\delta] \iff \ell(x) = C_\ell + \mathcal{O}\big(\beta(x)\big)$ as $t \to \infty$, $[C_\ell]$

where $C_{\mathcal{L}}, C_\ell$ are positive constants and functions $\alpha(x), \beta(x)$ are in $[\mathcal{L}_\nu]$ and $[\ell_\delta]$. Then

$$\mathsf{L}(t) = \frac{\ell(t)}{\mathcal{L}(t)} = C_{\mathsf{L}} + \mathcal{O}\left(\frac{\ell(t)}{t^\delta}\right) \quad \text{as} \quad t \to \infty, \tag{6}$$

since $\delta < \nu$, where $C_{\mathsf{L}} = C_\ell / C_{\mathcal{L}}$. This requirement for $\mathsf{L}(t)$ is quite possible. Especially, one can obtain an "excellent result" if $C_{\mathsf{L}} = |\gamma|$ is chosen. The following explicit form of $\pi(s) = \lim_{t\to\infty} e^{T(t)}\mathcal{P}(t;s)$ was found in [3]:

$$\pi(s) = \exp\left\{\frac{1}{(1-s)^{|\gamma|}} + \int_s^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1-u)^{1+|\gamma|}}\right] du\right\}. \tag{7}$$

Now the convergence rate of $e^{T(t)}\mathcal{P}(t;s)$ to $\pi(s)$ is determined in the following theorem.

Theorem 2.2. *Let $\gamma < 0$ and $C_{\mathsf{L}} = |\gamma|$ in (6). If $\mu := 2\delta - \nu > 0$ then*

$$e^{T(t)}\mathcal{P}(t;s) = \pi(s)\big(1 + \rho(t;s)\big), \tag{8}$$

where $\rho(t;s) \to 0$ as $t \to \infty$ uniformly in $s \in [0,r]$, $r < 1$, and the limiting GF $\pi(s)$ can be expressed in the form given in (7)*. In addition, if assumptions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold then*

$$\rho(t;s) = \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^{\mu}}\right) \quad \text{as} \quad t \to \infty \tag{9}$$

uniformly in $s \in [0,r]$, $r < 1$. Denoting the power series expansion of $\pi(s)$ by $\sum_{j \in \mathcal{S}} \pi_j s^j$, transition functions have the form

$$p_{ij}(t) = \pi_j \left(1 + \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^{\mu}}\right)\right) \quad \text{as} \quad t \to \infty, \tag{10}$$

and $\{\pi_j, j \in \mathcal{S}\}$ is an invariant measure for MBPI.

Remark 1. *The form of limiting GF $\pi(s)$ given in the first part of Theorem 2.2 is compatible with the results presented in [9] and [5] where the case $\max\{f''(1-), g'(1-)\} < \infty$ was considered. Thus, this theorem essentially strengthens last-mentioned results.*

Remark 2. *The conditions $C_L = |\gamma|$ and $\mu > 0$ in Theorem 2.2 are essential because they ensure the convergence of the integral in* (7)*. In fact, due to Basic assumptions and* (6) *the majorizing function for the integrand is $(1-u)^{\mu-1}$. Then function*

$$\mathcal{B}(s) := \exp\left\{\int_s^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1-u)^{1+|\gamma|}}\right] du\right\} \tag{11}$$

is bounded for $s \in [0,1]$.

The following result is a consequence of Theorem 2.2.

Corollary 1. *Under the conditions of Theorem 2.2*

$$e^{T(t)} p_{00}(t) = \mathcal{B}(0)\left(1 + \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^{\mu}}\right)\right) \quad \text{as} \quad t \to \infty,$$

where function $\mathcal{B}(s)$ is defined in (11)*.*

Remark 3. *Further reasoning imply that functions $\mathcal{L}(x)$ and $\ell(x)$ can be omitted in estimations of error terms of asymptotic relations in given above Theorems. Taking into account assertions $[C_{\mathcal{L}}]$ and $[C_\ell]$, these functions are asymptotically constant.*

3. Auxiliaries

In this section, some auxiliary assertions are provided. They are essential for the proof of theorems.

First, the asymptotic representation of GF of Markov branching processes $Z(t)$ without immigration is considered. Let $F(t;s) = \mathbb{E}\left[s^{Z(t)} \middle| Z(0) = 1\right]$ be GF of the process initiated by single individual. Let $R(t;s) := 1 - F(t;s)$. The following result called the Basic lemma of the theory of critical Markov branching processes [4]. It is presented in slightly different form below.

Lemma 1. *If condition $[f_\nu]$ holds then*

$$\frac{1}{R(t;s)} = \frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)} \cdot \left[1 + \frac{\mathcal{M}(s)}{t}\right]^{1/\nu} \tag{12}$$

for all $s \in [0,1)$, *where* $\mathcal{N}(x)$ *is* SV_∞ *such that*

$$\mathcal{N}^{\nu}(t) \cdot \mathcal{L}\left(\frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)}\right) \longrightarrow 1 \quad as \quad t \to \infty, \tag{13}$$

and $\mathcal{M}(s)$ *is GF of invariant measures of MBP that has the form*

$$\mathcal{M}(s) = \int\limits_{1}^{1/(1-s)} \frac{dx}{x^{1-\nu}\mathcal{L}(x)} .$$

Let us introduce the following function

$$\Lambda(y) := y^{\nu}\mathcal{L}\left(\frac{1}{y}\right) = \frac{f(1-y)}{y}$$

for $y \in (0,1]$. Let us note that function $y\Lambda(y)$ is positive, tends to zero and it has the monotone derivative so that $y\Lambda'(y)/\Lambda(y) \to \nu$ as $y \downarrow 0$ (see [2, p. 401]). Then it is natural to write

$$\frac{y\Lambda'(y)}{\Lambda(y)} = \nu + \delta(y), \tag{14}$$

where $\delta(y)$ is continuous and $\delta(y) \to 0$ as $y \downarrow 0$. Since $\Lambda(1) = \mathcal{L}(1) = a_0$ it follows from (14) that

$$\Lambda(y) = a_0 y^{\nu} \exp \int_1^y \frac{\delta(u)}{u} du.$$

Therefore

$$\mathcal{L}\left(\frac{1}{y}\right) = a_0 \exp \int_1^y \frac{\delta(u)}{u} du.$$

Substituting $u = 1/t$ in last integrand, one can obtain

$$\mathcal{L}(x) = a_0 \exp \int_1^x \frac{\varepsilon(t)}{t} dt,$$

where $\varepsilon(t) = -\delta(1/t)$ and $\varepsilon(t) \to 0$ as $t \to \infty$. Considering the last equation together with $[\mathcal{L}_\nu]$, one can obtain

$$\int_x^{\lambda x} \frac{\varepsilon(t)}{t} dt = \ln\left[1 + \mathcal{O}\big(\alpha(x)\big)\right] = \mathcal{O}\big(\alpha(x)\big) \quad as \quad x \to \infty$$

for each $\lambda > 0$. Applying the mean value theorem to the left-hand side of the last equality, we have that $\varepsilon(x) = \mathcal{O}(\alpha(x))$. Then condition $[\mathcal{L}_\nu]$ gives

$$\delta(y) = \mathcal{O}\left(\alpha\left(\frac{1}{y}\right)\right) \quad as \quad y \downarrow 0. \tag{15}$$

The following result is a modification of Lemma 1 and it is required in the subsequent discussions.

Lemma 2. *Let assumptions* $[f_\nu]$ *and* $[\mathcal{L}_\nu]$ *hold. Then*

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \mathcal{O}\big(\ln \nu(t;s)\big) \quad as \quad t \to \infty, \tag{16}$$

where $\nu(t;s) = \Lambda(1-s)\nu t + 1$.

Proof. One can write from (14) that

$$\frac{R\Lambda'(R)}{\Lambda(R)} = \nu + \delta(R) \tag{17}$$

since $R := R(t;s) \to 0$ as $t \to \infty$. Using the backward Kolmogorov equation $\partial F/\partial t = f(F)$ and considering representation $[f_\nu]$, relation (17) becomes

$$\frac{d\Lambda(R)}{dt} = -\frac{\Lambda(R)}{R} f(1-R)\big(\nu + \delta(R)\big) = -\Lambda^2(R)\big(\nu + \delta(R)\big).$$

Therefore

$$d\left[\frac{1}{\Lambda(R)} - \nu t\right] = \delta(R)\,dt. \tag{18}$$

Integrating (18) over $[0,t)$, the following equation is obtained

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \int_0^t \delta(R(u;s))du, \tag{19}$$

where $\delta(y)$ is in (14). Now one should take integral in (19). Considering (15),one can write

$$\int_0^t \delta(R(u;s))du = \int_0^t \mathcal{O}\big(\Lambda(R(u;s))\big)du. \tag{20}$$

One should mention that due to (12) $R(t;s) \to 0$ as $t \to \infty$ uniformly in $s \in [0,1)$. Therefore, since $\Lambda(y) \to 0$ as $y \downarrow 0$, the integral in the right-hand side of (20) is $o(t)$ as $t \to \infty$. Hence

$$\Lambda(R(t;s)) = \frac{1}{\lambda(t;s)} + o\left(\frac{1}{\lambda(t;s)}\right) \quad as \quad t \to \infty,$$

where $\lambda(t;s) = \nu t + \Lambda^{-1}(1-s)$. Therefore

$$\int_0^t \mathcal{O}\big(\Lambda(R(u;s))\big)du = \mathcal{O}\left(\int_0^t \Lambda(R(u;s))du\right) = \mathcal{O}\big(\ln \nu(t;s)\big) \quad as \quad t \to \infty.$$

Together with (19) and (20) this gives relation (16). □

Lemma 3. *Let $L(t)$ be SV_∞ with remainder $\varrho(t)$. Then for $\sigma > 0$*

$$\int_t^\infty y^{-(1+\sigma)} L(y)dy = \frac{1}{\sigma}\frac{1}{t^\sigma} L(t)\left(1 + \mathcal{O}\big(\varrho(t)\big)\right) \quad as \quad t \to \infty. \tag{21}$$

Proof. Undoubtedly $\int_1^\infty u^{-(1+\sigma)}du = 1/\sigma$. Considering this fact and making the substitution $y := ut$ in the integrand of (21), one can write

$$\int_t^\infty y^{-(1+\sigma)} L(y)dy = \frac{1}{\sigma}\frac{L(t)}{t^\sigma}\left[1 + \sigma \int_1^\infty \left[\frac{L(ut)}{L(t)} - 1\right] u^{-(1+\sigma)}du\right]. \tag{22}$$

By definition of SV_∞-function with remainder, the expression in brackets of the integrand on the right-hand side of (22) tends to 0 as $t \to \infty$ uniformly in $u > 1$ (by Uniform Convergence Theorem for SV_∞-functions [2, Theorem 1.5.2]) with the rate $\mathcal{O}\big(\varrho(t)\big)$. Thus relation (21) is obtained.

The Lemma is proved. □

Lemma 4. *Let conditions* $[\mathcal{L}_\nu]$ *and* $[\ell_\delta]$ *hold and* $\gamma > 0$. *Then*

$$\int_x^1 \frac{g(u)}{f(u)}\,du = \frac{1}{\gamma}\frac{g(x)}{\Lambda(1-x)}\big(1+\mathcal{O}\big(\Lambda(1-x)\big)\big) \quad as \;\; x \uparrow 1. \tag{23}$$

Proof. It follows from Basic assumption that

$$\mathcal{I}(x) := \int\limits_x^1 \frac{g(u)}{f(u)}\,du = -\int\limits_{1/(1-x)}^{\infty} y^{-(1+\gamma)} \mathsf{L}(y)dy, \tag{24}$$

where $\mathsf{L}(t) = \ell(t)/\mathcal{L}(t)$ as before. One can easily show that

$$\frac{\mathsf{L}(ut)}{\mathsf{L}(t)} - 1 = \mathcal{O}\left(\frac{\mathcal{L}(t)}{t^\nu}\right) \quad as \;\; t \to \infty$$

uniformly in $u > 0$. Considering the right-hand side of (24), one can directly use (21) with $t = 1/(1-x)$ and $r(t) = \mathcal{O}\left(\mathcal{L}(t)/t^\nu\right)$. Then

$$\mathcal{I}(x) = -\frac{1}{\gamma}\frac{\mathsf{L}(t)}{t^\gamma}\left[1+\mathcal{O}\left(\frac{\mathcal{L}(t)}{t^\nu}\right)\right] \quad as \;\; t \to \infty.$$

Now returning to primary designations, relation (23) is obtained.

The Lemma is proved. □

4. Proof of Theorems

In this final section the Main results are consistently proved.

Proof of Theorem 2.1. Let us rewrite (2) as follows

$$\mathcal{P}(t;s) = U(s)\exp\left\{\int\limits_1^{F(t;s)} \frac{g(u)}{f(u)}\,du\right\}, \tag{25}$$

where

$$U(s) = \exp\left\{\int_s^1 \frac{g(u)}{f(u)}\,du\right\}. \tag{26}$$

Considering (3), the integral in (25) converges for $s \in [0,1)$ and becomes 0 as $t \to \infty$. Therefore $\mathcal{P}(t;s)$ converges to $U(s)$ as $t \to \infty$ uniformly over compact subsets. Now, using the functional equation $F(t+\tau;s) = F\big(t;F(\tau;s)\big)$ (see [9, p. 134]), it follows that

$$\begin{aligned}\mathcal{P}(t+\tau;s) &= \mathcal{P}(\tau;s)\cdot\exp\left\{\int_\tau^{t+\tau} g\left(F(u;s)\right)du\right\} \\ &= \mathcal{P}(\tau;s)\cdot\exp\left\{\int_0^t g\left(F\big(u;F(\tau;s)\big)\right)du\right\} = \mathcal{P}(\tau;s)\cdot\mathcal{P}\big(t;F(\tau;s)\big).\end{aligned}$$

Taking limit as $t \to \infty$, one can obtain the following Schröder type functional equation

$$U\big(F(\tau;s)\big) = \frac{1}{\mathcal{P}(\tau;s)}U(s) \quad \text{for any} \;\; \tau \in \mathcal{T}. \tag{27}$$

Writing the power series expansion $U(s) = \sum_{j\in\mathcal{S}} u_j s^j$, equation (27) has an invariant property $u_j = \sum_{i\in\mathcal{S}} u_i p_{ij}(\tau)$. Obviously $U(1-) = 1$ and hence the function in (26) generates an invariant distribution $\{u_j, j \in \mathcal{S}\}$ for MBPI.

Let us prove now of of (4). Considering (25) and using (23), one can obtain

$$\mathcal{P}(t;s) = U(s)\exp\Big\{-I(t;s)\Big\}, \tag{28}$$

where

$$I(t;s) = \frac{1}{\gamma}\frac{g\big(F(t;s)\big)}{\Lambda\big(R(t;s)\big)}\Big(1+\mathcal{O}\Big(\Lambda(R(t;s))\Big)\Big) \quad \text{as} \quad t\to\infty. \tag{29}$$

Next, let us use the asymptotic expansion of $R(t;s)$. Relation (16) implies

$$\frac{1}{\Lambda\left(R(t;s)\right)} = \lambda(t;s)\left(1+\mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t;s)\right]}{\lambda(t;s)}\right)\right) \quad \text{as} \quad t\to\infty, \tag{30}$$

and therefore

$$R(t;s) = \frac{\mathcal{N}(t;s)}{\left(\lambda(t;s)\right)^{1/\nu}}\left(1+\mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t;s)\right]}{\lambda(t;s)}\right)\right) \quad \text{as} \quad t\to\infty, \tag{31}$$

where $\lambda(t;s) = \nu t + \Lambda^{-1}(1-s)$ and $\mathcal{N}(t;s) = \mathcal{L}^{-1/\nu}\big(1/R(t;s)\big)$. Let us note that $g(s)$ has the form of $[g_\delta]$. Using (30) and (31), one can obtain

$$\frac{g\big(F(t;s)\big)}{\Lambda\big(R(t;s)\big)} = -\frac{\mathcal{N}^\delta(t;s)}{\left(\lambda(t;s)\right)^{\gamma/\nu}}\ell\Big(\frac{1}{R(t;s)}\Big)\left(1+\mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t;s)\right]}{\lambda(t;s)}\right)\right) \tag{32}$$

as $t\to\infty$. It is easy to verify that the function $\mathcal{N}(t;s)$ is asymptotically equivalent to the SV_∞-function $\mathcal{N}(t)$ defined in Lemma 1.

Asymptotic formula (4) now follows from a combination of (28), (29) and (32). Equation (5) follows from the continuity theorem for power series.

The Theorem is proved. □

Proof of Theorem 2.2. Let us write

$$\begin{aligned} e^{T(t)}\mathcal{P}(t;s) &= \exp\left\{\big(\tau(t)\big)^{|\gamma|} + \int_0^t g\left(F(u;s)\right)du\right\} = \\ &= \exp\left\{\Delta(t;s) + \big(\tau(t;s)\big)^{|\gamma|} + \int_s^{F(t;s)} \frac{g(x)}{f(x)}\,dx\right\}, \end{aligned} \tag{33}$$

where $\Delta(t;s) = \big(\tau(t)\big)^{|\gamma|} - \big(\tau(t;s)\big)^{|\gamma|}$ and $\tau(t;s) = R^{-1}(t;s)$. Standard integration yields

$$\big(\tau(t;s)\big)^{|\gamma|} = \frac{1}{(1-s)^{|\gamma|}} + \int_s^{F(t;s)} \frac{|\gamma|}{(1-u)^{1+|\gamma|}}du.$$

Therefore, relation (33) can be written as follows

$$e^{T(t)}\mathcal{P}(t;s) = \pi(s)\cdot\exp\left\{\Delta(t;s) - \int_{F(t;s)}^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1-u)^{1+|\gamma|}}\right]du\right\}, \tag{34}$$

where $\pi(s)$ has the form of (7). An exponential factor in (34) defines the convergence rate $\rho(t;s)$ in (8). Let us first evaluate $\Delta(t;s)$ as $t \to \infty$. According to Lemma 1 $\mathcal{M}(0) = 0$ and,therefore, $\tau(t) = \tau(t;0)$. Hence, asymptotic representation (12) gives

$$\begin{aligned}\Delta(t;s) &= \big(\tau(t)\big)^{|\gamma|}\left[1-\left(1+\frac{\mathcal{M}(s)}{t}\right)^{|\gamma|/\nu}\right]\\ &\sim -|\gamma|\big(\tau(t)\big)^{|\gamma|}\frac{\mathcal{M}(s)}{\nu t} = -|\gamma|\frac{\mathcal{M}(s)}{(\nu t)^{\delta/\nu}\mathcal{N}^{|\gamma|}(t)} \quad \text{as} \quad t \to \infty.\end{aligned}$$

On the other hand $\mathcal{M}(s)$ is bounded for $s \in [0, r]$, $r < 1$. Thus

$$\Delta(t;s) = \mathcal{O}\left(\frac{\mathcal{L}_\gamma(t)}{t^{\delta/\nu}}\right) \quad \text{as} \quad t \to \infty, \tag{35}$$

uniformly in $s \in [0, r]$, $r < 1$, where $\mathcal{L}_\gamma(t) = \mathcal{N}^{-|\gamma|}(t)$.

Let us observe the integral in (34). Taking into account relations (3) and (6), the integrand in brackets becomes $\mathcal{O}\big((1-u)^{\mu-1}\ell\big(1/(1-u)\big)\big)$ in the neighbourhood of the point $u = 1$. Let us examine the integral $\int\limits_{F(t;s)}^{1} \big[(1-u)^{\mu-1}\ell\big(1/(1-u)\big)\big]\, du$ as $t \to \infty$. Substitution $y = (1-u)^{-1}$ gives the alternative form

$$\int\limits_{1/R(t;s)}^{\infty} y^{-(1+\mu)}\ell(y)dy.$$

The direct application of Lemma 3 transforms the last integral to the form

$$\int\limits_{1/R(t;s)}^{\infty} y^{-(1+\mu)}\ell(y)dy = \frac{1}{\mu}R^{\mu}(t;s)\ell\left(\frac{1}{R(t;s)}\right)\big(1+o(1)\big) \quad \text{as} \quad t \to \infty.$$

But $R(t;s) = \tau^{-1}(t;s)$ and $\tau(t;s)\tau^{-1}(t) \to 1$ as $t \to \infty$ uniformly in $s \in [0, 1)$. Thus

$$\int\limits_{F(t;s)}^{1}\left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1-u)^{1+|\gamma|}}\right]du = \mathcal{O}\left(\frac{\ell\left(\tau(t)\right)}{\left(\tau(t)\right)^{\mu}}\right) \quad \text{as} \quad t \to \infty. \tag{36}$$

Taking into account that $\mu < \delta$ and comparing relations (35) and (36), one can obtain that $\Delta(t;s)$ decreases to zero faster than last integral, i.e., $\Delta(t;s) = o\left(\ell\left(\tau(t)\right)/\left(\tau(t)\right)^{\mu}\right)$ as $t \to \infty$. So, asymptotic relation (8) with the error part $\rho(t;s)$ in form (9) is found from (34)–(36). Equation (10) follows from the continuity theorem for power series.

Finally, one can verify that function $\pi(s)$ satisfies equation (14). Therefore, denoting its power series representation by $\pi(s) = \sum\limits_{j\in\mathcal{S}} \pi_j s^j$, an invariant property $\pi_j = \sum\limits_{i\in\mathcal{S}} \pi_i p_{ij}(\tau)$ is obtained for any $\tau > 0$. Thus $\{\pi_j, j \in \mathcal{S}\}$ is an invariant measure for MBPI $X(t)$.

The Theorem is proved. □

Proof of Corollary 1. The statement follows immediately from (8) by setting $x = 0$. □

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Об оценке скорости сходимости к инвариантным мерам в марковских ветвящихся процессах с возможной бесконечной дисперсией и иммиграцией

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Аннотация. В работе исследуется марковский ветвящийся случайный процесс с непрерывным временем и с иммиграцией. Мы рассматриваем критический случай, в котором второй момент закона размножения частиц и первый момент закона иммиграции бесконечны. Предполагая, что нелинейные части соответствующих производящих функций правильно меняются в смысле Карамата, мы доказываем теоремы о сходимости переходных вероятностей процесса к инвариантным мерам. Мы определим скорости этой сходимости при условии, что медленно меняющиеся части являются функциями с остатком.

Ключевые слова: марковский ветвящийся процесс, производящие функции, иммиграция, переходные вероятности, медленно меняющаяся функция, инвариантные меры, скорость сходимости.

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Iterations and Groups of Formal Transformations

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Abstract. In this paper, we consider the problem of formal iteration. We construct an area preserving mapping which does not have any square root. This leads to a counterexample to Moser's existence theorem for an interpolation problem. We give examples of formal transformation groups such that the iteration problem has a solution for every element of the groups.

Keywords: iteration, formal transformations, functional equations.

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Introduction

Iterated functions are objects of study in computer science, fractals, dynamical systems and renormalization group physics [1–4]. Here we will consider continuous iterations of mappings. Let $\mathbb{K}$ denote either the set of real numbers or the set of complex numbers. Suppose we are given a local diffeomorphism u of a neighborhood of the origin $0 \in \mathbb{K}^n$ onto another and leaves 0 fixed. The problem of continuous iteration consists in finding a one-parameter family of mappings (a flow) $f(t,x) = f^t(x)$ such that

$$f^t \circ f^s = f^{t+s}, \qquad f^1 = u, \qquad f^0(x) = x \qquad \forall\, t, s \in \mathbb{R}. \tag{1}$$

The iteration problem was investigated by Koenigs, Lewis, Baker, Chen, Sternberg and others. Bibliographical references can be found in [4–6].

Every smooth flow f^t is defined by a system of ordinary differential equations

$$y' = X(y)$$

with initial condition $y(0) = x$. Thus the iteration problem is equivalent the following question. Given a a local diffeomorphism u, does there exist a system of ordinary differential equations such that $y(1) = u$? If the answer to this question is affirmative then we say that the map u is embedded in the flow f^t.

The problem is of great interest in the study of the exponential mapping of infinite-dimensional Lie algebras of vector fields [7–9]. Let $\exp(tX)$ denote an one-parameter group generated by a vector field X, then the map $\exp : X \mapsto \exp(X)$ is called the exponential map or time-one map. Let G be a group of smooth (or formal) maps, and we are given the mapping $u \in G$. The question which arises is this: under what conditions is there a vector field X such that $u = \exp(X)$? If such a vector field X exists, then it is called the logarithm of u. We will also say that the formal transformation u possesses a logarithm.

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Let us denote by $GS_n(\mathbb{K})$ the group of formal power series transformations [9]. Lewis [10] proved that if a transformation $u \in GS_n(\mathbb{K})$ satisfies so-called pseudo-incommensurable condition, then the iteration problem has a formal power series solution. This Lewis result has been repeatedly proved by different authors [5, 6, 9].

In this paper we discuss the iteration problem for some subgroups of the group $GS_n(\mathbb{K})$. It turns out that there are mappings u to which the problem does not even have formal solution, namely, we give an example of a polynomial mapping $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ preserving the area such that there does not exist a 2-tuple $g = (g_1, g_2)$ of formal power series $g_1, g_2 \in \mathbb{R}[[x, y]]$ with $g \circ g = u$. This is a counterexample to Moser's statement [3] about the existence of a solution to the iteration problem for area-saving mappings. We present sufficient conditions for the existence of a solution of the iteration problem. These conditions allow to indicate some groups of formal transformations such that any element of a group possesses a logarithm and the corresponding iteration problem has a formal solution.

1. Examples and condition for the existence of solutions

We begin with the case of a linear mapping

$$u(x) = Ux, \quad x \in \mathbb{K}^n,$$

where U is an invertible matrix. In this case, a solution of the iteration problem has the form

$$u^t(x) = U^t x = e^{t\ln(U)}x$$

whenever the matrix $\ln(U)$ is correctly defined. When $\mathbb{K} = \mathbb{C}$ the matrix $\ln(U)$ exists but in general it is not unique. If $\mathbb{K} = \mathbb{R}$ and U is positive definite then $\ln(U)$ is a real matrix. Some details of the linear case can be found in [10]. Sometimes a nonlinear problem (1) can be reduced to a linear one. This is true if an analytical map u is conjugate to a linear map. Some of the most known results in this direction are Poincaré and Siegel-Sternberg theorems [11–13].

We now consider the groups of formal transformations. Let $\mathbb{K}[[x]]$ denote the ring of formal power series in indeterminate $x_1, \dots, x_n$ with coefficients in $\mathbb{K}$. The ring has a maximal ideal $\mathfrak{M}_1$ and a ideal $\mathfrak{M}_2$ consisting of series without constant and linear terms. Denote by $\mathfrak{M}_i^n$ $(i = 1, 2)$ the n-ary Cartesian product of $\mathfrak{M}_i$. Obviously $\mathfrak{M}_1^n$ is a monoid under substitution of series. We denote by $GS_n(\mathbb{K})$ the set of all invertible elements of $\mathfrak{M}_1^n$. We shall call elements of $GS_n(\mathbb{K})$ formal transformations. It is clear that $GS_n(\mathbb{K})$ is a group. As usual, the general linear group of degree n over $\mathbb{K}$ is denoted by $GL_n(\mathbb{K})$.

Example 1. Let us consider the group $GS_1(\mathbb{C})$ and a polynomial map

$$u = e^{i\pi/3}z + z^7.$$

It is easy to see that there is no a formal power series

$$g = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots$$

such that

$$g \circ g = u. \tag{2}$$

Actually, comparing coefficients of z in (2), we have

$$c_1^2 = e^{i\pi/3}.$$

Then comparing coefficients of $z^2, \dots, z^6$ yields $c_2 = \dots = c_6 = 0$. Finally, comparing coefficients of z^7, we obtain

$$c_1 c_7 (c_1^6 + 1) = 1.$$

This is a contradiction, because $c_1^6+1=0$. This example shows that there is no one-parameter group passing through the polynomial $e^{i\pi/3}z+z^7$. If such a group f^t exists, then $f^{1/2}\circ f^{1/2}=u$. But it is not possible as we just proved. This example shows that polynomial map $u=e^{i\pi/3}z+z^7\in GS_1(\mathbb{C})$ does not possess a logarithm.

We remark that such examples have been known for a long time (see, for example [4,9]).

Example 2. Let $SS_n(\mathbb{K})$ denote the set $\{f\in GS_n(\mathbb{K}) : det(Df)=1\}$, where Df is the Jacobian matrix of f, i.e. $SS_n(\mathbb{K})$ is a group of volume preserving formal transformations. Consider an area preserving polynomial mapping $v\in SS_2(\mathbb{R})$ given by

$$\tilde{x}_1=x_1+x_2^{m+1},\quad \tilde{x}_2=x_2$$

and the rotation matrix

$$M=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix},$$

where $\alpha=2\pi/m$ and $m\geqslant 2$ is an even number. Thus $u=Mv$ is an area preserving mapping.

It is convenient to use the complex variables $z=x_1+ix_2$ and $\bar{z}=x_1-ix_2$. Then the mapping u has the form

$$u(z,\bar{z})=e^{\frac{2i\pi}{m}}\left(z+\left(\frac{z-\bar{z}}{2i}\right)^{m+1}\right). \tag{3}$$

Let us show that there does not exist a formal series

$$g(z,\bar{z})=c_{10}z+c_{01}\bar{z}+c_{20}z^2+c_{11}z\bar{z}+c_{02}\bar{z}^2+\ldots$$

satisfying the condition (2). We assume that such series exists and try to find his coefficients.

Collect all terms belonging to $z,\bar{z}$ in (2). Then we have two equations

$$e^{i\frac{2\pi}{m}}=c_{10}^2+|c_{01}|^2, \tag{4}$$

$$c_{01}(c_{10}+\bar{c}_{10})=0.$$

It follows that

$$c_{01}=0,\quad c_{10}=\pm\exp(i\pi/m).$$

Then comparing coefficients of $z^k\bar{z}^l$ ($1<k+l<m+1$) yields equation

$$c_{kl}(c_{10}+c_{10}^k\bar{c}_{10}^l)=0.$$

Obviously, the following inequality holds

$$c_{10}+c_{10}^k\bar{c}_{10}^l\neq 0$$

whenever $1<k+l<m+1$. Thus we have $c_{kl}=0$.

Finally, we collect all terms belonging to z^{m+1} and obtain equalities

$$\frac{\exp(2i\pi/m)}{(2i)^{m+1}}=c_{(m+1)0}c_{10}(1+c_{10}^m)=0,$$

since $c_{10}=\pm\exp(i\pi/m)$ and m is an even number. This contradiction proves our assertion.

This example implies that Moser's theorem [3] on the solvability of the iteration problem in the class of formal series is not true even for polynomial mappings. Moreover, it is impossible to find the square root of a area preserving mapping in the general case. This example shows that the polynomial map (3) does not possess a logarithm. We shall see that the above examples are related to resonances.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be characteristic values of a matrix $U \in GL_n(\mathbb{K})$. We recall that an identity of the form

$$\lambda_s = \lambda_1^{m_1} \cdots \lambda_n^{m_n}, \quad m_i \in \mathbb{N}, \quad \sum_{i=1}^{n} m_i > 1 \tag{5}$$

is called the resonance (induced by U). We say that the resonance (5) is not obstructive if

$$\lambda_s^t = \lambda_1^{tm_1} \cdots \lambda_n^{tm_n} \qquad \forall\, t \in \mathbb{R}. \tag{6}$$

It is easy to see that we have resonances of the form

$$\lambda = \lambda^{m+1},$$

in Examples 1 and 2 above. These resonances are obstructive since

$$\lambda^{\frac{1}{2}} \neq \lambda^{\frac{m+1}{2}}.$$

Using the theory of normal forms we proved the following statement in [14].

Lemma. *Let $u = Ux + g \in GS_n(\mathbb{C})$ be a formal transformation with $U \in GL_n(\mathbb{C})$ and $g \in \mathfrak{M}_2^n$. If any resonance induced by the matrix U is not obstructive then u possesses a logarithm.*

Now we show that the conditions (5), (6) are equivalent to Lewis's ones. Indeed, it follows from (5) that

$$\exp(\log \lambda_s) = \exp(m_1 \log \lambda_1 + \cdots + m_n \log \lambda_n).$$

The last equality is equivalent to

$$\log(\lambda_s) - \sum_{j=1}^{n} m_j \log(\lambda_j) \in 2\pi i \mathbb{Z}. \tag{7}$$

Similarly, the condition (6) yields

$$t(\log(\lambda_s) - \sum_{j=1}^{n} m_j \log(\lambda_j)) \in 2\pi i \mathbb{Z} \quad \forall\, t \in \mathbb{R}.$$

It follows that

$$\log(\lambda_s) = \sum_{j=1}^{n} m_j \log(\lambda_j). \tag{8}$$

Conversely, it is easy to see that the equality (8) gives (6) and (7) implies (5).

We recall that Lewis's condition means that any relation (7) implies the equality (8) (see [9, 10]).

One can apply Lemma to obtain subgroups G of $GS_n(\mathbb{K})$ such that any $u \in G$ possesses a logarithm. For example, consider subgroup B_l which consists of formal transformations

$$u = Ux + g, \quad g \in \mathfrak{M}_2^n,$$

where U is a lower triangular matrix with real positive eigenvalues.

Corollary. *Any formal transformation $u \in B_l$ possesses a logarithm.*

The analogous result holds for subgroup of formal transformations B^u with upper triangular matrices.

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Итерации и группы формальных преобразований

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Аннотация. В работе рассматривается задача формальной итерации. Строится сохраняющее площадь отображение, которое не допускает извлечения квадратного корня, что, в свою очередь, приводит к контрпримеру — к теореме Мозера для задачи интерполяции. Даны примеры групп формальных преобразований, для которых задача итерации имеет решение для произвольного элемента группы.

Ключевые слова: итерация, формальное преобразование, функциональное уравнение.

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Laurent-Hua Loo-Keng Series with Respect to the Matrix Ball from Space $\mathbb{C}^n\left[m \times m\right]$

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Abstract. The aim of this work is to obtain multidimensional analogs of the Laurent series with respect to the matrix ball from space $\mathbb{C}^n\left[m \times m\right]$. To do this, we first introduce the concept of a "layer of the matrix ball" from $\mathbb{C}^n\left[m \times m\right]$, then in this layer of the matrix ball we use the properties of integrals of the Bochner-Hua Loo-Keng type to obtain analogs of the Laurent series.

Keywords: matrix ball, Laurent series, holomorphic function, Shilov's boundary, Bochner-Hua Loo Keng integral, orthonormal system.

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1. Introduction and preliminaries

In classical complex analysis the Laurent expansions play an important role in studies (in the study) of holomorphic functions in a neighborhood of isolated singular points (in a ring). Analogs of Laurent series have already been constructed in multivariable complex analysis, for example, in the product of circular rings

$$\{z \in \mathbb{C}^n : r_\nu < |z_\nu - a_\nu| < R_\nu,\, \nu = 1, 2, \ldots, n\} \tag{1}$$

or in the domains of Hartogs

$$\{z = ('z, z_n) \in \mathbb{C}^n : 'z \in {'D}\,, r\left('z\right) < |z_n - a_n| < R\left('z\right)\}\,, \tag{2}$$

where $'D$ is a domain of $\mathbb{C}^{n-1}$ (Hartogs-Laurent series). Namely, any function f holomorphic in (1) can be represented as a multiple Laurent series

$$f\left(z\right) = \sum_{|k|=-\infty}^{\infty} c_k (z-a)^k, \tag{3}$$

where $k = (k_1, k_2, \ldots, k_n)$ integer vectors, and

$$c_k = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f\left(\zeta\right) d\zeta}{(\zeta - a)^{k+1}},$$

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$$\Gamma = \{\zeta \in \mathbb{C}^n : |\zeta_\nu - a_\nu| = \rho_\nu,\, r_\nu < \rho_\nu < R_\nu,\, \nu = 1, 2, \dots, n\}.$$

In this case, the domains of convergence of series (3) are relatively complete Reinhart domains.

In the works of É. Cartan [1], C. L. Siegel [2], Hua Loo-Keng [3], I. I. Pjateckiĭ-Šapiro [4], as well as in [5] the matrix approach of presenting the theory of multivariable complex analysis is widely used. It mainly deals with the classical domains and related questions of function theory and geometry. The importance of studying classical domains is that they are not reducible, i.e. these domains are, in a sense, model domains of multidimensional space.

Recently, scientists have obtained many significant results in the classical fields, and at the same time, a number of open problems have been formulated. For example, in [6] the regularity and algebraicity of mappings in classical domains are studied, and in [7] harmonic Bergman functions in classical domains are studied from a new point of view. In the paper [8], holomorphic and pluriharmonic functions are defined for classical domains of the first type, the Laplace and Hua Loo-Keng operators are studied also. A connection was found between these operators.

In addition, scientific works in matrix balls associated with classical domains from the space $\mathbb{C}^n\,[m \times m]$ are developing.

Consider the space of complex m^2 variables denoted by $\mathbb{C}^{m^2}$. In some questions, it is convenient to represent the point Z of this space in the form of a square $[m \times m]$ matrix, that is, in the form $Z = (z_{ij})_{i,j=1}^m$. With this representation of points, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m \times m]$. The direct product $\underbrace{\mathbb{C}[m \times m] \times \dots \times \mathbb{C}[m \times m]}_{n}$ that have n copies of $[m \times m]$ matrices we denote by $\mathbb{C}^n[m \times m]$.

Let $Z = (Z_1, \dots, Z_n)$ be a vector composed of square matrices Z_j of order m, considered over the field of complex numbers $\mathbb{C}$. Let us write the elements of the vector $Z = (Z_1, \dots, Z_n)$ as points z of the space $\mathbb{C}^{nm^2}$:

$$z = (z_{11}^{(1)}, \dots, z_{1m}^{(1)}, \dots, z_{m1}^{(1)}, \dots, z_{mm}^{(1)}, \dots, z_{11}^{(n)}, \dots, z_{1m}^{(n)}, \dots, z_{m1}^{(n)}, \dots, z_{mm}^{(n)}) \in \mathbb{C}^{nm^2}. \tag{4}$$

Hence, we can assume that Z is an element of the space $\mathbb{C}^n\,[m \times m]$, that is, we arrive at the isomorphism $\mathbb{C}^n\,[m \times m] \cong \mathbb{C}^{nm^2}$.

Let us define the matrix "scalar" product:

$$\langle Z, W \rangle = Z_1 W_1^* + \dots + Z_n W_n^*,$$

where W_j^* is the conjugate and transposed matrix for the matrix W_j.

It is known (see [9, 10]) that the matrix balls $\mathbb{B}_{m,n}^{(1)}$, $\mathbb{B}_{m,n}^{(2)}$ and $\mathbb{B}_{m,n}^{(3)}$ of the first, second and third types, respectively, have the form:

$$\mathbb{B}_{m,n}^{(1)} = \{(Z_1, \dots, Z_n) = Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z \rangle > 0\},$$

$$\mathbb{B}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z \rangle > 0, \quad \forall Z'_\nu = Z_\nu,\ \nu = 1, \dots, n\},$$

and

$$\mathbb{B}_{m,n}^{(3)} = \left\{(Z \in \mathbb{C}^n\,[m \times m] : I + \langle Z, Z \rangle > 0,\ \forall\, Z_\nu^{'} = -Z_\nu, \quad \nu = 1, \dots, n\right\}.$$

The skeletons (the Shilov boundaries) of matrix balls $\mathbb{B}_{m,n}^{(k)}$, denoted by $\mathbb{X}_{m,n}^{(k)}$, $\quad k = 1, 2, 3$, i.e.,

$$\mathbb{X}_{m,n}^{(1)} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z \rangle = I\},$$

$$\mathbb{X}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z \rangle = I, \quad Z'_v = Z_\nu,\ \nu = 1, 2, \dots, n\},$$

$$\mathbb{X}^{(3)}_{m,n} = \left\{ Z \in \mathbb{C}^n\,[m\times m] : I + \langle Z, Z\rangle = 0,\;\; Z'_\nu = -Z_\nu,\;\; \nu = 1,2,\ldots,n \right\}.$$

Note that, $\mathbb{B}^{(1)}_{1,1}$, $\mathbb{B}^{(2)}_{1,1}$ and $\mathbb{B}^{(3)}_{2,1}$ are unit disks, and $\mathbb{X}^{(1)}_{1,1}$, $\mathbb{X}^{(2)}_{1,1}$, and $\mathbb{X}^{(3)}_{2,1}$ are unit circles in the complex plane $\mathbb{C}$.

If $n = 1$, $m > 1$, then $\mathbb{B}^{(k)}_{m,1}$, $k = 1, 2, 3$ are the classical domains of the first, second and third type (according to the classification of E. Cartan (see [1])), and the skeletons $\mathbb{X}^{(1)}_{m,1}$, $\mathbb{X}^{(2)}_{m,1}$, and $\mathbb{X}^{(3)}_{m,1}$ are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

Note that the matrix balls $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ are complete circular convex bounded domains. In addition, the domains $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ and their skeletons $\mathbb{X}^{(1)}_{m,n}$, $\mathbb{X}^{(2)}_{m,n}$, $\mathbb{X}^{(3)}_{m,n}$ are invariant under unitary transformations (see [10, 12]).

The first type of matrix ball was considered by A. G. Sergeev in [11], and by G. Khudayberganov in [9]. In [10], formulas for the volume of a matrix ball of the first type and its skeleton are obtained, the holomorphic automorphisms for a matrix ball of the first type are described, and integral formulas for matrix balls of the second and third types are obtained. In [13] the volumes of the third type matrix ball and the generalized Lie ball are calculated. The total volumes of these domains are necessary to find the kernels of the integral formulas for these domains (Bergman, Cauchy–Szegő, Poisson kernels, etc. (see, for example, [14–17])). In addition, they are useful for the integral representation of functions holomorphic in these domains in the mean value theorem and other important concepts. In the papers [18, 19] analogs of Laurent series with respect to the classical Cartan domains of the first, second, and third types are obtained.

The aim of this work is to obtain analogs of the Laurent series[‡] with respect to the matrix ball from space $\mathbb{C}^n\,[m\times m]$. To do this, we first introduced the concept of a "layer of the matrix ball" from $\mathbb{C}^n\,[m\times m]$, then in this layer of the matrix ball, we used the properties of integrals of the Bochner–Hua Loo-Keng type to obtain analogs of the Laurent series.

1. Laurent-Hua Loo-Keng series with respect to the matrix ball $\mathbb{B}_{m.n}$

Let $\mathbb{B}_{m.n}$[§] be a matrix ball. For functions $f(Z) = f(z^{(1)}_{11},\ldots,z^{(1)}_{1m},\ldots,z^{(n)}_{m1},\ldots,z^{(n)}_{mm})$ holomorphic in $\mathbb{B}_{m.n}$ and continuous on $\overline{\mathbb{B}}_{m,n}$ ($\overline{\mathbb{B}}_{m,n} = \mathbb{B}_{m,n} \cup \partial\mathbb{B}_{m,n}$) the Bochner–Hua Loo-Keng integral formula is valid [10, 20]:

$$f(Z) = \int_{\mathbb{X}_{m,n}} \det{}^{-mn}\left(I^{(m)} - \langle Z, U\rangle\right) f(U)\,d\mu, \tag{5}$$

where $f(U)$ is an integrable function, $d\mu$ is the Haar measure on $\mathbb{X}_{m,n}$.

Let

$$\mathbb{B}^{-}_{m,n} = \left\{ Z \in \mathbb{C}^n\,[m\times m] :\; I - \langle Z, Z\rangle < 0 \right\}.$$

The integral (5) of Bochner–Hua Loo-Keng type makes sense in each of the domains $\mathbb{B}_{m,n}$ and $\mathbb{B}^{-}_{m,n}$ ([21]).

Let us write the elements of vector $Z = (Z_1,\ldots,Z_n) \in \mathbb{B}_{m,n}$ in the form (4) and by $z^{[\alpha]}$ we will denote a vector with components

$$\sqrt{\frac{|\alpha|!}{\alpha_1!\alpha_2!\ldots\alpha_{nm^2}!}}\left(z^{(1)}_{11}\right)^{\alpha_1}\cdots\left(z^{(1)}_{1m}\right)^{\alpha_m}\cdots\left(z^{(n)}_{mm}\right)^{\alpha_{nm^2}},\;\; |\alpha| = \sum_{i=1}^{nm^2}\alpha_i,\;\; \alpha_i \geqslant 0. \tag{6}$$

[‡]In what follows, we will call these series Laurent-Hua Loo-Keng series.

[§]For convenience, we denote $\mathbb{B}^{(1)}_{m,n}$ by $\mathbb{B}_{m,n}$, and $\mathbb{X}^{(1)}_{m,n}$ by $\mathbb{X}_{m,n}$.

The dimension of the subspace generated by the vector $z^{[\alpha]}$ is equal to the dimension of the direct sum of subspaces with dimensions (see [3, 22, 23])

$$q(\alpha_1, \alpha_2, \ldots, \alpha_m) = N(\alpha_1, \alpha_2, \ldots, \alpha_m) \cdot N(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0)$$

and it is equal to

$$\sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_m=|\alpha| \\ \alpha_1\geqslant\alpha_2\geqslant\ldots\geqslant\alpha_m\geqslant 0}} N(\alpha_1, \alpha_2, \ldots, \alpha_m) \cdot N(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0) = \frac{(nm^2 + |\alpha| - 1)!}{\alpha!(nm^2 - 1)!},$$

where

$$N(\alpha_1, \alpha_2, \ldots, \alpha_m) = \frac{D(\alpha_1 + m - 1, \alpha_2 + m - 2, \ldots, \alpha_{m-1} + 1, \alpha_m)}{D(m - 1, m - 2, \ldots, 1, 0)},$$

$$D(\alpha_1, \alpha_2, \ldots, \alpha_m) = \prod_{1\leqslant i<j\leqslant m} (\alpha_i - \alpha_j), \quad m \geqslant 2.$$

Obviously, (6) contains all monomials of degree α, that is, any polynomial in

$$z_{11}^{(1)}, \ldots, z_{1m}^{(1)}; \ldots; z_{m1}^{(1)}, \ldots, z_{mm}^{(1)}; \ldots; z_{11}^{(n)}, \ldots, z_{1m}^{(n)}; \ldots; z_{m1}^{(n)}, \ldots, z_{mm}^{(n)}$$

is a linear combination of expressions like (6), if α takes values $0, 1, 2, \ldots$.

Let us denote by

$$\varphi^{(p)}_{\alpha_1,\alpha_2,\ldots,\alpha_m}(Z), \ \ p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m)$$

the components of the vector $z^{[\alpha]}$.

In [24] it was proved that the system of functions

$$(\rho_\alpha)^{-\frac{1}{2}} \varphi^{(p)}_\alpha(Z), \quad p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m), \ \ \alpha = 0, 1, 2, \ldots$$

is an orthonormal system in the domain $\mathbb{B}_{m,n}$, where

$$\rho_\alpha = \int_{\mathbb{B}_{m,n}} \left|\varphi^{(p)}_\alpha(Z)\right|^2 d\nu, \quad d\nu = \prod_{k=1}^{n} \prod_{1\leqslant i\leqslant j\leqslant m} dx^{(k)}_{i,j} dy^{(k)}_{i,j}$$

and the system of functions

$$(\delta_\alpha)^{-\frac{1}{2}} \varphi^{(p)}_\alpha(U), \quad p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m), \ \ \alpha = 0, 1, 2, \ldots. \tag{7}$$

forms a complete orthonormal system on $\mathbb{X}_{m,n}$, where $\varphi^{(p)}_\alpha(U)$, $p = 1, 2, \ldots, q(\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\alpha = 0, 1, 2, \ldots$ are components of the vector $u^{[\alpha]}$ ($u = (u^{(1)}_{11}, \ldots, u^{(1)}_{1m}, \ldots, \ldots, \ldots, u^{(n)}_{m1}, \ldots, u^{(n)}_{mm})$) and

$$\delta_\alpha = \int_{\mathbb{X}_{m,n}} \left|\varphi^{(p)}_\alpha(U)\right|^2 d\mu.$$

Theorem 1.1 (see [24]). *Let $f(U)$ be an integrable function in $\mathbb{X}_{m,n}$ and let*

$$a^p_\alpha = \frac{V(\mathbb{X}_{m,n})}{\sqrt{\delta_\alpha}} \cdot \int_{\mathbb{X}_{m,n}} f(U)\overline{\varphi^{(p)}_\alpha(U)} d\mu \tag{8}$$

be the Fourier coefficients of this function with respect to the orthonormal system (7)*. Then, in* $\mathbb{B}_{m,n}$ *the integral* (5) *represents a holomorphic function that expands in this domain in a series*

$$\sum_{\alpha\geqslant 0}\sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}}. \tag{9}$$

If we denote

$$F^{\pm}(Z)=\begin{cases} F(Z), Z\in\mathbb{B}_{m,n}, \\ F(Z), Z\in\mathbb{B}_{m,n}^{-}, \end{cases}$$

then, by Theorem 1.1, for all $Z\in\mathbb{B}_{m,n}$ we have

$$F^{+}(Z)=\sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}}, \tag{10}$$

with coefficients (8). Therefore, $F^{+}(Z)\in\mathcal{O}(\mathbb{B}_{m,n})$, i.e. $F^{+}(Z)$ is holomorphic in $\mathbb{B}_{m,n}$. Now let $Z\in\mathbb{B}_{m,n}^{-}$. Then we have

$$F^{-}(Z)=\int\limits_{X_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)}-\left\langle (\langle Z,Z\rangle)^{-1}Z,U\right\rangle\right)}d\mu. \tag{11}$$

The Cauchy-Szegő kernel has the following form (see [24]):

$$C(\widetilde{Z},U)=\frac{1}{V(\mathbb{X}_{m,n})}\det{}^{-mn}(I^{(m)}-\left\langle \widetilde{Z},U\right\rangle=\sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)}\varphi_{i,j}^{(\alpha)}(\widetilde{Z})\overline{\varphi_{i,j}^{(\alpha)}(U)}.$$

Using the equalities from [3, p.114] we obtain

$$\varphi_{\alpha_1,\ldots,\alpha_m}^{(p)}(U)=\varphi_{\alpha_1-\alpha_m,\alpha_2-\alpha_m,\ldots,\alpha_{m-1}-\alpha_m,0}^{(p)}(U)(\det U)^{\alpha_m},$$

and noticing that $\widetilde{Z}=(\langle Z,Z\rangle)^{-1}Z$, we have

$$\frac{1}{\det^{mn}\left(I^{(m)}-\left\langle (\langle Z,Z\rangle)^{-1}Z,U\right\rangle\right)}=V(\mathbb{X}_{m,n})\sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)}\varphi_{i,j}^{(\alpha)}\left((\langle Z,Z\rangle)^{-1}Z\right)\overline{\varphi_{i,j}^{(\alpha)}(U)}.$$

Multiplying the last expression by $f(U)$ and integrating term by term against the measure $d\mu$, we obtain

$$\begin{aligned} F^{-}(Z)&=\int_{\mathbb{X}_{m,n}} f(U)\,V(\mathbb{X}_{m,n})\sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)}\varphi_{i,j}^{(\alpha)}\left((\langle Z,Z\rangle)^{-1}Z\right)\overline{\varphi_{i,j}^{(\alpha)}(U)}d\mu= \\ &=\sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)}\varphi_{i,j}^{(\alpha)}\left((\langle Z,Z\rangle)^{-1}Z\right)\left[V(\mathbb{X}_{m,n})\int_{\mathbb{X}_{m,n}} f(U)\overline{\varphi_{i,j}^{(\alpha)}(U)}d\mu\right]= \\ &=(-1)^{mn}\sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)} a_{-\alpha_m-m,\ldots,-\alpha_1-m}^{(p)}\frac{\varphi_{-\alpha_m-m,\ldots,-\alpha_1-m}^{(p)}\left((\langle Z,Z\rangle)^{-1}Z\right)}{\sqrt{\delta_\alpha}}. \end{aligned} \tag{12}$$

Therefore, in the domain $\mathbb{B}_{m,n}^{-}$ the integral (11) represents a holomorphic function of $(\langle Z,Z\rangle)^{-1}Z$, which has an expansion

$$F^{-}(Z)=(-1)^{mn}\sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)}a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left((\langle Z,Z\rangle)^{-1}Z\right)}{\sqrt{\delta_\alpha}}. \qquad (13)$$

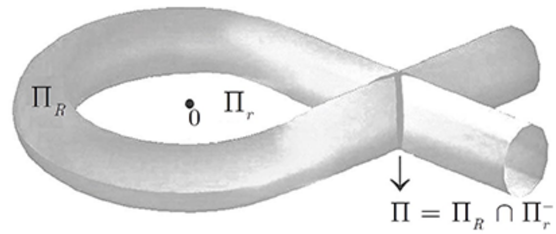
Consider the matrix domains of the form:

$$\Pi_R=\left\{Z\in\mathbb{C}^n\left[m\times m\right]: R^2I^{(m)}-\langle Z,Z\rangle>0\right\},$$

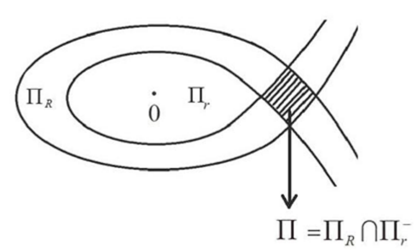
$$\Pi_r=\left\{Z\in\mathbb{C}^n\left[m\times k\right]: r^2I^{(m)}-\langle Z,Z\rangle<0\right\},$$

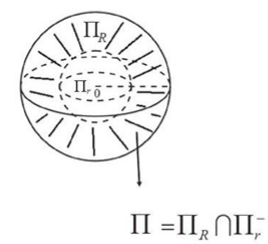
where R, r are real numbers such that $0<r<R<\infty$. We denote $\Pi=\Pi_R\cap\Pi_r^-$ and the sets Π will call the "*layer of the matrix ball*".

The following diagram shows a layer of a matrix ball from the space $\mathbb{C}^n\left[m\times m\right]$



and, in particular, for $n=1$ and $m=1$ we have:





The following theorem holds

Theorem 1.2. *If $F(Z)\in\mathcal{O}(\Pi)\cap C\left(\bar{\Pi}\right)$, then for $Z\in\Pi$ the Laurent-Hua Loo-Keng expansion*

$$F(Z)=F^{+}(Z)+F^{-}(Z).$$

where the coefficients in (13) are calculated by the formula

$$a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}=V\left(\mathbb{X}_\rho\right)\int_{\mathbb{X}_\rho}f\left(U\right)\overline{\varphi^{(\alpha)}_{i,j}}(U)d\mu, \qquad (14)$$

and

$$\mathbb{X}_\rho=\left\{U\in\mathbb{C}^n\left[m\times m\right]:\ \langle U,U\rangle=\rho^2I^{(m)}, r<\rho<R.\right\}.$$

Proof. Fix an arbitrary point $Z \in \Pi$ and construct a layer Π' such that $Z \in \Pi' \subseteq \Pi$ ($\Pi' = \Pi'_R \cap \Pi'_r, r < r' < R' < R$). Then, by virtue of the Bochner-Hua Loo-Keng integral formula (5) the following expansion takes place:

$$F\left(Z\right) = \int_{\Gamma'} \frac{f\left(U\right)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu + \int_{-\gamma'} \frac{f\left(U\right)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu, \tag{15}$$

where

$$\Gamma' = \left\{U \in \mathbb{C}^n\left[m \times m\right] : \langle U, U\rangle = \left(R'\right)^2 I^{(m)}\right\},$$

and

$$\gamma' = \left\{U \in \mathbb{C}^n\left[m \times m\right] : \langle U, U\rangle = \left(r'\right)^2 I^{(m)}\right\}.$$

Therefore, by virtue of (10)

$$\int_{\Gamma'} \frac{f\left(U\right)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu = \sum_{\alpha_1 \geqslant \ldots \geqslant \alpha_m \geqslant 0} \sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}} = F^+\left(Z\right), \tag{16}$$

where

$$a_\alpha^p = \frac{V(\Gamma')}{\sqrt{\delta_\alpha}} \cdot \int_{\Gamma'} f(U) \overline{\varphi_\alpha^{(p)}(U)} d\mu.$$

Assuming

$$\varphi_{\alpha_1,\ldots,\alpha_m}^{(p)}(U) = \varphi_{\alpha_1-\alpha_m,\alpha_2-\alpha_m,\ldots,\alpha_{m-1}-\alpha_m,0}^{(p)}(U)(\det U)^{\alpha_m},$$

for any $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m$ from (13) we get

$$\int_{-\gamma'} \frac{f\left(U\right)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} =$$

$$= (-1)^{mn} \sum_{\alpha_1 \geqslant \ldots \geqslant \alpha_m \geqslant 0} \sum_{p=1}^{q(\alpha)} a_{-\alpha_m - m,\ldots,-\alpha_1 - m}^{(p)} \frac{\varphi_{-\alpha_m - m,\ldots,-\alpha_1 - m}^{(p)}\left(\left(\langle Z, Z\rangle\right)^{-1} Z\right)}{\sqrt{\delta_\alpha}} = F^-(Z), \tag{17}$$

where

$$a_{-\alpha_m - m,\ldots,-\alpha_1 - m}^{(p)} = V\left(\gamma'\right) \int_{-\gamma'} f\left(U\right) \overline{\varphi_{i,j}^{(\alpha)}\left(U\right)} d\mu.$$

Now, substituting (16) and (17) into (15), we obtain the required expansion

$$F(Z) = F^+(Z) + F^-(Z).$$

It remains to note that by Cauchy's homotopy theorem in formulas for calculating the coefficients a_α^p and $a_{-\alpha_m - m,\ldots,-\alpha_1 - m}^{(p)}$ can be replaced with any

$$\mathbb{X}_\rho = \left\{U \in \mathbb{C}^n\left[m \times m\right] : \langle U, U\rangle = \rho^2 I^{(m)}, r < \rho < R\right\},$$

and then these formulas will take the form (14). The theorem is proved. □

Corollary 1. *In Theorem 1.2, when $n = 1$ the expansion (14) coincides with the Laurent expansion in the "matrix ring" defined in the Cartan classical domains (in the space $\mathbb{C}\left[m \times m\right]$):*

$$\Pi = \Pi_R \cap \Pi_r^-,$$

where matrix domains

$$\Pi_R = \left\{ Z \in \mathbb{C}\,[m \times k] : R^2 I^{(m)} - ZZ^* > 0 \right\},$$

$$\Pi_r = \left\{ Z \in \mathbb{C}\,[m \times k] : r^2 I^{(m)} - ZZ^* < 0 \right\},$$

and R, r are real numbers such that $0 < r < R$ (see [18]).

Corollary 2. *When $m = 1$ then the expansion (14) coincides with Laurent expansion of holomorphical function in a ball layer (in the space $\mathbb{C}^n$):*

$$\Pi = \{z \in \mathbb{C}^n : r < |z| < R\}$$

(by Severi's theorem, this expansion coincides with the Taylor expansion in the ball $\mathbb{B}_{1,n} = \{z \in \mathbb{C}^n : |z| < 1\}$).

Corollary 3. *When $m = n = 1$ we obtain the Laurent expansion on the complex plane.*

2. Open problems

We present some unsolved problems related to the matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$, associated with the classical domains of the second and third types:

1. Obtain analogs of the expansion of the Laurent–Hua Loo-Keng series for the matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$.

$2_{1,2,3}$. Describe domains of convergence of Laurent series with respect to matrix balls $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$.

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Ряды Лорана-Хуа Ло-кена относительно матричного шара из пространства $\mathbb{C}^n\left[m \times m\right]$

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Аннотация. Целью данной работы является получение аналогов ряда Лорана относительно матричного шара из пространства $\mathbb{C}^n\left[m \times m\right]$. Для этого сначала введены понятие "слоя матричного шара" из $\mathbb{C}^n\left[m \times m\right]$, затем в этом слое матричного шара использовались свойства интегралов типа Бохнера-Хуа Ло-кена для получения аналогов ряда Лорана.

Ключевые слова: матричной шар, ряд Лорана, голоморфная функция, граница Шилова, интеграл Бохнера-Хуа Ло-кена, ортонормальная система.

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On the Zeta-Function of Zeros of an Entire Function

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Abstract. This article is devoted to the study of the properties of the zeta-function of zeros of an entire function. We obtain an explicit expression for the kernel of the integral representation of the zeta-function in one case.

Keywords: zeta-function of zeros, integral representation, canonical product.

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Introduction

The purpose of this article is to correct a mistake in the work [1]. Namely, in the article [1] an incorrect statement was given that for an entire function f, satisfying some additional conditions, the following equality holds on the positive part of the real axis

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2x}. \tag{1}$$

It is easy to see that for any entire function f this equality cannot be true on the whole positive semiaxis. Indeed, the function $\dfrac{f'(z)}{f(z)}$ is meromorphic in the whole complex plane. By virtue of the uniqueness theorem, the equality (1) holds not only on $\mathbb{R}^+$, but also in $\mathbb{C} \setminus \{0\}$. However, the function

$$\frac{\sqrt{\pi}}{2\sqrt{z}} - \frac{1}{2z}$$

is not meromorphic in a neighborhood of the origin.

Our article is devoted to correcting the relation (1) and some of its consequences. Note that this result is related to the study of a generalized zeta-function constructed by zeros of some entire function.

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1. Auxiliary results

Let $f(z)$ be an entire function of order ρ in $\mathbb{C}$. Consider the equation

$$f(z) = 0. \tag{2}$$

Denote by $N_f = f^{-1}(0)$ the set of all solutions to (2) (we take every zero as many times as its multiplicity). The numbers of roots is at most countable.

The zeta-function $\zeta_f(s)$ of Eq. (2) is defined in the following way:

$$\zeta_f(s) = \sum_{z_n \in N_f} (-z_n)^{-s},$$

where $s \in \mathbb{C}$.

In [2], using the residue theory, V. I. Kuzovatov and A. A. Kytmanov obtained two integral representation for the zeta-function constructed by zeros of an entire function of finite order on the complex plane. With the help of these representations, they described a domain which the zeta-function can be extended to.

Theorem 1.1 ([2]). *Let $f(z)$ be an entire function of the zero order in $\mathbb{C}$ and satisfy the condition*

$$\frac{f'(z)}{f(z)} - \omega_0 = O\left(\frac{1}{|z|}\right), \quad |z| \to \infty.$$

Suppose that $0 < \operatorname{Re} s < 1$. *Then*

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \int_0^\infty \left(\frac{f'(x)}{f(x)} - \omega_0\right) x^{-s}\, dx, \tag{3}$$

where ω_0 is the limit value of $\dfrac{f'(x)}{f(x)}$ at infinity.

The method of proof of Theorem 1.1 shows that the statement remains valid in the case when $f(z)$ is an entire function of order less than 1.

Now we will give an integral representation for the zeta-function $\zeta_f(s)$ of zeros z_n of f which are $z_n = -q_n + is_n$, $q_n > 0$. Let us denote

$$F(f, x) = \sum_{n=1}^{\infty} e^{z_n x}. \tag{4}$$

We will assume that $\operatorname{Re} s = \sigma > 1$ and the following conditions hold:

$$\varliminf_{n \to \infty} \frac{q_n}{n} > 0, \tag{5}$$

$$\text{the series} \quad \sum_{n=1}^{\infty} \left(\frac{1}{q_n}\right)^{\sigma - 1} \quad \text{converges.} \tag{6}$$

For the convergence of the series (4), using condition (5), it is necessary and sufficient (for real x) that $x > 0$ [2].

Theorem 1.2 ([2]). *Suppose that the conditions* (5) *and* (6) *are satisfied and* $\operatorname{Re} s > 1$. *Then*

$$\zeta_f(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F(f, x)\, dx,$$

where $F(f, x)$ is defined by formula (4), *and $\Gamma(s)$ is the Euler gamma-function.*

Our goal is to obtain an explicit expression for the kernel of the integral representation (3) in case $z_n = -\pi n^2$. This choice of zeros z_n is due to the fact that for series

$$F(f,x) = \sum_{n=1}^{\infty} e^{z_n x} = \sum_{n=1}^{\infty} e^{-\pi n^2 x} := \psi(x)$$

for $x > 0$ it is known (see, for example, [3, Chapter II, S. 6]) that

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left\{ 2\psi\left(\frac{1}{x}\right) + 1 \right\}.$$

2. The main result

Theorem 2.1. *Let $f(z)$ be an entire function of order $\rho < 1$ with zeros $z_n = -\pi n^2$. Then for real $x \in (0; +\infty)$ the following holds*

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} \operatorname{cth} \sqrt{\pi x} - \frac{1}{2x}.$$

Proof. Since the order of f is less than 1, it has the form

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \tag{7}$$

The representation (7) is true, for example, for entire functions of order less than 1 or for entire functions of the first order with the additional condition, i.e. the series $\sum\limits_{n=1}^{\infty} \frac{1}{|z_n|}$ is convergent. In particular, the representation (7) is true for functions of the zero genus.

It is easy to show that in this case we obtain

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \tag{8}$$

if $z \neq z_n$.

Since the order of the canonical product (7) is equal to the index of convergence ρ_1 of its zeros and for given values of z_n

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln |z_n|} = \frac{1}{2},$$

representations (7) and (8) are true for considered function $f(z)$.

To further prove the assertion of the theorem, we use the standard decomposition (see, for example, [4, formula 5.1.25.4])

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cth} \pi a.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cth} \pi a.$$

Thus

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{\infty} \frac{1}{x+\pi n^2} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2+x/\pi} = \frac{1}{\pi}\left(-\frac{1}{2x/\pi} + \frac{\pi}{2\sqrt{x/\pi}}\operatorname{cth}\pi\sqrt{x/\pi}\right) =$$
$$= -\frac{1}{2x} + \frac{\sqrt{\pi}}{2\sqrt{x}}\operatorname{cth}\sqrt{\pi x}. \qquad \square$$

Corollary 1. *Suppose that the conditions of Theorem 2.1 are satisfied. If* ω_0 *is the limit value of* $\frac{f'(x)}{f(x)}$ *at infinity, i.e.*

$$\omega_0 = \lim_{x\to+\infty}\frac{f'(x)}{f(x)},$$

then $\omega_0 = 0$.

Proof. To prove the statement, we note that

$$\lim_{x\to+\infty}\operatorname{cth}x = \lim_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to+\infty}\frac{e^x\left(1+e^{-2x}\right)}{e^x\left(1-e^{-2x}\right)} = 1. \qquad \square$$

Remark 1. If f is an arbitrary entire function of order $1 \leqslant \rho < \infty$, with zeros $z_n = -\pi n^2$, then the ratio can be represented as

$$\frac{f(z)}{\prod\limits_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)} = e^{g(z)},$$

where $g(z)$ is an entire function. Since $1 \leqslant \rho < \infty$, $g(z)$ is a polynomial, $\deg g = \rho$, and $\rho \in \mathbb{N}$ [5]. Therefore,

$$f(z) = \Pi(z)e^{g(z)}, \quad \Pi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right),$$

and

$$\frac{f'(z)}{f(z)} = \frac{\Pi'(z)e^{g(z)} + \Pi(z)e^{g(z)}g'(z)}{\Pi(z)e^{g(z)}} = \frac{\Pi'(z)}{\Pi(z)} + g'(z).$$

Consequently in this case we take

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}}\operatorname{cth}\sqrt{\pi x} - \frac{1}{2x} + g'(x), \quad 1 \leqslant \rho < \infty.$$

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О дзета-функции нулей целой функции

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Аннотация. Данная статья посвящена исследованию свойств дзета-функции нулей целой функции. Получено явное выражение для ядра интегрального представления дзета-функции в одном случае.

Ключевые слова: дзета-функция нулей, интегральное представление, каноническое произведение.

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On Pairs of Additive Subgroups Associated with Intermediate Subgroups of Groups of Lie Type over Nonperfect Fields

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Abstract. The author has previously (Trudy IMM UrO RAN, 19(2013), no. 3) described the groups lying between twisted Chevalley groups $G(K)$ and $G(F)$ of type ${}^2A_l$, ${}^2D_l$, ${}^2E_6$, ${}^3D_4$ in the case when the larger field F is an algebraic extension of the smaller nonperfect field K of exceptional characteristic for the group $G(F)$ (characteristics 2 and 3 for the type ${}^3D_4$ and only 2 for other types). It turned out that apart from, perhaps, the type ${}^2D_l$, such intermediate subgroups are standard, that is, they are exhausted by the groups $G(P)H$ for some intermediate subfield P, $K \subseteq P \subseteq F$, and of the diagonal subgroup H normalizing the group $G(P)$. In this note, it is established that intermediate subgroups are also standard for the type ${}^2D_l$.

Keywords: groups of Lie type, nonperfect field, intermediate subgroups, carpet of additive subgroups.

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1. Introduction and preliminaries

Groups of Lie type $G(F)$ over the field F consist of Chevalley groups of type $\Phi = A_l$, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2 and twisted Chevalley groups of type ${}^n\Phi = {}^2A_l$, ${}^2D_l$, ${}^3D_4$, ${}^2E_6$, ${}^2B_2$, ${}^2G_2$, ${}^2F_4$. The number of fundamental reflections generating the Weyl group associated with the group $G(F)$ is called its Lie rank. Groups of type A_1, ${}^2A_2$, ${}^2B_2$, ${}^2G_2$ constitute all groups of Lie rank 1. Groups of type ${}^2A_l$, ${}^2D_l$, ${}^3D_4$, ${}^2E_6$ are also called by Steinberg groups, groups of type ${}^2B_2$ are called by Suzuki groups, groups of type ${}^2G_2$, ${}^2F_4$ are called by Ree groups, in honor of their discoverers.

The exceptional characteristics of the ground field F for the group $G(F)$ are usually:

– characteristic 2 for types B_l, C_l, F_4, ${}^2A_l$, ${}^2D_l$ and ${}^2E_6$;

– characteristics 2 and 3 for types G_2 and ${}^3D_4$.

This is due to the fact that the Coxeter graph associated with the group $G(F)$ has edges of multiplicity 2 or 3.

In what follows, everywhere the field F is an algebraic extension of the field K. The intermediate subgroups between the groups $G(K)$ and $G(F)$ are described in the author's papers [1–3]. For exceptional characteristics, the description depends on whether the field K is perfect. By definition, a field K of characteristic $p > 0$ is called perfect if $K^p = K$.

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In 1983, the following result was obtained in [1]. If the Lie rank of the group $G(F)$ is greater than one and it is different from the group Re of type ${}^2F_4$, and in the exceptional characteristics for $G(F)$ the field K is perfect, then the groups lying between the groups $G(K)$ and $G(F)$, are exhausted by the groups $G(P)H$ for some intermediate subfield P and a diagonal subgroup H normalizing the group $G(P)$. We call such intermediate subgroups *standard*.

In [2], the groups that lie between the Chevalley groups $G(K)$ and $G(F)$ of type B_l, C_l, F_4, G_2 are described in the case of an nonperfect field K that is exceptional characteristics for the group $G(F)$. It turned out that in each of these cases, except type G_2 in characteristic 2, nonstandard subgroups appear and they are parameterized by two additive subgroups of the field F. Moreover, if $G(K)$ is of type F_4 or G_2, then both additive subgroups are fields, and if $G(K)$ is of type B_l $(l \geqslant 3)$ or C_l $(l \geqslant 3)$, then one additive subgroup is a field. The paper [4] contains examples of non-standard intermediate subgroups for types B_l $(l \geqslant 3)$ and C_l $(l \geqslant 3)$, which are parameterized by two additive subgroups, one of which is not a field, and for the type $B_2 = C_2$ both such additive subgroups may not be fields.

In [3], the groups lying between twisted Chevalley groups $G(K)$ and $G(F)$ of type ${}^2A_l$, ${}^2D_l$, ${}^2E_6$, ${}^3D_4$ are described in the case of nonperfect fields K of exceptional characteristic for the group $G(F)$. It turned out that except, perhaps, the type ${}^2D_l$, the intermediate subgroups are standard.

In this note, we classify pairs of additive subgroups that parameterize non-standard subgroups between the groups $G(K)$ and $G(F)$ (Section 2) and prove the standardness of such intermediate subgroups for the type ${}^2D_l$ (Section 3). Thus, non-standard groups lying between the groups $G(K)$ and $G(F)$ appear only for Chevalley groups of normal type B_l, C_l, F_4 and G_2 over the nonperfect field F of characteristic 2 and, respectively 3. Note also that if we remove the condition of algebraicity of the extension of a larger field over a smaller one, then the description of intermediate subgroups becomes immeasurable for Chevalley groups associated with Coxeter graphs without multiple connections [5, 6].

2. Pairs of additive subgroups associated with intermediate subgroups of Chevalley groups of type B_l, C_l, F_4 и G_2

Let Φ be a reduced indecomposable root system, $\Phi(F)$ be a Chevalley group of type Φ over the field F generated by the root subgroups

$$x_r(F) = \{x_r(t) \mid t \in F\}, \quad r \in \Phi.$$

Following V. M. Levchuk [7], by *a carpet of type* Φ *over* F, we mean a family of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of the field F with the condition

$$C_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js}, \quad r, s, ir + js \in \Phi, \quad i, j > 0, \tag{1}$$

where $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$, and constants $C_{ij,rs}$ are equal to ± 1, ± 2 or ± 3. Inclusions (1) come from the Chevalley commutator formula

$$\big[x_s(u), x_r(t)\big] = \prod_{i,j>0} x_{ir+js}\big(C_{ij,rs}(-t)^i u^j\big), \quad r, s, ir + js \in \Phi. \tag{2}$$

Every carpet $\mathfrak{A}$ defines a *carpet subgroup* $\Phi(\mathfrak{A})$ generated by the subgroups $x_r(\mathfrak{A}_r)$, $r \in \Phi$. A carpet $\mathfrak{A}$ is called *closed* if its carpet subgroup $\Phi(\mathfrak{A})$ has no new root elements, i.e., if

$$\Phi(\mathfrak{A}) \cap x_r(F) = x_r(\mathfrak{A}_r).$$

Summing Theorems 3.1 and 4.1 from [2], we obtain the following result.

Theorem 1 ([2])**.** *Let F be an algebraic extension of an nonperfect field K of characteristic p and M be a group lying between Chevalley groups $\Phi(K)$ and $\Phi(F)$ of type $\Phi = B_l$ $(l \geqslant 2)$, C_l $(l \geqslant 2)$, F_4, G_2. Let $p = 2$ for $\Phi = B_l$, C_l, F_4 and $p = 3$ for $\Phi = G_2$. Then M is the product of the carpet subgroup $\Phi(\mathfrak{A})$ and some diagonal subgroup H_M normalizing $\Phi(\mathfrak{A})$. The carpet $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ is closed and*

$$\mathfrak{A}_r = \begin{cases} P, & \text{if } r \text{ is a short root,} \\ Q, & \text{if } r \text{ is a long root,} \end{cases}$$

for some additive subgroups P and Q of the field F with the conditions

$$R \leqslant P^p \leqslant Q \leqslant P \leqslant K.$$

Moreover, depending on the type of the Chevalley group $\Phi(K)$, the following refinements hold for the additive subgroups P and Q of the field F and the diagonal subgroup H_M:
a) if $\Phi = B_l$ and $l \geqslant 3$, then Q is a field;
b) if $\Phi = C_l$ and $l \geqslant 3$, then P is a field;
c) if $\Phi = F_4, G_2$, then both additive subgroups P and Q are fields and H_M is the unit subgroup.

Here, for any additive subgroup A of some field, by definition

$$A^p = \{t^p \mid t \in A\},$$

$$A^{-1} = \{0\} \cup \{t \in A \mid t^{-1} \in A\}.$$

For $\Phi = F_4, G_2$, the structure of the additive subgroups P and Q is clear, they are fields. The next proposition clarifies their structure for $\Phi = B_l, C_l$. For any root $r \in \Phi$ and any t from the multiplicative group F^* of the fields F by definition

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t),$$

$$h_r(t) = n_r(t)n_r(-1).$$

Proposition 1. *Let M, P and Q be the same as in Theorem 1 and $p = 2$. Then the additive subgroups P and Q satisfy the following conditions:*

A1) $1 \in P \cap Q$;

A2) $PQ \leqslant P$;

A3) $P^2Q \leqslant Q$;

A4) $P^2P \leqslant P$;

A5) $Q^2Q \leqslant Q$;

A6) $P^{-1} = P$;

A7) $Q^{-1} = Q$.

Moreover, P^2 and Q^2 are fields, P and Q are P^2-modules, and the subgroup M contains all diagonal elements of the form $h_r(tu)$, $t, u \in P \setminus \{0\}$ (respectively, $t, u \in Q \setminus \{0\}$), if r is a short root (respectively, if r is a long root).

Proof. Condition A1) follows from definition of the subgroup M. Conditions A2) and A3) follow from the commutator formula (2) and the carpet condition for the subgroup M. In [1, p. 535] it was established that for any $t \in P/\{0\}$ (respectively $t \in Q/\{0\}$) the polynomial ring $K[t]$ (respectively $K^2[t]$) lies in P (respectively, in Q). Hence, since the extension F/K is algebraic, we obtain equalities A6) and A7). For any short root r and any $t, u \in P/\{0\}$, equality A6) implies that $h_r(t)h_r(u) = h_r(tu) \in M$. Similarly, for any long root r and any $t, u \in Q/\{0\}$ from A7), we obtain the inclusion $h_r(tu) \in M$. Conjugating the subgroup $x_r(\mathfrak{A}_r)$, $r \in \Phi$, by these diagonal elements, we obtain the inclusions A4) and A5). It follows from A4) and A5) that P^2 and Q^2 are fields. Finally, from A3) and A4) we obtain that P and Q are P^2-modules. The proposition is proved. □

In [4, Sec. 7] for types B_l $(l \geqslant 2)$ and C_l $(l \geqslant 2)$, examples of subgroups P and Q from Theorem 1, one of which is not a field, and for the type $B_2 = C_2$ both of which are not fields, are given. Therefore, the inclusion of diagonal elements of the form $h_r(tu)$, $t, u \in P \setminus \{0\}$ (respectively, $t, u \in Q \setminus \{0\}$) if r is a short root (respectively, if r is a long root) into the subgroup M, despite the fact that the product tu may not lie in the subgroup P (respectively, in Q).

Any algebraic extension of a perfect field is perfectly [8, p. 217] and any finite field is perfect, so the results of the paper [1] say that there are no finite additive subgroups that are not fields that parameterize intermediate subgroups in groups of Lie type. The next proposition asserts that they do not exist even under weaker constraints.

Proposition 2. *If the characteristic of the field F is equal to 2 and its finite additive subgroup P satisfies the conditions* A1) *and* A4)*, then P is finite field.*

Proof. The inclusions A1) and A4) imply the inclusion $P^2 \leqslant P$, and since squaring is an isomorphism of any field of characteristic 2, taking into account the finiteness of P, we obtain the equality $P^2 = P$. Hence and again in view of A4), P is a ring and, therefore, a field, since any finite integral domain is a field. The proposition is proved. □

3. Groups lying between twisted Chevalley groups

Let A be a subset of the field F. The sets A^n and A^{-1} have the same meaning as in Section 2. The Steinberg group $G(F)$ of type ${}^nX_l$ is associated with an automorphism σ of order n of the fields F. By F_σ we denote the subfield of fixed elements of the automorphism σ. By definition, we set $\sigma(u) = \bar{u}$, $\overline{A} = \{\bar{u} \mid u \in A\}$ and $A_\sigma = A \cap F_\sigma$. The groups lying between the Steinberg groups $G(K)$ and $G(F)$, where F is an algebraic extension of a nonperfect field K of exceptional characteristic p, are described by the author in [3].

Theorem 2 ([3])**.** *Let M be a group lying between the Steinberg groups $G(K)$ and $G(F)$ of type ${}^2A_l$, $l \geqslant 4$, ${}^2D_l$, $l \geqslant 3$, ${}^2E_6$ or ${}^3D_4$, where F is an algebraic extension of an nonperfect field K of characteristic p, and $p = 2$ or 3 if $G(F)$ is of type ${}^3D_4$, and $p = 2$ otherwise. Then:*

1) If $G(F)$ is of type ${}^2A_l$, $l \geqslant 4$, ${}^2E_6$ or ${}^3D_4$, then $M = G(P)H_M$ for some intermediate subfield P, $K \subseteq P \subseteq F$, and some diagonal subgroup H_M normalizing the group $G(P)$.

2) If $G(F)$ is of type ${}^2D_l$, $l \geqslant 3$, then $M = G(P, Q)H_M$ for some diagonal subgroup H_M normalizing the group $G(P, Q)$ which is generated by intersections

$$M \cap x_R(K) = x_r(\mathfrak{A}_r),\ r \in {}^2D_l,$$

where

$$\mathfrak{A}_r = \begin{cases} P, & \text{if } r \text{ short root,} \\ Q, & \text{if } r \text{ long root,} \end{cases}$$

P and Q are subgroups of the additive group of the field F containing the subfield K and respectively K_σ, and they satisfy the following conditions: $PQ \subseteq P$, $P^2P \subseteq P$, $P^{-1} = \overline{P} = P$, $u\bar{u}, u+\bar{u} \in Q$ for all $u \in P$, and if $l \geqslant 4$, then Q is a field.

The next proposition axiomatizes the properties of the additive subgroups P and Q from Theorem 2 for $G(F)$ of type ${}^2D_l$, $l \geqslant 3$.

Proposition 3. *Let M, P and Q be the same as in Theorem 2 for $G(F)$ of type ${}^2D_l$, $l \geqslant 3$. Then the additive subgroups P and Q satisfy the following conditions:*

B1)$1 \in P \cap Q$, $P \nleqslant F_\sigma$ *и* $Q \leqslant F_\sigma$*;*

B2) $PQ \leqslant P$*;*

B3) $u\overline{u}t$, $\overline{u}v + u\overline{v} \in Q$ *for any* $u, v \in P$ *and* $t \in Q$*;*

B4) $P^2P \leqslant P$*;*

B5) $Q^2Q \leqslant Q$*;*

B6) $P^{-1} = P$*;*

B7) $Q^{-1} = Q$.

Next, we need the following technical lemma on algebraic extensions fields.

Lemma 1. *Let F be an algebraic extension of the field K, the field F has an automorphism σ, and F_σ and K_σ be centralizers of the automorphism σ in the fields F and K, respectively. Then the extension F_σ/K_σ is also algebraic.*

Proof. Let f be an arbitrary nonzero element from F_σ. Since the extension F/K is algebraic, there exists a smallest natural number m such that

$$f^m + k_{m-1}f^{m-1} + \cdots + k_1 f + k_0 = 0$$

for some simultaneously non-zero elements k_i from the field K. But then

$$f^m + \sigma(k_{m-1})f^{m-1} + \cdots + \sigma(k_1)f + \sigma(k_0) = 0.$$

Subtracting the second equality from the first, we obtain

$$(k_{m-1} - \sigma(k_{m-1}))f^{m-1} + \cdots + (k_1 - \sigma(k_1))f + (k_0 - \sigma(k_0)) = 0.$$

Hence, either for some $i \geqslant 1$ the difference $(k_i - \sigma(k_i))$ is nonzero, which is impossible due to the minimality of m, or all the differences $(k_i - \sigma(k_i))$ are zero, and then the element f is algebraic over the field K_σ, as required. The lemma is proved. □

Proposition 4. *Suppose that a field F of characteristic 2 has an automorphism σ of order 2, P and Q are its additive subgroups satisfying conditions* B1)–B7)*. Then the subgroups P and Q are fields, and $Q = P_\sigma$.*

Proof. Since $Q \leqslant F_\sigma$, the inclusion $Q \leqslant P_\sigma$ follows from B1) and B2). Since $P \nleqslant F_\sigma$, then there is an element $t \in P$ such that the sum $t+\bar{t}$ is nonzero and due to B3) lies in Q, and in force B7) $\dfrac{1}{t+\bar{t}} \in Q$. Hence and by virtue of B2) the element $u = \dfrac{t}{t+\bar{t}}$ lies in P, and $u+\overline{u} = 1$. Let $v \in P_\sigma$. Then by virtue of B3) the subgroup Q contains the element $u\overline{v} + \overline{u}v = uv + \overline{u}v = (u+\overline{u})v = v$. Therefore, $P_\sigma \leqslant Q$. So $Q = P_\sigma$. Now B2) implies the inclusion $QQ \leqslant Q$. Therefore, Q is a ring. By virtue of Lemma 1, the extension F_σ/K_σ is algebraic, and since the ring Q is enclosed between K_σ and F_σ, it is a field.

Let us show that P is a field. Since the extension F/K is algebraic, it suffices to show that for any two elements of P their product lies in P. So, let $u, v \in P$. If one of the elements u or v lies in P_σ, then by condition B2) and the equality $Q = P_\sigma$ proved above, we obtain the inclusion $uv \in P$. Let both u and v not lie in Q. Then they are the roots of the irreducible polynomials $x^2 + (u + \overline{u})x + u\overline{u}$ and, accordingly, $x^2 + (v + \overline{v})x + v\overline{v}$ of degree 2 over the field Q. The polynomial ring $Q[u]$ is a field and, by B2) and B4), lies in P. If $v \in Q[u]$, then $uv \in P$. If $v \notin Q[u]$, then the polynomial ring $Q[u, v]$ is an algebraic extension of degree 4 of the field Q and again, by B2) and B4), lies in P. Therefore, in any case, $uv \in P$. The proposition is proved. □

Combining Theorem 2 and Proposition 4, we obtain the following theorem, which gives a uniform and standard description of intermediate subgroups for Steinberg groups over nonperfect fields in exceptional characteristics.

Theorem 3. *Let M be a group lying between the Steinberg groups $G(K)$ and $G(F)$ of type ${}^2A_l$, $l \geqslant 4$, ${}^2D_l$, $l \geqslant 3$, ${}^2E_6$ or ${}^3D_4$, where F is an algebraic extension of an nonperfect field K of characteristic p, and $p = 2$ or 3 if $G(F)$ is of type ${}^3D_4$, and $p = 2$ otherwise. Then $M = G(P)H_M$ for some intermediate subfield P, $K \subseteq P \subseteq F$, and some diagonal subgroup H_M normalizing the group $G(P)$.*

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О парах аддитивных подгрупп, ассоциированных с промежуточными подгруппами групп лиева типа над несовершенными полями

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Аннотация. Ранее (Труды ИММ УрО РАН, 19(2013), № 3) автор описал группы, лежащие между скрученными группами Шевалле $G(K)$ и $G(F)$ типа ${}^2A_l$, ${}^2D_l$, ${}^2E_6$, ${}^3D_4$ в случае, когда большее поле F является алгебраическим расширением меньшего несовершенного поля K исключительной характеристики для группы $G(F)$ (характеристики 2 и 3 для типа ${}^3D_4$ и только 2 для остальных типов). Оказалось, что кроме, быть может, типа ${}^2D_l$, такие промежуточные подгруппы стандартны, то есть они исчерпываются группами $G(P)H$ для некоторого промежуточного подполя P, $K \subseteq P \subseteq F$ и диагональной подгруппы H, нормализующей группу $G(P)$. В данной заметка установлено, что промежуточные подгруппы являются стандартными и для типа ${}^2D_l$.

Ключевые слова: группы лиева типа, несовершенное поле, промежуточные подгруппы, ковер аддитивных подгрупп.

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Features in the Resonance Behavior of Magnetization in Arrays of Triangular and Square Nanodots

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Abstract. Collective modes of the gyrotropic motion of a magnetic vortex core in ordered arrays of triangular and square ferromagnetic film nanodots have been theoretically investigated. The dispersion relations have been derived. The dipole–dipole interaction of the magnetic moments of the magnetic vortex cores of elements has been taken into account in the approximation of a small shift from the equilibrium position. It is shown that the effective rigidity of the magnetic subsystem of triangular elements is noticeably higher than that of the subsystem of square elements of the same linear sizes. The potential application of the polygonal film nanodisks as nanoscalpels for noninvasive tumor cell surgery is discussed.

Keywords: differential equation, Cauchy problem, split, stability, convergence.

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Introduction

Ordered arrays and suspensions of ferromagnetic nanodots have a great potential for application in new spintronic devices and noninvasive cell nanosurgery of malignant tumors in medicine [1, 2]. The requirements for the magnetic moment of a nanoobject used as a magnetic-filed-driven nanoscalpel for cell destruction are contradictory: an increase in the magnetic moment facilitates the cell destruction, but is accompanied by an undesirable effect of agglomeration

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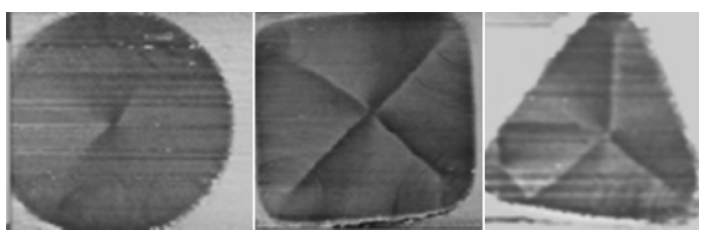
of nanoparticles in a suspension. To resolve this contradiction, it is necessary to search for a compromise situation where the initial magnetic moment of a nanoscalpel is small, but significantly increases upon switching on a magnetic field at the instant of destruction.

The aim of this work, along with studying the features in the resonance behavior of the magnetization of ordered arrays of triangular and square nanodots, is to clarify whether the problem of nanoparticle agglomeration in a cell surgery suspension can be solved by changing the nanoparticle geometry.

The point is that the disk is not an optimal nanoscalpel configuration. In ferromagnetic nanoparticles suspended in a liquid, the magnetic flux tends to close inside an element with the formation of magnetic vortices. In a disk-shaped particle, a single vortex is formed (Fig. 1a), in which the magnetization is circularly oriented in the plane at the periphery of a particle and is out-of-plane at its center (the vortex core). The magnetic moment of an object is induced mainly by the core. The external field causes a reversible displacement of the core at the almost invariable value of the magnetic moment.

For the analysis, we chose square and triangle configurations as presumably promising. As in disks, the magnetic flux in square and triangular nanodots is almost completely closed within an element. A quasi-vortex with a core at the center is formed (Fig. 1 b and c). A core is the magnet region with a size of $\delta_0 \approx 10$ nm in which the magnetization is out-of-plane (perpendicular to the magnet plane) due to the competition between the exchange and demagnetizing energies: $\delta_0 \approx \sqrt{A/(\mu_0 M_S)}$ (A is the exchange constant and M_S is the saturation magnetization).



a) b) c)

Fig. 1. Equilibrium magnetic structures of (a) circular, (b) square, and (c) triangular permalloy film nanospots [3]. The quasi-vortices of the square and triangular spots represent closed domain structures with a vortex core at the center

Under the action of any factors (external fields, spin-polarized currents, anisotropy field gradients, stresses, etc.), a magnetic vortex moves along a curvilinear trajectory, being driven by the Magnus forces [4–8]. In analytical calculations, the well-proven rigid vortex approximation is often used. In the model used, it is assumed that the magnetization configuration in the region covered by the vortex distribution remains unchanged upon displacement of the core from the equilibrium position. In this case, the vortex dynamics should be described by the method of collective variables, which are the core coordinate and velocity. Then, the equations of motion take the well-known form (the Thiele equation [9]):

$$\mathbf{G} \times \mathbf{v} + \widehat{D}\mathbf{v} + \nabla W = 0. \tag{1}$$

Here $\mathbf{G}$ is the gyrovector [7, 9, 10], perpendicular to the magnet plane, $\mathbf{v}$ is the core velocity, W is the potential energy of the vortex, and $\widehat{D}$ is the tensor of effective coefficients of the force of friction. The potential energy W includes the terms responsible for its growth due to the exit of the magnetic subsystem of elements from the metastable state (the shift of the core from the equilibrium position), the terms describing the pairing energy of interaction between the magnetic moments of different elements, and the terms describing external factors (fields).

Below, we consider specific equations for square and triangular elements using the models that are simple, but make it possible to compare the resonance behavior of disks with different shapes and the effective rigidity of their magnetic subsystems.

1. Effective potential energy of a polygonal ferromagnetic element

In this Section, we estimate the increment of the potential energy W of magnetic elements upon displacement of the magnetic vortex core from the equilibrium position. A rigorous analytical solution to this problem faces great computational difficulties. Therefore, the numerical modeling is frequently used (see, for example, [11–14]). This calculation is necessary to determine the parameters of vortex motion using Eq. (1).

Fig. 2. presents the models of ferromagnetic elements. Each element has a domain magnetization structure with a vortex at the center of a magnet. To perform the estimation, we assume that, upon displacement of the core, the energy of the magnetic subsystem changes mainly due to an increase in the energy of the magnetostatic interaction of domains. We calculate this energy using the dipole–dipole approximation. It can be shown that the energy of the magnetostatic interaction of uniformly magnetized triangular regions (domains) can be approximately presented as the energy of interaction between dipoles located in the medians at a distance of one third of the height from the base to which the magnetization is parallel (quasi-dipoles). In Fig. 2, these positions are shown by closed circles with arrows. The value of magnetic moment $\mathbf{M}$ of each dipole is determined by the magnetic moment of the corresponding magnetic domain.

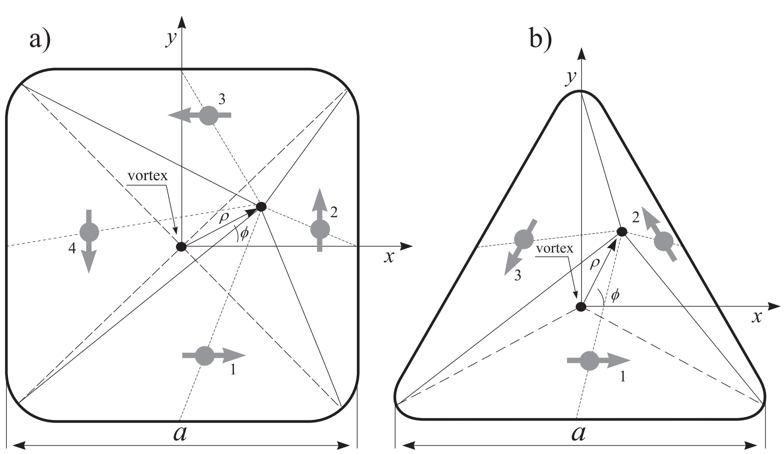


Fig. 2. Models of ferromagnetic nanoelements in the form of a square and a regular triangle

When the core is shifted from the element centers, the domain structure configuration changes, which is reflected in a change in the domain size and, consequently, in the energy of the interaction between domains. This process can be considered as a variation in the energy of interaction between magnetic dipoles (quasi-dipoles), which, in this case, shifted and changed the absolute values of their magnetic moments.

Let us enumerate domains as shown in Fig. 2. For the energy of interaction between the n-th and m-th dipoles, we write

$$W_{nm} = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{M}_n\mathbf{M}_m}{r_{nm}^3} - 3\frac{(\mathbf{M}_n\mathbf{r}_{nm})(\mathbf{M}_m\mathbf{r}_{nm})}{r_{nm}^5}\right). \tag{2}$$

Here $\mathbf{r}_{nm}$ is the radius vector connecting the dipoles and $\mathbf{M}_n$ and $\mathbf{M}_m$ are their magnetic moments. For the magnetic moments, we can write

$$|\mathbf{M}_n| \approx M_S V_n = \frac{1}{2}M_S b S_n = \frac{1}{2}M_S b a h_n. \tag{3}$$

Here M_S is the saturation magnetization, V_n is the domain (prism) volume, S_n is the domain square (triangle), h_n is the altitude of a triangular domain plotted from the core position to the outer side of the element, b – is the element thickness, and a is the side of a regular polygon.

The vortex core position is specified by length ρ of its radius vector and azimuth angle ϕ. Obviously, the dipole positions and absolute values of the magnetic moments are determined by the ρ and ϕvalues. Solving a simple geometric problem, we can express the parameters included in energy (2) through ρ and ϕ. The results of the calculation are given in Tab. 1.

Table 1. Parameters of quasi-dipoles as functions of ρ and ϕ according to the numeration in Fig. 2. The absolute values of the magnetic moments are calculated using Eq. (3)

Element shape	Quasi-dipoles Cartesian coordinates	Quasi-dipole magnetic moment
Square	$x_1 = \frac{1}{3}\rho\cos(\phi)$ $x_2 = \frac{1}{3}(a+\rho\cos(\phi))$ $x_3 = x_1$ $x_4 = -\frac{1}{3}(a-\rho\cos(\phi))$ $y_1 = -\frac{1}{3}(a-\rho\sin(\phi))$ $y_2 = \frac{1}{3}\rho\sin(\phi)$ $y_3 = \frac{1}{3}(a+\rho\sin(phi)$ $y_4 = y_2.$	$M_1 = \frac{1}{2}M_S ba\left(\frac{a}{2}+\rho\sin(\phi)\right)$ $M_2 = \frac{1}{2}M_S ba\left(\frac{a}{2}-\rho\cos(\phi)\right)$ $M_3 = \frac{1}{2}M_S ba\left(\frac{a}{2}-\rho\sin(\phi)\right)$ $M_4 = \frac{1}{2}M_S ba\left(\frac{a}{2}+\rho\cos(\phi)\right)$
Triangle	$x_1 = \frac{1}{3}\rho\cos(\phi)$ $x_2 = \frac{1}{3}\left(\frac{a}{2}+\rho\cos(\phi)\right)$ $x_3 = -\frac{1}{3}\left(\frac{a}{2}-\rho\cos(\phi)\right)$ $y_1 = -\frac{1}{3}\left(\frac{a\sqrt{3}}{3}-\rho\sin(\phi)\right)$ $y_2 = \frac{1}{3}\left(\frac{a\sqrt{3}}{6}+\rho\sin(\phi)\right)$ $y_3 = y_2$	$M_1 = \frac{1}{2}M_S ba\left(\frac{a\sqrt{3}}{6}+\rho\sin(\phi)\right)$ $M_2 = \frac{1}{4}M_S ba\left(\frac{a\sqrt{3}}{3}-\sqrt{3}\rho\cos(\phi)-\rho\sin(\phi)\right)$ $M_3 = \frac{1}{4}M_S ba\left(\frac{a\sqrt{3}}{3}+\sqrt{3}\rho\cos(\phi)-\rho\sin(\phi)\right)$

Taking into account the data from Tab. 1, we obtain the square element energy

$$W_{sq}(\rho) = W_{12}+W_{13}+W_{14}+W_{23}+W_{24}+W_{34} = \frac{27}{128}\frac{\mu_0 M_S^2 b^2}{\pi a}\left(\rho^2 - \frac{a^2}{2}\right), \tag{4}$$

and the triangular element energy

$$W_{tr}(\rho) = W_{12} + W_{13} + W_{32} = \frac{15}{2}\frac{27}{128}\frac{\mu_0 M_S^2 b^2}{\pi a}\left(\rho^2 - \frac{a^2}{3}\right). \tag{5}$$

Importantly, according to the calculation, the energies of both a square and triangular element are independent of the azimuth angle; i.e. the potential of the vortex core has a cylindrical symmetry. This is a safety signal, since the potential does not reflect the shape of a magnet. This is true until the area covered by the vortex reaches the boundary of a magnet. If the shift is so large that the core appears in the vicinity of the boundary ($\rho \approx a$), then the vortex distribution will lead to the occurrence of noticeable magnetostatic charges on the lateral surface. The contribution of the terms related to these charges to the magnetostatic energy will ensure the dependence of the total potential energy on the lateral surface shape. In this case, the symmetry of a magnet will be reflected in the functional dependence of the energy on azimuth angle ϕ [15–18]. Thus, Eqs. (4) and (5) are valid as long as the core shift does not lead to the exit of the magnetization from the lateral surface of a magnet.

The analytical form of Eqs. (4)–(5) allows us to estimate the initial susceptibilities of nanoelements. To do that, we obtain the dependence of the total magnetic moment of an element on applied dc magnetic field H. For simplicity, we assume that the magnetic field is parallel to the polygon side (directed along the OX axis for both the square and the triangle).

When the external field is applied, the energy of the magnetic subsystem should be added with the Zeeman energy

$$W_{tot} = W(\rho) - \mathbf{MH}. \tag{6}$$

Switching on the field $\mathbf{H}$, along the positive direction of the x axis (see Fig. 2), leads to the displacement of the core from the magnet center in the positive direction of the y axis ($\phi = \pi/2$, $q = +1$). or in the opposite direction ($\phi = -\pi/2$) if the magnetization in elements rotates clockwise ($q = -1$)rather than counterclockwise.

The total magnetic moment of an element can be calculated as follows (see Fig. 3). A domain with the magnetization co-directed with the field grows by the expense of domains the magnetization of which has the energetically unfavorable direction. Then, for the magnetic moment of a square element, obtain

$$|\mathbf{M}| = \frac{1}{2}M_S ba\left(\frac{a}{2} + \rho\right) - \frac{1}{2}M_S ba\left(\frac{a}{2} - \rho\right) = M_S ba\rho. \tag{7}$$

The search for the equilibrium shift ρ is conventionally made with regard to Eq. (7):

$$\frac{dW_{tot}}{d\rho} = \frac{54}{128}\frac{\mu_0 M_S^2 b^2}{\pi a}\rho - M_S baH = 0. \tag{8}$$

Then, for the equilibrium position of the core, obtain

$$\rho = \frac{128}{54}\frac{\pi a^2}{\mu_0 M_S b}H. \tag{9}$$

Taking into account (9), for magnetic moment (7) obtain

$$M = \chi_{sq}H, \qquad \chi_{sq} = \frac{128}{54}\frac{\pi a^3}{\mu_0}. \tag{10}$$

The calculation for triangular elements, analogously to (7)–(10) yields

$$\chi_{tr} = \frac{24}{135}\frac{\pi a^3}{\mu_0}. \tag{11}$$

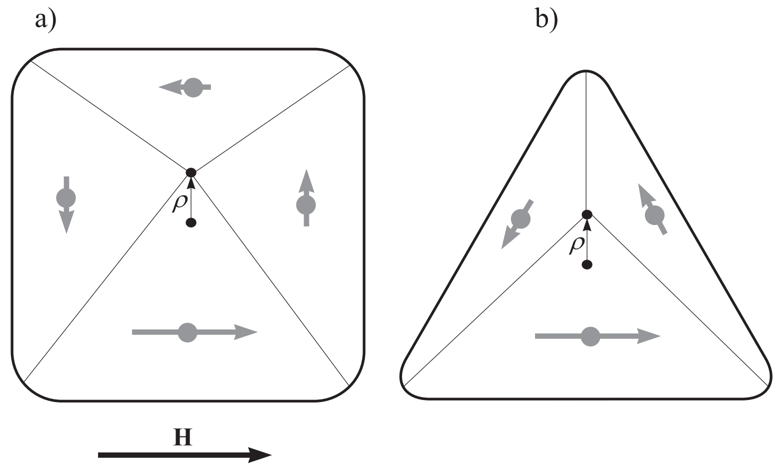


Fig. 3. Occurrence of the magnetic moment in (a) a square and (b) triangular element

2. Collective modes in an ordered array of polygons

The above analytical expressions for energy $W(\rho)$ allow us to analyze the gyrotropic motion of vortex cores in the arrays of elements arranged in a certain order, for example, forming a square lattice (see Fig. 4 for triangular elements).

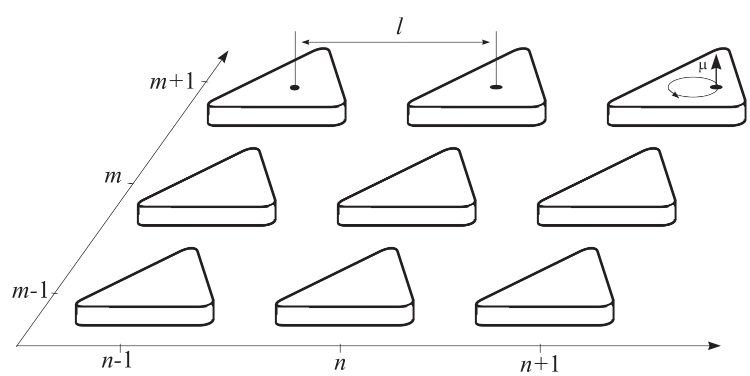


Fig. 4. Example of an ordered array of triangular elements. The distance between the centers of elements is l. Example of orientation of the magnetization of the core and trajectory of its motion

Let us continue the discussion of the case when the vortex cores in elements remain fairly distant from the element edges during the motion induced by, e.g., an ac magnetic field. In this case, the magnetic subsystems of triangles interact magnetostatically only due to the presence of a magnetic moment of cores at the center of the vortices. At the vortex center, the magnetization is perpendicular to the element surface and, depending on the p polarity, can be conditionally oriented upward ($p = 1$) or downward ($p = -1$). The example of the trajectory of the core motion in the centrosymmetric potential is shown in Fig. 4 for one of the elements. The direction of the gyrotropic rotation is determined by the sign of the product of polarity and chirality: $q = \pm 1$.

The energy of interaction of the magnetic moments of the cores of all elements must be included in the total energy of a system. The energy of the pairwise interaction of the core of some selected element with polarity p and another element with polarity p_{nm}, which belongs to the column with number n in the row with number m can be presented in the form

$$W_{nm}^{(dip)} = pp_{nm}\frac{\mu_0}{4\pi}\frac{\mu^2}{r_{nm}^3}. \tag{12}$$

Here $r_{nm} = \sqrt{(nl + (x_{nm} - x))^2 + (ml + (y_{nm} - y))^2}$ is the distance between the cores, μ is the magnetic moment of the core vortex, x and y are the core shifts along the x and y axes of the chosen element, respectively, and x_{nm} and y_{nm} are the core shifts along the x and y axes of the second element with coordinates (n, m), respectively. The summation is made over all n and m except for the case $n = m = 0$. For the total energy of the interaction between the chosen element and the rest matrix, we write

$$W^{(dip)} = \frac{p\mu_0\mu^2}{4\pi}\sum_{n,m}\frac{p_{nm}}{r_{nm}^3}. \tag{13}$$

Then, taking into account (4) or (5), the force acting on the core is

$$\begin{aligned}\mathbf{f} = -\nabla(W(\rho) + W^{(dip)}) = \mathbf{e}_x\left(-\kappa x + \xi p\sum_{n,m}\frac{p_{nm}\left(4n^2 - m^2\right)(x - x_{nm})}{(n^2 + m^2)^{\frac{7}{2}}}\right) + \\ + \mathbf{e}_y\left(-\kappa y + \xi p\sum_{n,m}\frac{p_{nm}\left(4m^2 - n^2\right)(y - y_{nm})}{(n^2 + m^2)^{\frac{7}{2}}}\right).\end{aligned} \tag{14}$$

The designations used here are $\kappa = 54\mu_0 M_S^2 b^2/(128\pi a)$ for squares or $\kappa = 405\mu_0 M_S^2 b^2/(128\pi a)$ for triangles; $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors of the x and y axes, respectively; and $\xi = 3\mu_0\mu^2/(4\pi l^5)$. In addition, the relations $\rho^2 = x^2 + y^2$, $x, y, x_{nm}, y_{nm} \ll l$. were used. It should be noted that Eqs. (14) are valid at the symmetric distribution of the polarities of elements in an infinite array. For instance, at $p_{nm} = \pm 1$, regardless of n and m, we have either $p_{nm} = (-1)^{n+m}$ or $p_{nm} = (-1)^n$, etc., since, in these cases, the core of a selected element is not affected by constant forces from the side of its neighbors and has an equilibrium position at the center of this element.

According to Eq. (14), we write equation of motion (1) in the components

$$\begin{cases} Gv_y + Dv_x + \left(\kappa + \xi\sum\limits_{n,m}\frac{pp_{nm}\left(4n^2 - m^2\right)}{(n^2 + m^2)^{\frac{7}{2}}}\right)x - \xi\sum\limits_{nm}\frac{pp_{nm}\left(4n^2 - m^2\right)x_{nm}}{(n^2 + m^2)^{\frac{7}{2}}} = 0, \\ -Gv_x + Dv_y + \left(\kappa + \xi\sum\limits_{n,m}\frac{pp_{nm}\left(4m^2 - n^2\right)}{(n^2 + m^2)^{\frac{7}{2}}}\right)y - \xi\sum\limits_{nm}\frac{pp_{nm}\left(4m^2 - n^2\right)y_{nm}}{(n^2 + m^2)^{\frac{7}{2}}} = 0. \end{cases} \tag{15}$$

It is reasonable to search for solutions of system of equations (15) in the form of waves

$$x_{nm}(\mathbf{r}, t) = Xe^{i(\mathbf{kr} - \Omega t)}, \qquad y_{nm}(\mathbf{r}, t) = iYe^{i(\mathbf{kr} - \Omega t)}. \tag{16}$$

Here $i = \sqrt{-1}$ is the imaginary unit, $\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y$ is the wave vector, $\mathbf{r} = nl\mathbf{e}_x + ml\mathbf{e}_y$ is the radius vector connecting the centers of the chosen element and the element for the column with number n and the row with number m, and X and Y are the amplitudes of the core shift along the x and y axes, respectively.

Substituting the trial solutions of (16) into system of equations (15), we obtain the system of algebraic equations of variables X and Y

$$\begin{cases} G\Omega Y - (iD\Omega - \kappa - \xi S_x(k_x, k_y))\,X = 0, \\ G\Omega X - (iD\Omega - \kappa - \xi S_y(k_x, k_y))\,Y = 0. \end{cases} \tag{17}$$

Here, we used the designations

$$\begin{aligned} S_x(k_x,k_y) &= \sum_{nm} \frac{4n^2-m^2}{(n^2+m^2)^{\frac{7}{2}}}\left(1-\cos(k_x nl)\cos(k_y ml)\right), \\ S_y(k_x,k_y) &= \sum_{nm} \frac{4m^2-n^2}{(n^2+m^2)^{\frac{7}{2}}}\left(1-\cos(k_x nl)\cos(k_y ml)\right). \end{aligned} \tag{18}$$

Equating the determinant built on the coefficients at X and Y to zero, we obtain the quadratic equation of variable Ω, the solution of which is

$$\Omega = \pm\omega - i\delta, \tag{19}$$

where

$$\omega = \left[\frac{(\kappa+\xi S_x(k_x,k_y))\,(\kappa+\xi S_y(k_x,k_y))}{G^2+D^2} - \frac{\kappa^2 D^2}{(G^2+D^2)^2}\right]^{\frac{1}{2}}, \tag{20}$$

$$\delta = \frac{\kappa D}{G^2+D^2}. \tag{21}$$

The real part of Eq. (19) — ω — determines the angular velocity of the vortex core rotation (the gyrotropic frequency). The imaginary part δ determines the effective damping parameter.

In the long-wavelength limit $(k_x, k_y \to 0)$, the sums turn to zero and we obtain the well-known expressions for the frequency and damping parameter in a single element [19,20]. Indeed, in this case, the cores of all disks of the array move synchronously and the dipole–dipole interaction forces acting on a selected element are compensated pairwise from neighbors with numbers n and $-n$, similar to m and $-m$. In other words, the interaction between the elements in this case does not affect the gyrotropic frequency. In other cases, Eq. (20) specifies the dispersion law $\omega(k_x, k_y)$.

When creating ordered arrays of ferromagnetic elements, nature chooses the energetically most favorable state, which corresponds to a checkboard pattern of the polarity signs, i.e., the state given by the expression $p_{nm} = (-1)^{n+m}$. In the opposite limit (the least favorable distribution), the polarities of all elements have the same sign (the magnetization at the center of all vortices is identically directed); i.e., p is independent of numbers n and m. Such a state can be created by applying a magnetic field exceeding the saturation field perpendicular to the array plane. The dispersion surfaces for these extreme cases are shown in Fig. 5. The linear sizes of the cores are, as a rule, smaller than the sizes of elements; therefore, the inequality $\xi \ll \kappa$ can be considered valid.

Basing on Fig. 5, we should emphasize an important feature. At sufficiently large ξ values, in the model with the alternating p signs, there are the wavenumber ranges in which ω becomes imaginary, which is indicative of the aperiodic motion of the cores. This can be qualitatively explained as follows. The alternation of the polarity sign leads to a decrease in the effective rigidity of the potential in which the core moves. This obviously follows from Eqs. (15) or (20), where the rigidity κ is added with the correction the value and sign of which depend on the values of polarities p_{nm}. At different polarity signs, the attraction between the cores of neighboring elements arises (on the contrary to the repulsion at the same p), which competes with the restoring force determined by the rigidity κ. of the magnetic subsystem. As a result, the restoring force acting on the core becomes so small that it leads to the imaginary ω value (20).

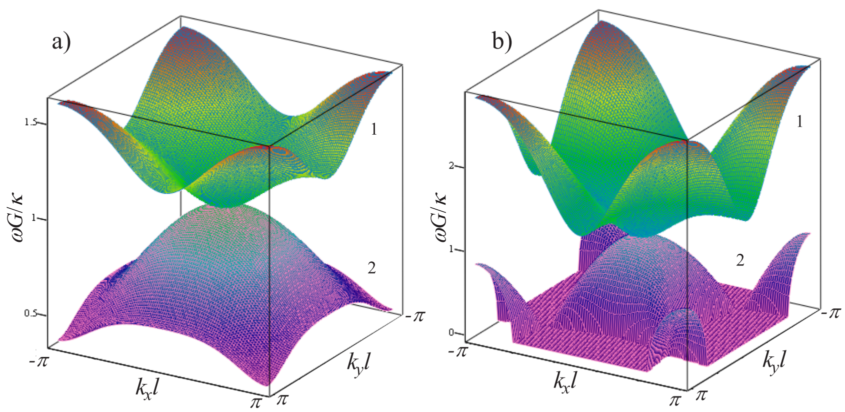


Fig. 5. Dispersion laws $\omega(k_x, k_y)$ based on Eq. (20). The frequency in units of κ/G is laid off along the vertical axis. (a) The case of $\delta = 0.1\omega$ and $\xi = 0.1\kappa$ and (b) the case of $\xi = 0.3\kappa$. The surfaces with number 1 correspond to the case of the same polarity of vortices of all elements and the surfaces with number 2, to the case of a checkboard pattern of the polarity signs

3. Discussion

To sum up the comparison of the properties of arrays of square and triangular elements, we can emphasize some circumstances that can play a significant role in selecting the objects that are candidates for use as functional tools for medicine or various spintronic devices. First of all, it should be noted that the effective rigidities of the magnetic subsystems of square and triangular elements of the same linear sizes are significantly different. This feature is responsible for the difference between the initial susceptibilities of the arrays of these elements by almost an order of magnitude. According to Eqs. (10) and (11), we have

$$\frac{\chi_{sq}}{\chi_{tr}} = \frac{128}{54}\frac{135}{24} \approx 13. \tag{22}$$

This phenomenon was confirmed in our experiment. The arrays of square, triangular, and circular elements for the experiment were formed by explosive lithography from a continuous film using high-vacuum thermal sputtering of an 80HXC alloy onto a silicon substrate coated with a photoresist. The required morphology was formed on the substrate surface using an AZ Nlof 2035 negative photoresist. The magnetic structure and morphology of the obtained elements were examined on a Veeco MultiMode NanoScope IIIa scanning probe microscope.

The surface morphology of the investigated arrays is shown in Fig. 6. The images were obtained on an atomic force microscope operating in a tapping mode [21, 22]. A cantilever was brought to the surface so that in the lower half-period of the oscillations, the sample surface was touched. The interaction of the cantilever with the surface in the tapping mode was ensured by the van der Waals forces, which, at the instant of touching, are added with the elastic force acting on the cantilever from the surface side.

Fig. 7 shows a typical scan of the magnetic structure obtained using a two-pass technique in the cantilever frequency modulation mode. The return passage height is $z_0 = 50$ nm. The obtained images allow us to conclude that, in square elements, an equilibrium structure with the

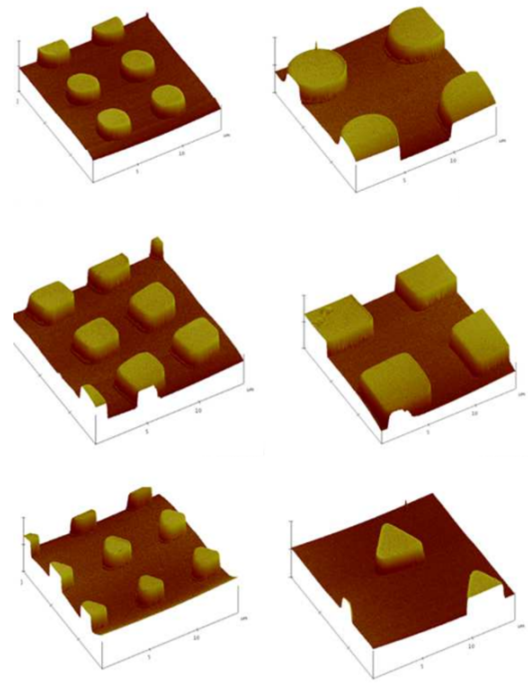


Fig. 6. Atomic force microscopy images of the surface relief of the arrays of different elements. The linear sizes of the elements are the same (the diameter is 3 micrometers and the thickness is 12 nm)

closed magnetic flux (quasi-vortex) of four domains separated by 90-degree Neel walls is most often implemented, while in triangular elements, the structure is formed by three domains with 120-degree walls. At the center, at the intersection of the diagonals, there is a core similar to that at the center of circular elements.

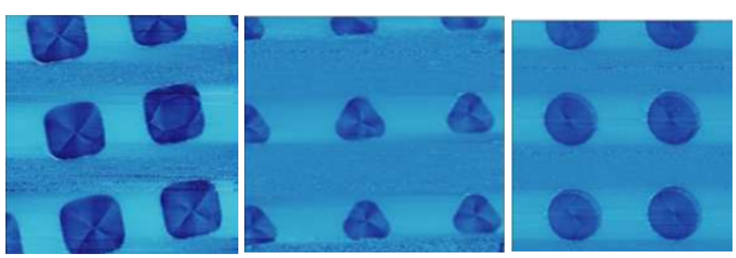


Fig. 7. Atomic force microscopy images of the magnetic structure of different elements

The hysteresis loops of the arrays of squares and triangles were obtained on a NanoMOKE facility. The result is presented in Fig. 8. It can be seen that the slopes of the initial portions of the curves for square and triangular elements differ by more than an order of magnitude, which is in good agreement with estimate (22).

Such a significant dependence of the effective rigidity of the magnetic subsystems on the element shape should be taken into account in designing the basic components, for example, storage devices, field sensors, etc. For medicine, square-shaped elements seem to be preferable, since they more readily respond to an external field, which simplifies the control of their motion in suspensions and biological liquids. In zero external field, they have a weak total magnetic moment, since the flux is closed inside an element and the magnetic moment of the central part of the vortex is small due to the small size of the core.

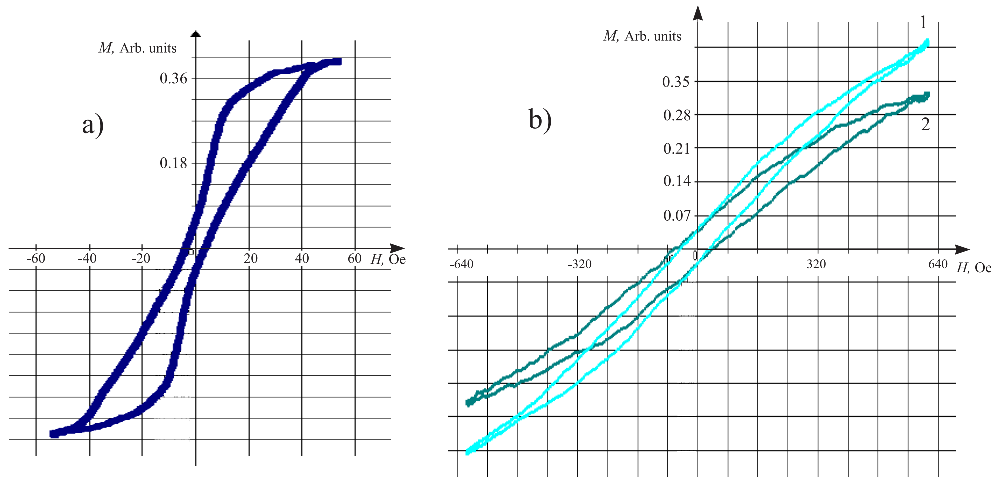


Fig. 8. Hysteresis loops obtained on an array of (a) square and (b) triangular elements. The loop in (a) was obtained by applying a field along the square side. Plot 1 in (b) was obtained by applying a field along the triangle edge and plot 2, by applying a field along the triangle height

In addition, a significant difference in the rigidity of the system will affect the resonance properties of the arrays. The frequencies of the gyrotropic motion of arrays of square elements were about 400 MHz 400 МГц [15], while for triangular elements one should expect multiply higher frequencies. Features of the collective modes for different shapes of elements will be objects of further investigations.

As intermediate results, in this work, the analytical expressions for the potential energy of the magnetic vortex core and the dispersion relations were obtained.

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Особенности резонансного поведения намагниченности в массивах треугольных и квадратных наноточек

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Аннотация. Теоретически исследуются коллективные моды гиротропного движения ядра магнитного вихря в упорядоченных массивах ферромагнитных пленочных наноточек треугольной и квадратной форм. Получены дисперсионные соотношения. Учитывается диполь-дипольное взаимодействие магнитных моментов ядер магнитных вихрей элементов в приближении малого смещения от положения равновесия. Показано, что эффективная жесткость магнитной подсистемы в треугольных элементах заметно больше, чем в квадратных при одинаковых линейных размерах. Обсуждается перспектива использования пленочных нанодисков-многоугольников в качестве "наноскальпелей" для неинвазивной клеточной хирургии опухолей.

Ключевые слова: дифференциальные уравнения, задача Коши, расщепление, устойчивость, сходимость.

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A List of Integral Representations for Diagonals of Power Series of Rational Functions

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Abstract. In this paper we present integral representations for the diagonals of power series. Such representations are obtained by lowering the multiplicity of integration for the previously known integral representation. The procedure for reducing the order of integration is carried out in the framework of the Leray theory of multidimensional residues. The concept of the amoeba of a complex analytic hypersurface plays a special role in the construction of new integral representations.

Keywords: multidimensional power series, complex integral, integral representation, amoeba, Taylor series, diagonal of a power series.

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Introduction

A range of problems associated with branching of parametric integrals is concerned with a study of the diagonals of power series [1, 2] and [3]. It should be noted that much earlier the concept of the diagonal of a power series was used by A. Poincare [4] to study the anomalies of planetary motion.

The diagonal of a Laurent power series

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \tag{1}$$

is defined as the generating function of a subsequence of coefficients $\{c_\alpha\}_{\alpha \in L}$ numbered by elements α of some sublattice $L \subset \mathbb{Z}^n$ (see [1] and [5]). Such diagonals are called complete. Diagonals are graded according to the dimension (rank) of the sublattice.

Following [1], we describe the specifics of the problem on the properties of the diagonals of series for rational functions of n variables

$$F(z) = \frac{P(z)}{Q(z)} = \frac{P(z_1, \ldots, z_n)}{Q(z_1, \ldots, z_n)}, \tag{2}$$

where P and Q are irreducible polynomials. Consider an arbitrary Laurent series for F centered at zero:

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}.$$

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It is known that such a series converges in domain $\mathrm{Log}^{-1}(E)$, where E is a connected component of the complement $R^n \backslash A_Q$ of amoeba of the denominator Q [6]. Recall that *amoeba* A_Q of the polynomial Q or of the algebraic hypersurface

$$V = \{z \in (\mathbb{C}\backslash 0)^n : Q(z) = 0\}$$

is called the image of V under the mapping $\mathrm{Log} : (\mathbb{C}\backslash 0)^n \to \mathbb{R}^n$, defined by the formula

$$\mathrm{Log} : (z_1, \dots, z_n) \to (\log|z_1|, \dots, \log|z_n|).$$

Sometimes instead of the designation A_Q we write A_V. According to the result of the article [7], there is an injective order function

$$\nu : E \to \mathbb{Z}^n \bigcap N_Q,$$

mapping each connected component E of the complement $\mathbb{R}^n \backslash A_Q$ to integer vector $\nu = \nu(E)$, belonging to the Newton polytope N_Q of the polynomial Q. Thus, all connected components can be indexed as $\{E_\nu\}$, where ν runs over a some subset of integer points from N_Q. For example, the Taylor series of a rational function $\dfrac{P}{Q}$, $Q(0) \neq 0$ converges in the component E_0.

Let us consider in more detail the p-dimensional diagonal of the series (1). Consider a p-dimensional sublattice $l \subset L$, with a basis $q^{(1)}, \dots, q^{(p)}$. We assume that this basis can be extended of L by $n-p$ integer vectors $q^{(p+1)}, \dots, q^{(n)}$ (this assumption equivalent to say that the totality of all $(p \times p)$-minors of the matrix $\tilde{A} = (q^{(1)}, \dots, q^{(p)})$ are mutually prime) (see [8] or [9, Proposition 4.2.13]). Obviously the matrix

$$A = (q^{(1)}, \dots, q^{(n)})$$

is unimodular, and we can assume it's determinant equals 1. Directions $q^{(1)}, \dots, q^{(p)}$ define a diagonals subsequence $\{c_{lq}\}_{l \in \mathbb{Z}_+^p}$, where $l \cdot q$ means the product of the $(1 \times p)$-matrix l and $(p \times n)$-matrix $\tilde{A}$: $lq = l_1 q^{(1)} + \dots + l_p q^{(p)}$.

The generating function

$$d_q(t) = \sum_{l \in \mathbb{Z}_+^p} c_{l_q} t_1^{l_1} \cdots t_p^{l_p}$$

of the subsequence $\{c_{lq}\}_{l \in \mathbb{Z}_+^n}$ is called *the one-sided q-diagonal* of the series (1).

We assume that the denominator Q in (2) is not zero at $z = 0$, so the origin $O \in \mathbb{Z}^n$ belongs to the Newton polytope and there is nonempty component E_0 of $\mathbb{R}^n \setminus A_Q$. We start by the Laurent series for the function (1) in $\mathrm{Log}^{-1}(E_0)$, which is in fact the Taylor series of (1) at $z = 0$. It is not hard to prove the following. If $\rho \in \mathrm{Log}^{-1}(E_0)$, then $d_q(t)$ admits the integral representation

$$d_q(t) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\rho} F(z) \frac{z^{q^{(1)}} \cdots z^{q^{(p)}}}{(z^{q^{(1)}} - t_1) \dots (z^{q^{(p)}} - t_p)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n}, \tag{3}$$

where z^q is a monomial $z_1^{q_1} \dots z_n^{q_n}$, and cycle

$$\Gamma_\rho = \{z \in \mathbb{C}^n : |z_1| = e^{\rho_1}, \dots, |z_n| = e^{\rho_n}\}$$

is chosen so that

a) poles of $F(z)$ don't intersect the closed polydisc

$$\overline{U_\rho} = \{z \in \mathbb{C}^n : |z_1| \leqslant e^{\rho_1}, \dots, |z_n| \leqslant e^{\rho_n}\};$$

b) parameters $t = (t_1, \dots, t_p)$ satisfy the inequalities $|t_i| < e^{\langle q_i, \rho \rangle}$, $i = 1, \dots, p$.

Integration loop Γ_ρ is a preimage $Log^{-1}\rho$ of the point ρ from the connected component E_0 of the amoeba A_Q complements. Here we prove that the integral which represent the diagonal $d_q(t)$ admits a decrease of the order of integration while preserving the rationality of the integrand.

We will assume that $N_Q \subset R^n_u$, and the image $A^{-1}(N_Q) \subset R^n_v$, here R^n_v and R^n_u are the n-dimensional real variable spaces u и v respectively. Let us denote by N' projection of the polyhedron $A^{-1}N_Q$ on the coordinate $(n-p)$-dimensional plane $\{v \in \mathbb{R}^n : v_1 = 0, \dots, v_p = 0\}$, and by $Q'(t, w')$ Laurent polynomial $Q[(t, w')^{A^{-1}}]$ from variables $w' = (w_{p+1}, \dots, w_n)$, wherein $t_1, \dots, t_p$ are parameters.

Theorem 1. *Diagonal $d_q(t)$ in (3) is represented by an integral in the $(n-p)$-dimensional complex algebraic torus $(\mathbb{C}\backslash 0)^{n-p}$ of variables $w_{p+1}, \dots, w_n$ according to the formula*

$$d_q(t) = \frac{1}{(2\pi i)^{n-p}} \int\limits_{\mathrm{Log}^{-1}(\rho')} F[(t_1, \dots, t_p, w_{p+1}, \dots, w_n)^{A^{-1}}] \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}, \tag{4}$$

where

$$\rho' = ((A\rho)_{p+1}, \dots, (A\rho)_n)$$

belongs to the connected component E_0' of the amoeba $A_{Q'}$ supplement of hypersurface $V' = \{w' \in (\mathbb{C}\backslash 0)^{n-p} : Q'(t, w') = 0\}$.

Proof of the theorem

Under the conditions of the theorem it is assumed that the diagonal (3) is considered for the Taylor series of the rational function $F = \dfrac{P}{Q}$. Therefore, it is automatically assumed that means $Q(0) \neq 0$, and that means that the origin 0 is a vertex of the Newton polytope.

Because the determinant of the integer matrix A is equal to one, inverse matrix to it

$$A^{-1} = \begin{pmatrix} b_1^{(1)} & \dots & b_n^{(1)} \\ \vdots & \ddots & \vdots \\ b_1^{(n)} & \dots & b_n^{(n)} \end{pmatrix} = (b_j^{(i)})$$

is an integer and its elements $b_j^{(i)}$ are algebraic complements to the elements $q_j^{(i)}$. The rows and columns of this matrix will be denoted by $b^{(i)}$ and b_j respectively. Let us make in the integral (3) the change of variables

$$z = w^{A^{-1}} = (w^{b_1}, \dots, w^{b_n}),$$

or, in a more detail:

$$(z_1, \dots, z_n) = (w_1^{b_1^{(1)}} \cdots w_n^{b_1^{(n)}}, w_1^{b_2^{(1)}} \cdots w_n^{b_2^{(n)}}, \dots, w_1^{b_n^{(1)}} \cdots w_n^{b_n^{(n)}}).$$

First, note that z^{q^i} will pass to w_i

$$z^{q^{(i)}} = z_1^{q_1^{(i)}} \dots z_n^{q_n^{(i)}} = \left(w_1^{b_1^{(1)}} \dots w_n^{b_1^{(n)}}\right)^{q_1^{(i)}} \dots \left(w_1^{b_n^{(1)}} \dots w_n^{b_n^{(n)}}\right)^{q_n^{(i)}} =$$

$$= w_1^{\langle b^{(1)}, q^{(i)} \rangle} \cdots w_n^{\langle b^{(n)}, q^{(i)} \rangle} = w_i,$$

since $\langle b^{(i)}, q^{(j)} \rangle = \delta_{ij}$ is the Kronecker symbol.

Applying our change of variables to the logarithmic differentials, we obtain

$$\frac{dz_i}{z_i} = \frac{d(w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}})}{w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}}} = \frac{\sum_{k=1}^{n} b_i^{(k)} w_1^{b_i^{(1)}} \dots w_k^{b_i^{(k)}-1} \dots w_n^{b_i^{(n)}} dw_k}{w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}}}.$$

Multiplying the obtained expressions for the logarithmic differentials $\frac{dz_i}{z_i}$ (taking into account the properties of the external product of differentials: $dw_i dw_i = 0$ и $dw_i dw_j = -dw_j dw_i$), we get

$$\frac{|A^{-1}| w_1^{\sum_{i=1}^n b_i^{(1)}-1} \dots w_n^{\sum_{i=1}^n b_i^{(n)}-1} dw_1 \dots\ dw_n}{w_1^{\sum_{i=1}^n b_i^{(1)}} \dots w_n^{\sum_{i=1}^n b_i^{(n)}}} = \frac{dw_1 \wedge \dots \wedge dw_n}{w_1 \dots w_n}.$$

Let us apply the formula for change of variables to the integral (3):

$$d_q(t) = \frac{1}{(2\pi i)^n} \int\limits_{\varphi_\sharp(\Gamma_\rho)} F[(w_1, \dots, w_n)^{A^{-1}}] \frac{w_1 \cdots w_p}{(w_1 - t_1) \dots (w_p - t_p)} \frac{dw_1 \dots dw_n}{w_1 \dots w_n}, \tag{5}$$

where $\varphi_\sharp$ is the homomorphism induced by the mapping $\varphi : z \to w = z^A$.

The cycle Γ_ρ is parameterized in the form

$$\mathrm{Log}^{-1}(\rho) = \{z = e^{\rho + iA^{-1}\theta} : \theta \in A([0, 2\pi)^n)\}.$$

Hence,

$$\varphi_\sharp(\Gamma_\rho) = \{w = z^A : z \in \Gamma_\rho\} = \{w = e^{A\rho + AiA^{-1}\theta}\} = \mathrm{Log}^{-1}(A\rho).$$

In this way,

$$\varphi_\sharp(\Gamma_\rho) = \{w : |w_1| = e^{(A\rho)_1}, \dots, |w_n| = e^{(A\rho)_n}\},$$

where $(A\rho)_i$ is the i-th component of the vector $A\rho$.

By the Cauchy formula

$$d_q(t) = \frac{1}{(2\pi i)^n} \int\limits_{\varphi_\sharp(\Gamma_\rho)} F[(w_1, \dots, w_n)^{A^{-1}}] \frac{dw_1}{w_1 - t_1} \cdots \frac{dw_p}{w_p - t_p} \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}$$

we get

$$d_q(t) = \frac{1}{(2\pi i)^{n-p}} \int\limits_{\mathrm{Log}^{-1}(\rho')} F[(t_1, \dots, t_p, w_{p+1}, \dots, w_n)^{A^{-1}}] \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}, \tag{6}$$

where

$$\rho' = ((A\rho)_{p+1}, \dots, (A\rho)_n)$$

belongs to the connected component E_0' of the complement of the amoeba $A_{Q'}$ of the hypersurface $V' = \{w' \in (\mathbb{C}\backslash 0)^{n-p} : Q'(t, w') = 0\}$.

The theorem is proved.

Let me make the following comment on the reduction of the formula (3) to (6). It is not difficult to see that the integrand in (3) admits representation in the form

$$\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_p}{f_p} \wedge \psi,$$

where $\psi = \psi_p$ is a rational differential form of degree $n - p$, and $f_i = z^{q^{(i)}} - t_i$. The system of binomial equations $f_1 = 0, \ldots, f_p = 0$ defines an $(n-p)$-dimensional complex torus $\mathbb{T}^{n-p}$ (embedded in the torus $\mathbb{T}^n = (\mathbb{C} \setminus 0)^n$). In this case, the real torus Γ_ρ is a p-fold tube over a real torus $\gamma \subset \mathbb{T}^{n-p}$(in the coordinates w, it is $Log^{-1}(A\rho'))$. Thus, we are in the conditions of the multiple Leray residue formula (see [10, 11]), according to that the integrals in (3) and (6) coincide.

Example

Consider the example of applying of the theorem to find the integral representation of the diagonal defined by the vectors $q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ of the Taylor series of the function $F(z) = \dfrac{1}{1 + z_1 + z_2 + z_3 + z_2 z_3}$. Newton polytope of the denominator of the function F has the form shown in Fig. 1.

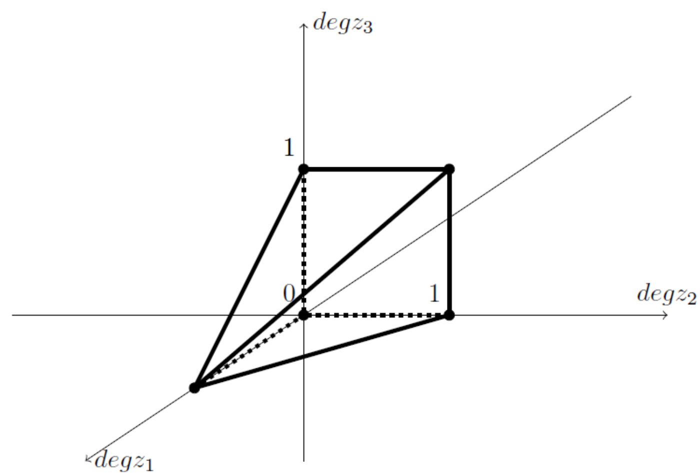

Fig. 1. Newton polytope $1 + z_1 + z_2 + z_3 + z_2 z_3$

For the two-dimensional diagonal $d_{q_1,q_2}(t_1, t_2) = \sum\limits_{l \in \mathbb{Z}^2_+} c_{l_1 q^{(1)} + l_2 q^{(2)}} t_1^{l_1} t_2^{l_2}$ in the set $\mathrm{Log}^{-1}(E_0)$, one has the following integral representation

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^3} \int\limits_{\Gamma_\rho} \frac{1}{1 + z_1 + z_2 + z_3 + z_2 z_3} \cdot \frac{z_1 z_2 z_3 * z_1 z_2^2 z_3^2}{(z_1 z_2 z_3 - t_1)(z_1 z_2^2 z_3^2 - t_2)} \cdot \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}, \qquad (7)$$

where the cycle

$$\Gamma_\rho = \{z \in \mathbb{C}^n : |z_1| = e^{\rho_1}, |z_2| = e^{\rho_2}, |z_3| = e^{\rho_3}\}$$

is chosen so that

a) poles of $F(z)$ don't intersect the closed polydisc

$$\overline{U_\rho} = \{z \in \mathbb{C}^n : |z_1| \leqslant e^{\rho_1}, |z_2| \leqslant e^{\rho_2}, |z_3| \leqslant e^{\rho_3}\};$$

b) parameters $t = (t_1, t_2)$ satisfy the inequalities $|t_i| < e^{\langle q_i, \rho\rangle}$, $i = 1, 2$.

Now let's form the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. Using replacement $z^{A^{-1}} = w$, get $z_1 = w_1^2 w_2^{-1}$; $z_2 = w_1^{-1} w_2^1 w_3^{-1}$; $z_3 = w_3$. The denominator of the function F after replacement is converted to $1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1$ and Newtonian polytope is shown in Fig. 2

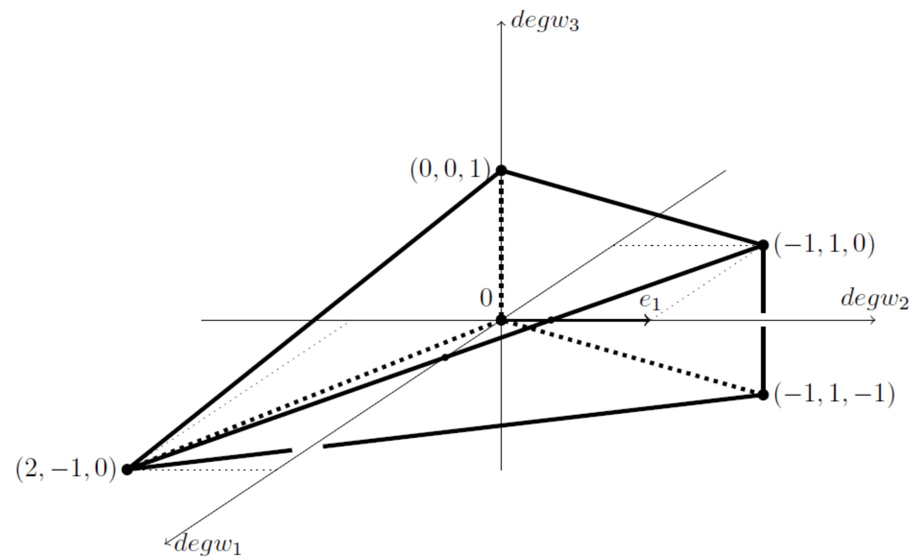

Fig. 2. Newton polytope $1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1$

The integral after replacement $z^{A^{-1}} = w$ looks like this

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^3} \int\limits_{\Gamma_\rho} \frac{1}{1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1} \cdot \frac{w_1 * w_2}{(w_1 - t_1)(w_2 - t_2)} \cdot \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3}.$$

After integration by the Cauchy formula with the variable w_1 we obtain the following form of the diagonal

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_{\rho'}} \frac{1}{1 + t_1^2 w_2^{-1} + t_1^{-1} w_2^1 w_3^{-1} + w_3 + t_1^{-1} w_2^1} \cdot \frac{w_2}{w_2 - t_2} \cdot \frac{dw_2}{w_2} \frac{dw_3}{w_3}.$$

We construct the Newton polytope of the denominator of the function $F(t_1, w_2, w_3)$ (Fig. 3). After integrating with the variable w_2 we obtain

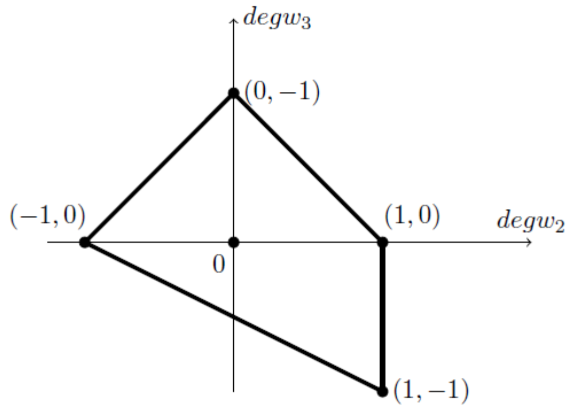

Fig. 3. Newton polytope $1 + t_1^2 w_2^{-1} + t_1^{-1} w_2^1 w_3^{-1} + w_3 + t_1^{-1} w_2^1$

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)} \int\limits_{\Gamma_{\rho''}} \frac{1}{1 + t_1^2 t_2^{-1} + t_1^{-1} t_2^1 w_3^{-1} + w_3 + t_1^{-1} t_2^1} \cdot \frac{dw_3}{w_3}. \tag{8}$$

Therefore the integral (7) admits a reduction to the one-dimensional integral (8) with rational integrand. It is known ([12], Section 10.2) that such integral is an algebraic function in variables t_1, t_2.

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Список интегральных представлений для диагонали степенного ряда рациональной функции

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Аннотация. В работе приводятся интегральные представления для диагоналей степенных рядов. Такие представления получаются понижением кратности интегрирования для известного ранее интегрального представления. Процедура понижения кратности реализуется в рамках многомерной теории вычетов Лере. Особую роль в конструкции новых интегральных представлений играет понятие амебы комплексной аналитической гиперповерхности.

Ключевые слова: многомерные степенные ряды, комлексный интеграл, интегральное представление, амеба, ряд Тейлора, диагональ степенного ряда.

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Modeling of Anisotropy Dynamics of the Proton Pitch Angle Distribution in the Earth's Magnetosphere

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Abstract. Last years the attention to research of anisotropy of the charged particle pitch angle distribution has considerably increased. Therefore for research of anisotropy dynamics of the proton pitch angle distribution is used the two-dimensional Phenomenological Model of the Ring Current (PheMRC 2-D), which includes the radial and pitch angle diffusions with consideration of losses due to wave-particle interactions. Experimental data are collected on the Polar/MICS satellite during the magnetic storm on October 21–22, 1999. Solving the non-stationary two-dimensional equation of pitch angle and radial diffusions, numerically was determined the proton pitch angle distribution anisotropy index (or parameter of the proton pitch angle distribution) for the pitch angle of 90 degrees during the magnetic storm, when the geomagnetic activity Kp-index changed from 2 in the beginning of a storm up to 7+ in the end of a storm. Dependence of the perpendicular proton pitch angle distribution anisotropy index with energy $E = 90$ keV during the different moments of time from the McIlwain parameter L ($2.26 < L < 6.6$) is received. It is certain at a quantitative level for the magnetic storm on October 21–22, 1999, when and where on the nightside of the Earth's magnetosphere (MLT = 2300) to increase in the geomagnetic activity Kp-index there is a transition from normal (pancake) proton pitch angle distributions to butterfly proton pitch angle distributions. That has allowed to determine unequivocally and precisely the anisotropy dynamics of the proton pitch angle distribution in the given concrete case. It is shown, that with increase of the geomagnetic activity Kp-index the boundary of isotropic proton pitch angle distribution comes nearer to the Earth, reaching $L \approx 3.6$ at $Kp = 7+$.

Keywords: magnetosphere, pitch angle distribution, anisotropy, data of the Polar/MICS satellite, proton flux.

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Introduction

The literature on charged particle pitch angle distributions and an anisotropy of the pitch angle distributions is enough extensive. For example, modeling the pitch angle distribution on the dayside of the Earth's magnetosphere was considered in [1], and on the nightside of the magnetosphere — in [2]. In work [3] it has been offered two-dimensional Phenomenological Model of Ring Current dynamics in the Earth's magnetosphere (PheMRC 2-D). In these three works the non-stationary equation of pitch angle and radial diffusions numerically was solved in a range of pitch angles from 0° up to 180°. In [4] has been presented the statistical analysis of pitch angle distribution of radiation belt energetic electrons near the geostationary orbit: CRRES observations with definition of an pitch angle distribution anisotropy index. A survey of the anisotropy of the outer electron radiation belt during high-speed-stream-driven storms is presented in [5].

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An empirical model of pitch angle distributions for energetic electrons (REPAD) in the Earth's outer radiation belt has been offered in [6]. In [7] it is in detail considered statistically measuring the amount of pitch angle scattering that energetic electrons undergo as they drift across the plasmaspheric drainage plume at geosynchronous orbit. The proton and electron radiation belts at geosynchronous orbit: Statistics and behavior during high-speed stream-driven storms are presented in [8]. In [9] it is certain the inner magnetosphere ion composition and local time distribution over a solar cycle with 2001 on 2013 with the indication of an anisotropy index. And in [10] other an empirical model of radiation belt electron pitch angle distributions based on Van Allen probes measurements with examples of different typical pitch angle distributions is offered.

From the review for last years it is visible, that statistical and empirical models an charged particle pitch angle distribution anisotropy are, and the mathematical models based on the physics and describing an charged particle pitch angle distribution anisotropy index, possibly, no.

The purpose of the given work is more exact quantitative research of anisotropy dynamics of the proton pitch angle distribution during the magnetic storm in the Earth's magnetosphere. Therefore it was used the two-dimensional Phenomenological Model of the Ring Current (PheMRC 2-D) [3], based on the physics and describing the perpendicular proton pitch angle distribution anisotropy index depending from the McIlwain parameter L and the geomagnetic activity Kp-index.

1. The mathematical model

The offered model, PheMRC 2-D, is based on the general two-dimensional Fokker-Planck equation for phase space density, which describes the radial and pitch angle diffusions and losses due to charge exchange and wave-particle interactions. It can be expressed by the following equation [3]

$$\begin{aligned}\frac{\partial f}{\partial t} = L^2 \frac{\partial}{\partial L}\left(L^{-2} D_{LL} \frac{\partial f}{\partial L}\right) + \frac{1}{\sin\alpha}\frac{\partial}{\partial \alpha}\left(D_{\alpha\alpha}\sin\alpha\frac{\partial f}{\partial \alpha} + \sin\alpha\frac{d\alpha}{dt} f\right) - \\ -\lambda\cdot f - \frac{f}{T_{wp}} + f S_{\perp}\sin^2\alpha.\end{aligned} \tag{1}$$

Here, f is the phase space density (or distribution function); t is the time; L is the McIlwain parameter; α is the local pitch angle; D_{LL} is the radial diffusion coefficient; $D_{\alpha\alpha}$ is the pitch angle diffusion coefficient; $d\alpha/dt$ is the pitch angle velocity; λ is the rate of loss due to proton neutralization by exchange of charges; T_{wp} is the lifetime due to wave-particle interactions; $S_{\perp}$ is the perpendicular coefficient of the particle source function ($\alpha = 90°$).

Equation (1) describes the radial diffusion in the "conventional" space with losses due to charge exchange and the pitch angle diffusion in the velocity space with losses due to the wave-particle interactions. Therefore, a corresponding diffusion coefficient in the velocity space (namely, the pitch angle diffusion coefficient) is needed. The loss function is conditioned by the fall of charged particles in the so-called "loss cone" as a result of wave-particle interactions. The particle source function can be related, for example, to charged particles that move from the tail of the magnetosphere toward the Earth when affected by magnetospheric convection.

Equation (1) is a non-stationary, two-dimensional, second-order, partial differential equation. Its solution should be sought as a function of L, α, and t. We use this solution to determine the evolution of the pitch angle distribution of the Earth's ring current protons and to find the perpendicular ($\alpha = 90°$) proton pitch angle distribution anisotropy index as a function of the McIlwain parameter L during a given magnetic storm.

The initial and boundary conditions are given for the following ranges of the McIlwain parameter L and pitch angle α: $2.26 \leqslant L \leqslant 6.6$, $0° \leqslant \alpha \leqslant 180°$. We also use the relationship between the differential particle flux j and the phase space density f: $j = 2mEf$. In detail in [3].

Thus, non-stationary two-dimensional partial differential equation (1) is solved numerically by a finite-element projection method with initial and variable boundary conditions.

2. Results of calculations

Next, we study the dynamics of ring current protons during a magnetic storm. We consider the case of the magnetic storm of October 21–22, 1999, considered by Ebihara et al. [11]. The initial time was 0613 UT of October 21, 1999, which corresponds to 0000 RT (Run Time — time of modeling) (magnetically quiet conditions). The time interval taken for calculations was 2030 RT (the final time — 0243 UT of October 22, 1999) (the main phase of the magnetic storm). At the initial time (0000 RT) $Kp = 2$ and then increases up to $Kp = 7+$. All model calculations were conducted for the nightside of the Earth's magnetosphere (2300 MLT) for protons with energy of $E = 90$ keV.

To accurately calculate the perpendicular ($\alpha = 90°$) anisotropy index $\gamma_\perp$ of the pitch angle distribution of charged particles at any time, one should use the formula [3]

$$\gamma_\perp = -\frac{1}{j_\perp}\left(\frac{d^2 j}{d\alpha^2}\right)_\perp . \tag{2}$$

Or approximately this can be done if the second derivative $\left(\frac{d^2 j}{d\alpha^2}\right)_\perp$ is determined with good accuracy from experimental data.

We approximate the initial pitch angle distribution of protons (0000 RT) by dependence (the data measured on the Polar/MICS satellite, in detail in [3] $L = 5$, MLT = 2300, $E = 90$ keV)

$$j_0(\alpha) = j_{\perp 0} \sin^{\gamma_{\perp 0}} \alpha \approx 36948 \sin^{0.75} \alpha \tag{3}$$

with the dimension $[j_0(\alpha)] = (\text{cm}^2 \text{ s sr keV})^{-1}$.

Before the storm, the pitch angle distribution is pancake-like but becomes butterfly-like in the main phase of the storm. The same tendency was noted by Ebihara et al. [11] and is in detail confirmed in [3].

The model proton fluxes were compared with the data measured on the Polar/MICS satellite during the magnetic storm of October 21–22, 1999. There is good agreement between simulated fluxes and experimental data [3].

Thus, the model PheMRC 2-D (1) quantitatively describes the model evolution of proton pitch angle distributions during the magnetic storm on October 21–22, 1999, which is associated with the concurrent effect of the physical mechanisms of radial diffusion, pitch angle diffusion, ,charge exchange, wave-particle interactions, splitting of drift shells of the electric field, and particle injection and drift.

Using the formula (2), dependence of a perpendicular ($\alpha = 90°$) proton pitch angle distribution anisotropy index $\gamma_\perp$ with energy $E = 90$ keV at the different moments of time (Fig. 1) from the McIlwain parameter L ($2.26 \leqslant L \leqslant 6.6$) is received. It is certain at a quantitative level (Fig. 1) for the magnetic storm on October 21–22, 1999, when and where on the nightside of the Earth's magnetosphere (MLT = 2300) with increase in an geomagnetic activity Kp-index there is a transition from normal or pancake-like proton pitch angle distributions ($\gamma_\perp > 0$) to butterfly-like proton pitch angle distributions ($\gamma_\perp < 0$).

At $\gamma_\perp = 0$ the proton pitch angle distribution is isotropic. Therefore from figure 1 it is visible, that with increase in an geomagnetic activity Kp-index the boundary of isotropic proton pitch angle distribution comes nearer to the Earth, reaching $L \approx 3.6$ at $Kp = 7+$.

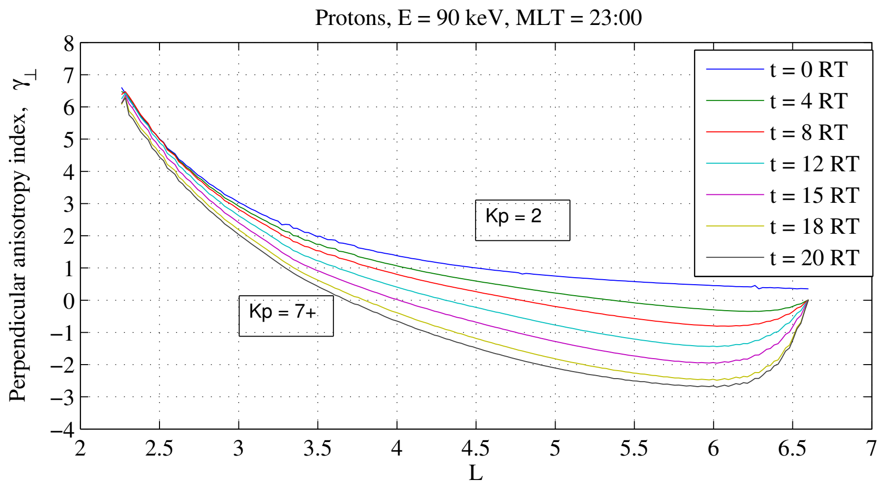

Fig. 1. Dependence of a perpendicular ($\alpha = 90^\circ$) proton pitch angle distribution anisotropy index $\gamma_\perp$ with energy $E = 90$ keV, MLT $= 2300$ at the different moments of time from the McIlwain parameter L ($2.26 \leqslant L \leqslant 6.6$) and the geomagnetic activity Kp-index

Thus, the lead calculations have allowed to determine unequivocally and precisely the anisotropy dynamics of the proton pitch angle distribution in the given concrete case, using the proton pitch angle distribution anisotropy index for the pitch angle of 90 degrees $\gamma_\perp$ (Fig. 1).

Conclusion

1. The anisotropy dynamics of the ring current proton pitch angle distribution ($E = 90$ keV) in the inner Earth's magnetosphere ($2.26 \leqslant L \leqslant 6.6$, MLT $= 2300$) with variable boundary conditions during the magnetic storm of October 21–22, 1999, was investigated with the use of two-dimensional Phenomenological Model of the Ring Current (PheMRC 2-D) (1).

2. PheMRC 2-D takes into account the radial and pitch angle diffusions and describes the losses due to charge exchange and wave-particle interactions.

3. The model proton fluxes were compared with the data measured on the Polar/MICS satellite during the magnetic storm of October 21–22, 1999. There is good agreement between simulated fluxes and experimental data.

4. We confirmed the experimentally revealed tendency that the pitch angle distribution is pancake-like before the magnetic storm and that it becomes butterfly-like in the main phase of the storm.

5. With increase in an geomagnetic activity Kp-index the boundary of isotropic proton pitch angle distribution comes nearer to the Earth, reaching $L \approx 3.6$ at $Kp = 7+$.

6. It is necessary to develop and specify in the further, for example, empirical (semiempirical) models of differential flux definition of charged particles at a pitch angle of $\alpha = 90^\circ$ $j_\perp$ and the anisotropy index of the pitch angle distribution of charged particles at a pitch angle of $\alpha = 90^\circ$ $\gamma_\perp$ under different geophysical conditions, especially for magnetically quiet conditions.

7. With appropriate experimental data, the model PheMRC 2-D can be used to simulate the anisotropy dynamics of charged particles pitch angle distribution in the Jovian and Saturn magnetospheres.

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Моделирование динамики анизотропии питч-углового распределения протонов в магнитосфере Земли

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Аннотация. В последние годы внимание к исследованию анизотропии питч-углового распределения заряженных частиц значительно возросло. Поэтому для исследования динамики анизотропии питч-углового распределения протонов используется двумерная феноменологическая модель кольцевого тока PheMRC 2-D (two-dimensional Phenomenological Model of the Ring Current), которая включает радиальную и питч-угловую диффузии с учетом потерь вследствие взаимодействий волна-частица. Экспериментальные данные собраны на спутнике Polar/MICS во время магнитной бури 21–22 октября 1999 г. Решая нестационарное двумерное уравнение питч-угловой и радиальной диффузий, численно определяли индекс анизотропии питч-углового распределения протонов (или показатель питч-углового распределения протонов) для питч-угла 90 градусов во время магнитной бури, когда Kp-индекс геомагнитной активности изменялся от 2 в начале бури до 7+ в конце бури. Получена зависимость перпендикулярного индекса анизотропии питч-углового распределения протонов с энергией $E = 90$ кэВ в разные моменты времени от параметра МакИлвейна L ($2.26 < L < 6.6$). Определено на количественном уровне для магнитной бури 21–22 октября 1999 г., когда и где на ночной стороне магнитосферы Земли (MLT = 2300) с увеличением Kp-индекса геомагнитной активности имеется переход от нормальных (блиноподобных) питч-угловых распределений протонов к бабочкоподобным питч-угловым распределениям. Это позволило определить однозначно и точно динамику анизотропии питч-углового распределения протонов в данном конкретном случае. Показано, что с увеличением Kp-индекса геомагнитной активности граница изотропного питч-углового распределения протонов приближается к Земле, достигая $L \approx 3.6$ при $Kp = 7+$.

Ключевые слова: магнитосфера, питч-угловое распределение, анизотропия, данные спутника Polar/MICS, поток протонов.

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On Problem of Finding all Maximal Induced Bicliques of Hypergraph

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Abstract. The problem of finding all maximal induced bicliques of a hypergraph is considered in this paper. Theorem on connection between induced bicliques of the hypergraph H and corresponding vertex graph $L_2(H)$ is proved. An algorithm for finding all maximal induced bicliques is proposed. Results of computational experiments with the use of the proposed algorithm are presented.

Keywords: hypergraph, maximal induced bicliques, search algorithm.

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A hypergraph is an extension of classical graph in which an edge of a graph can join any number of vertices. Traditionally hypergraphs have found practical application in the development of relational databases and combinatorial chemistry [1, 2]. Ability to combine multiple vertices in one edge provides a powerful tool to study processes in various networks. Thus, hypergraphs are actively used in modelling road and telecommunication networks [3, 4]. They are also used for constructing semantic networks when processing texts in natural languages [5, 6].

Many problems in studies of such networks are reduced to problems of determining various configurations. Configuration means any system of subsets of a finite set [2]. Of particular interest are problems of enumeration type [2], in which the existence of configurations is beyond doubt but there are two problems: find the number of configurations and the method of their representation. Considering that networks are representable by hypergraphs the search for configurations can be conveniently formulated in the language of $(0, 1)$-matrices [3]. In this case, the problem of finding all maximal induced bicliques of hypergraph can be reduced to the problem of constructing complete submatrices [4]. Let us note that problems of finding such configurations are $\sharp P$-complete [5].

The problem of finding all maximal induced bicliques for each given hypergraph, which is called Maximal Induced Biclique Generation Problem for Hypergraphs (MIBGP for Hypergraphs) is studied in the paper. This problem arises in various applications connected with data mining in many fields. For example, in telecommunication networks maximal bicliques used for route organizing and defining subnets for marking them [3, 7]. In metabolic and genetic networks maximal bicliques are used to represent the interrelation between organisms and different external conditions [8–10]. In marketing maximal bicliques allow one to form social

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recommendations and product bundling [8–10]. Maximal bicliques are used for clustering data in various fields [11].

A new algorithm for finding all maximal induced bicliques in hypergraph is proposed in the paper. The theorem on the interrelation between induced bicliques of hypergraph and corresponding special vertex graph is proved. The theorem on time complexity and correctness of the proposed algorithm is also proved.

1. Statement of problem of finding all maximal induced bicliques of hypergraph

Let a hypergraph $H = (X, U)$ be given, where X is a finite set of vertices and U is a finite family of hyperedges of hypergraph at the same time $|X| \geqslant 1$, $|U| \geqslant 1$, and any hyperedge of hypergraph is a subset of the set X. Let us assume that $X(u)$ is a set of all vertices that incident to the hyperedge $u \in U$, and $U(x)$ is a set of all hyperedges which incident to the vertex $x \in X$. One of the ways to define hypergraph is incidence $(0,1)$-matrix I, where 1 is put in the case when hyperedge contains vertex and 0 otherwise. The degree of hyperedge $u \in U$ is the cardinality of set $|X(u)|$. Let us introduce definition that is necessary for further presentation [12].

Definition 1.1. *A hypergraph $H' = (X', U')$ is called the subhypergraph induced by the set of vertices X', where $U' = \{u' \colon X(u') = X(u) \cap X' \neq \oslash, u \in U\}$.*

Note that in Definition 1.1 $|X(u) \cap X'| \geqslant 2$ and $|u'| = 1$ for U'.

The following definition of bipartiteness of hypergraph is known, which is similar to 2-coloring. A hypergraph $H = (X, U)$ is called the bipartite when the set of vertices X can be divided into two sets S_0 and S_1 in such a way that $S_0 \cup S_1 = X$, $S_0 \cap S_1 = \oslash$ and $|X(u) \cap S_0| = 1$ is true for any hyperedge $u \in U$ [13]. To prove the main theorem, the following definition of bipartition is introduced.

Definition 1.2. *The subhypergraph $H' = (X', U')$ induced by the set of vertices X' is bipartite if there exists such partition $S_0 \cup S_1 = X'$ that $S_0 \cap S_1 = \oslash$ and $|S_0 \cap X(u')| \leqslant 1$, $|S_1 \cap X(u')| \leqslant 1$ is true for all $u' \in U'$.*

Definition 1.3. *A vertex graph of hypergraph $H = (X, U)$ is called the graph $L_2(H) = (X, E)$ which set of vertices is equal to the set of vertices X of hypergraph H while two vertices of $L_2(H)$ are adjacent if and only if corresponding vertices of hypergraph H are adjacent.*

Theorem 1.1. *The subhypergraph $H' = (X', U')$ is bipartite if and only if vertex graph $L_2(H)$ of hypergraph H contains a bipartite subgraph induced by the set of vertices X'.*

Proof. Let us prove the sufficiency. If hypergraph H contains a bipartite subgraph $H' = (X', U')$ then $L_2(H)$ contains a bipartite subgraph induced by the set X'.

The proof follows directly from Definitions 1.1–1.3. Hyperedge $u \in U$ of hypergraph H generates hyperedge u' in hypergraph H'. Then, according to Definition 1.1, the degree of hyperedge u can be greater then u', that is, $|u| \geqslant |u'|$. However, for bipartite subhypergraph $H' = (X', U')$ the degree of any hyperedge $u' \in U'$ does not exceed two. It follows from Definition 1.2. This is because if $|X(u')| > 2$ then requirement $|S_0 \cap X(u')| \leqslant 1$, $|S_1 \cap X(u')| \leqslant 1$ is violated, where $S_0 \cup S_1 = X'$ and $S_0 \cap S_1 = \oslash$. Thus, if subhypergraph $H' = (X', U')$ is bipartite then by Definition 1.3 vertex graph $L_2(H')$ is bipartite as well. On the other hand, vertex graph $L_2(H')$ is the subgraph of $L_2(H)$ induced by the set X'. Therefore, $L_2(H')$ requires bipartite subgraph of vertex graph $L_2(H)$ of hypergraph H.

Let us first prove the necessity. If vertex graph $L_2(H)$ of the hypergraph H contains the bipartite subgraph induced by the set X' then there exists the bipartite subhypergraph $H' = (X', U')$.

Let bipartite subgraph with set of vertices X' with parts S_0 and S_1 exists in graph $L_2(H)$, where $S_0 \cup S_1 = X'$, $S_0 \cap S_1 = \oslash$. Let us consider the subhypergraph $H' = (X', U')$ induced by the set X' in the hypergraph $H = (X, U)$. According to Definition 1.1, a set of hyperedges has form $U' = \{u' \colon X(u') = X(u) \cap X' \neq \oslash, u \in U\}$ in subhypergraph H'. It follows from Definition 1.3 that for any hyperedge $u \in U$ a set $X(u)$ forms a complete subgraph in the vertex graph $L_2(H)$. It is known that any bipartite graph has no complete subgraphs with number of vertices more then two [14]. Hence, any hyperedge $u' \in U'$ satisfies the inequalities $|X(u') \cap S_0| \leqslant 1$ and $|X(u') \cap S_1| \leqslant 1$ and $|u'| \leqslant 2$. □

Definition 1.4. *A bipartite graph is called the complete bipartite graph (biclique) if each vertice of one part is connected with all vertices from the second part.*

This definition can be formulated differently. If bipartite graph contains all possible edges that do not violate the bipartiteness condition then such graph is called the complete bipartite graph. A number of graph problems which belong to the class of $\sharp P$-complete or NP-complete are reduced to the search of bicliques [15]. In the general case a number of maximal bicliques depends exponentially on the size of the graph [16].

Definition 1.5. *A subhypergraph $H' = (X', U')$ induced by the set of vertices X' such that $S_0 \cup S_1 = X'$, $S_0 \cap S_1 = \oslash$ and $U' = u : s_0, s_1 \in X(u), s_0 \in S_0, s_1 \in S_1$ is called bipartite induced subhypergraph of hypergraph H.*

In what follows, by an induced bicliques of hypergraph is meant a complete bipartite induced subhypergraph of hypergraph H in the sense of Definition 1.5.

Definition 1.6. *A biclique that can not be extended with additional adjacent vertices is called the maximal induced biclique. It means that there is no another biclique which completely includes the maximal biclique.*

Definition 1.6 is true for both graphs and hypergraphs.

The problem of finding maximal biclique is well known in graph theory. There are two variants of this problem: find a maximal biclique with maximal number of vertices and find a maximal biclique with maximal number of edges. These problems arise in detection of anomalies in data, in analysing gene structures and social structures [10]. Both variants are NP-hard for general graphs [17].

Another problem associated with maximal biclique is Maximal Biclique Generation Problem (MBGP). It consists in finding all maximal bicliques for a graph. It is known that MBGP cannot be solved in polynomial time with respect to the size of input since the size of output set can be exponentially large [17]. The complexity of this problem is comparable with the complexity of problem of searching of one maximal biclique which is NP-hard [10]. The problem of maximal biclique generation can be extended to hypergraph. Such case was investigated for bihypergraphs [13].

In this paper the problem of Maximal Induced Biclique Generation Problem for Hypergraphs (MIBGP for Hypergraphs) is studied.

MIBGP for Hypergraphs. A hypergraph $H = (X, U)$ without double hyperedges is given. It is necessary to find a set of all maximal induced bicliques.

Note that problem of finding maximal induced bicliques is connected with searching of matrices of special form [18]. Let us consider interrelation between the problem of finding maximal induced bicliques and the problem of finding maximal complete submatrices of $(0, 1)$-matrix.

An adjacency matrix of the vertex graph $L_2(H)$ of hypergraph H is denoted as A. Let us show the form of $(0, 1)$ adjacency matrix of $L_2(H')$ for corresponding bipartite subhypergraph $H' = (X', U')$. Here S_0, S_1 are parts of hypergraph H with cardinality c and d, respectively. Since $L_2(H')$ is also the bipartite graph then adjacency matrix has form

$$A' = \begin{pmatrix} O_c & B' \\ B'^T & O_d \end{pmatrix}, \tag{1}$$

where O_c, O_d are zero matrices of sizes c and d, respectively, and B' is the matrix of size $c \times d$ which represents the adjacency of vertices between parts S_0 and S_1. Obviously, when subhypergraph H' is biclique then matrix B is a complete submatrix of matrix A. It is easy to show that any submatrix can be chosen from the set of all maximally complete submatrices in linear time with respect to the size of submatrix. Thus, in order to find induced bicliques in the hypergraph $H = (X, U)$ it is required to find such complete submatrices B' of adjacency matrix A of vertex graph $L_2(H)$ for which there is a submatrix of form (1). This is related to the problem of finding all maximally complete submatrices of the $(0, 1)$-matrix.

Maximal complete submatrices can represent various combinatorial objects [19]. The problem of finding such submatrices is enumerative and belongs to the complexity class of $\sharp P$-complete problems [5, 19]. Algorithms for finding all maximal complete submatrices have high complexity with respect to the size of input matrix.

2. Algorithm of finding all maximal induced bicliques of hypergraph

Let both set of vertices and set of hyperedges of hypergraph H are lexicographically ordered.

In proposed algorithm a transition from hypergraph H to vertex graph $L_2(H)$ is considered. Adjacency matrix of $L_2(H)$ is represented as hypergraph $\Phi = (X_\Phi, U_\Phi)$. Let us introduce a definition of l-layer of hypergraph Φ with square matrix. Let us consider a subhypergraph Φ' induced by a set of vertices $S_0 \subseteq X_\Phi$ and a set of hyperedges $S_1 \subseteq U_\Phi$. If matrix of Φ' satisfy form (1) then Φ' is called induced biclique, and it is denoted by (S_0, S_1). The set of all induced bicliques (S_0, S_1) where $|S_0| = l$ is called the l-layer of hypergraph Φ.

The main idea of the algorithm involves generation of all induced bicliques for all l-layers and then choosing from them maximal induced bicliques. The HFindMIB algorithm diagram is shown in Fig. 1. Finding all maximal induced bicliques of hypergraph H is required in MIBGP for Hypergraphs. Proposed HFindMIB algorithm solves this problem in three stages. The transition from input hypergraph H to vertex graph $L_2(H)$ is realized on the initialization stage. All l-layers for adjacency matrix of vertex graph $L_2(H)$ represented as hypergraph Φ are generated on the generation stage. Finally, all maximal induced bicliques are selected from all generated l-layers n the filtration stage. This provides all maximal induced bicliques for hypergraph H. Let us shown that HFindMIB algorithm solves MIBGP for Hypergraphs correctly and estimate its time complexity. The proof of the following theorem is constructive with respect to the structure of the algorithm.

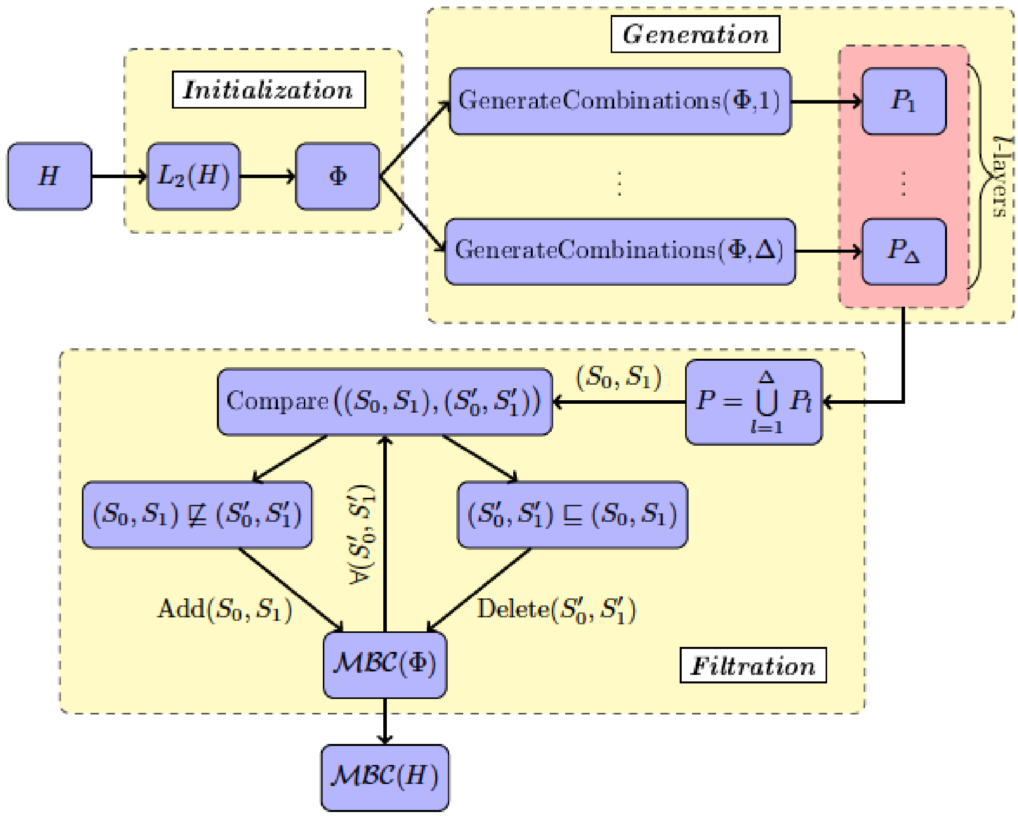

Fig. 1. Diagram of the HFindMIB algorithm

Theorem 2.1. *Let Δ be the maximum vertex degree of hypergraph $H = (X, U)$. Then HFindMIB algorithm correctly finds all maximal induced bicliques of the hypergraph, and it requires time that is not more than* $\mathcal{O}\left(2^{2\Delta} \cdot \Delta \cdot \left(|\mathcal{MBC}| + \Delta^3 \cdot \log(2^{2\Delta})\right) + |X|^2\right)$.

Proof. The HFindMIB algorithm consists of three stages. Let us consider and evaluate each stage sequentially.

Initialization stage. The HFindMIB algorithm requires a hypergraph $H = (X, U)$ with maximum vertex degree Δ as input data. The algorithm produces the vertex graph $L_2(H)$ in time $\mathcal{O}\left(|X|^2\right)$ [20]. In order to construct all induced bicliques of hypergraph H at generation phase the ability to quickly access any subset that consists of l rows in adjacency matrix of vertex graph $L_2(H)$ is needed. To achieve this artificial method is proposed. Let us fix numeration of vertices for vertex graph $L_2(H)$. Let us define a new hypergraph $\Phi = (X_\Phi, U_\Phi)$ as follows. The set of vertices X_Φ coincides up to numbering with the set of vertices of the graph $L_2(H)$. The set of hyperedges U_Φ is a system of subsets from X_Φ and it is constructed in accordance with the columns of adjacency matrix of the vertex graph $L_2(H)$. Let us note that hypergraph Φ does not allow any renumbering of vertices or hyperedges. This is necessary for one-to-one correspondence between hypergraph Φ and adjacency matrix of the vertex graph $L_2(H)$. The time requirement of the initialization stage is $\mathcal{O}(|X|^2)$.

Generation stage. Sets of all l-layers $P_l = \{(S_0, S_1) \colon S_0 \cap S_1 = \varnothing\}$, $l = 1, \ldots, \Delta$ are obtained after performing the generation stage. These sets contain all induced bicliques of the hypergraph H. Generation is carried out as follows. Sets P_l are formed for the corresponding l-layers, where $l = 1, \dots, \Delta$. For each value of l function $GenerateCombinations(\Phi, l)$ are executed. This function generates all possible subsets $X' \subseteq X_\Phi(u)$ for each hyperedge $u \in U_\Phi$ such that $|X'| = l$ and X' satisfy (1). Form (1) ensures that all vertices of X' are not adjacent with each other. The set of all such subsets of l-layer of hyperedge $u \in U_\Phi$ is denoted as C_u^l. Each generated set X' is considered as a part S_0 of biclique. Since hypergraph Φ represents adjacency matrix of $L_2(H)$ then any $u \in U_\Phi$ can be treated as vertex of the hypergraph H. A part S_1 for corresponding set $X' \in C_u^l$ is formed from hyperedges $u \in U_\Phi$ such that they do not violate (1). If addition of u to part S_1 violates (1) then current biclique is split in two (S_0, S_1) and $(S_0, S_1 \sqcup u)$, where $S_1 \sqcup u$ is the union of elements of S_1 with u such that they are not adjacent and satisfy (1). Thus, set P_l contains all induced bicliques for which $|S_0| = l$.

Let us evaluate complexity of the generation stage. Since hypergraph $\Phi = (X_\Phi, U_\Phi)$ corresponds to the adjacency matrix of the graph $L_2(H)$ then cardinality of any $X_\Phi(u)$ does not exceed Δ, where $u \in U_\Phi$. This is because maximal degree of the vertice of hypergraph H is equal to Δ. Hence, number of all possible subsets $X' \in C_u^l$ is less than C_Δ^l. Cardinality of parts S_0, S_1 of biclique does not exceed Δ for the same reason. Therefore, the number of possible parts S_1 for any part S_0 can be estimated at 2^Δ. Obviously that this estimate much higher than the real number of possible induced bicliques because of form (1). So number of all induced bicliques for l-layer does not exceed $C_\Delta^l \cdot 2^\Delta$. For all l-layers the following estimation is true

$$2^\Delta \cdot C_\Delta^1 + \dots + 2^\Delta \cdot C_\Delta^\Delta = 2^\Delta \cdot \left(C_\Delta^1 + \dots + C_\Delta^\Delta\right) = 2^\Delta \cdot 2^\Delta = \mathcal{O}\left(2^{2\Delta}\right).$$

Let us note that any subset $X' \in C_u^l$ is lexicographically ordered so any X_1' and X_2' from C_u^l are comparable. This allows one to store and refresh sets P_l with tree structures like red-black tree. Subsets X_1' and X_2' can be compared in a time $\mathcal{O}(\Delta)$ because $|X'| \leqslant \Delta$. Search and addition of elements in red-black tree can be done in a time $\mathcal{O}\big(\Delta \cdot \log(2^{2\Delta})\big)$ [21]. Checking form (1) is required for split operation that can be done in a time $\mathcal{O}(\Delta^2)$ and search for adjacent vertices in part S_1 can be done in a time $\mathcal{O}(\Delta)$. So generation phase can be done in time that does not exceed $\mathcal{O}\big(2^{2\Delta} \cdot \Delta^4 \cdot \log\left(2^{2\Delta}\right)\big)$.

Filtration stage. A set of all induced bicliques P is formed from sets P_l which are l-layers of the hypergraph. It was shown that sets P_l contain all induced bicliques with parts S_0, S_1 that $|S_0| = l$ and $|S_1| \leqslant \Delta$. Union of sets P_l into set P can be done in a time $\mathcal{O}(1)$. Filtration stage cleans set P from redundant and embedded induced bicliques. Bicliques (S_0, S_1) and (S_0', S_1') where $S_0 = S_1'$ and $S_1 = S_0'$ so (S_0, S_1) and (S_0', S_1') are generated according to the specifics of generation. To find all maximal induced bicliques it is required to determine such bicliques that are embedded in others. Complexity of this process depends on the size of output set of all maximal induced bicliques $\mathcal{MBC}(\Phi)$. Comparison and detection of embedded bicliques is done in $Compare\big((S_0, S_1), (S_0', S_1')\big)$ procedure. This procedure is called for each element of P and compares it with all elements of $\mathcal{MBC}(\Phi)$. Let us define $(S_0, S_1) \sqsubseteq (S_0', S_1')$ as follows. If $S_0 \subseteq S_0', S_1 \subseteq S_1'$ or $S_1 \subseteq S_0', S_0 \subseteq S_1'$ then biclique (S_0, S_1) is embedded in (S_0', S_1'). If induced biclique $(S_0, S_1) \not\sqsubseteq (S_0', S_1')$, where $(S_0', S_1') \in P$, then it is considered as maximal and it is added into the set $\mathcal{MBC}(\Phi)$. Such operation takes $4 \cdot \Delta$ operations. It is required to filter $2^{2\Delta}$ elements of set P and compare them with the number of elements $|\mathcal{MBC}(\Phi)|$. Hence, filtration stage can be done in a time $\mathcal{O}\left(2^{2\Delta} \cdot \Delta \cdot |\mathcal{MBC}(\Phi)|\right)$ time. According to Theorem 1.1 a set $\mathcal{MBC}(\Phi)$ is equivalent to the set of all maximal induced bicliques of hypergraph H. After filtration stage only maximal induced bicliques is extracted from the set P so the HFindMIB algorithm correctly solves MIBGP for Hypergraphs.

Combining the complexity of each stage of the HFindMIB algorithm, we obtain that resulting time does not exceed

$$\mathcal{O}\Big(2^{2\Delta} \cdot \Delta \cdot \big(|\mathcal{MBC}| + \Delta^3 \cdot \log(2^{2\Delta})\big) + |X|^2\Big).$$

□

Complexity of the HFindMIB algorithm given in Theorem 2.1 depends on the size of set $\mathcal{MBC}$. This is feature of MIBGP for Hypergraphs which is an enumeration problem. Besides the estimate depends on the value of Δ. However, it is overestimated since some of the subsets at each of the l-layers does not form a part of biclique.

3. Computational experiments

To evaluate the effectiveness of solution of the MIBGP for Hypergraphs with the use of the proposed HFindMIB algorithm computational experiments were performed. Hypergraphs $H = (X, U)$ with various numbers of vertices $|X|$ and hyperedges $|U|$ and with various maximum vertex degree Δ were used in experiments. Multiple hypergraphs with the same number of vertices and with maximum degree of vertexes Δ were generated. Number of vertices of such hypergraphs was constant but number of hyperedges was varied. Computational experiments were performed on a PC with an AMD Ryzen 5 3600 6-Core Processor 3.60 GHz and 16 GB of RAM. Averaged results of experiments for generated hypergraphs are presented in Table 1.

Table 1. Results of computational experiments

Δ	$\|X\|$	$\|U\|$	$\|\mathcal{MBC}\|$	t, clocks
3	100	88,5	192	27,7
	500	438,8	965,1	446,9
	1000	870,1	1943,3	1803,2
	2500	2186,6	4845,4	12726,8
5	100	78,9	443,8	101,1
	500	371,5	2304,3	2601
	1000	732,4	4624,1	11321,1
	2500	1830	11615,9	74750,4
7	100	73,5	818,2	367,5
	500	330	4285,9	9904,3
	1000	648,3	8633,6	55452,6
	2500	1586,5	21813,7	239831

Column marked with $|U|$ represents average number of hyperedges for hypergraph with equal number of vertices and maximal degree of vertices Δ. Columns marked with $|\mathcal{MBC}|$ and t represent an average cardinality of the set of maximal induced bicliques and an average search time of the HFindMIB algorithm, respectively. The time is represented in CPU cycles. One CPU cycle time was 0,001 seconds. As can be seen from Table 1 execution time of the HFindMIB algorithm essentially depends on cardinality of set $\mathcal{MBC}$. This is typical for the problem of finding all maximal induced bicliques since their number can exponentially depend on the size of the hypergraph. Other algorithms for finding all maximal induced bicliques have similar time complexity [22, 23].

Conclusion

The problem of finding all maximal induced bicliques of hypergraph is studied in the paper. The HFindMIB algorithm for solution of this problem is proposed. The HFindMIB algorithm is based on the theorem on equivalence of induced bicliques of the hypergraph H and the vertex graph $L_2(H)$. The proof of the theorem is presented in the paper. It was shown that the proposed HFindMIB algorithm has time complexity which is not in excess of $\mathcal{O}\Big(2^{2\Delta} \cdot \Delta \cdot \big(|\mathcal{MBC}| + \Delta^3 \cdot \log(2^{2\Delta})\big) + |X|^2\Big)$, where Δ is the maximum vertex degree of hypergraph H.

The structure of the HFindMIB algorithm allows the use of parallel computing technologies to speed up the performance of the its algorithm. Refinement of the theoretical estimate of the time complexity of the algorithm is the subject for future research.

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О задаче перечисления всех максимальных индуцированных биклик гиперграфа

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Аннотация. В работе рассматривается задача поиска всех максимальных индуцированных биклик гиперграфа. Доказана теорема о связи индуцированных биклик гиперграфа H и вершинного графа $L_2(H)$. Предложен алгоритм нахождения всех максимальных индуцированных биклик. Приведена теоретическая оценка сложности предлагаемого алгоритма и доказательство его корректности. Приведены вычислительные эксперименты.

Ключевые слова: гиперграф, максимальные индуцированные биклики, алгоритм поиска.

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Connecting Homomorphism and Separating Cycles

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Abstract. We discuss the construction of a long semi-exact Mayer–Vietoris sequence for the homology of any finite union of open subspaces. This sequence is used to obtain topological conditions of representation of the integral of a meromorphic n-form on an n-dimensional complex manifold in terms of Grothendieck residues. For such a representation of the integral to exist, it is necessary that the cycle of integration separates the set of polar hypersurfaces of the form. The separation condition in a number of cases turns out to be a sufficient condition for representing the integral as a sum of residues. Earlier, when describing such cases (in the works of Tsikh, Yuzhakov, Ulvert, etc.), the key was the condition that the manifold be Stein. The main result of this article is the relaxation of this condition.

Keywords: Mayer–Vietoris sequence, Grothendieck residue, separating cycle.

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Introduction

In the theory of functions of one complex variable the Cauchy residue of a function f at an isolated singular point a is represented in a local coordinate z by the integral over the cycle $\gamma^{(a)} = \{|z-a| = \varepsilon\}$ in a sufficiently small punctured neighborhood $U_a \setminus \{a\}$. The cycle $\gamma^{(a)}$ is called *local* cycle at a. By Cauchy's theorem the definition of the residue does not depend on the choice of a local cycle (choice of the local coordinate and the radius ε). It's usually not difficult to represent the integral

$$\int_\gamma f\,dz$$

of a meromorphic function f over a cycle γ lying outside the polar set of the function as a sum of residues: it suffices to know the homological expansion of the cycle γ in terms of local cycles.

The multidimensional analogue of the Cauchy residue is the *Grothendieck residue* of a meromorphic differential n-forms ω given on an n-dimensional complex-analytic manifold. This residue in turn is represented by the integral over a local n-cycle in a neighborhood of an isolated intersection point of polar hypersurfaces of ω. In this case, it is possible to show that in order for the integral of a meromorphic form to be represented in terms of residues, it is necessary that the cycle of integration in a certain sense *separates* the set of polar hypersurfaces of the form. The most complete results on the characterization of such separating cycles and their relationship to

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local cycles in Stein manifolds are presented by Tsikh and Yuzhakov (see [5, 9]) (these results have recently been complemented in [6, 7]). In this article, we develop a method for studying separating cycles, which makes it possible to weaken the homological conditions for the manifold and a family of polar hypersurfaces of integrating form, abandoning the Stain property of the manifold.

Our main tool is a generalization of the well-known Mayer–Vietoris long exact sequence. All the necessary information about the homology of the union of open subspaces (the homology of the topological space with open cover $\mathfrak{U}$) is discussed in Section 1. The case of the union of more than two subspaces leads to the study of the double complex of a cover and the *Mayer–Vietoris spectral sequence* (see [3]).

In Section 2 we construct a connecting homomorphism that allows us to get the long Mayer–Vietoris sequence for any finite cover. This sequence is not exact in the general case, but it is semi-exact. In obtaining our results, ideas from Gleason's article [4] are essentially used. The notion of the *resolution for a cycle* associated with an open covering of the space (the $\mathfrak{U}$-resolution) in [4] is not standard, but it is very appropriate in our opinion.

The main results are presented in Section 3. Note that our results are formulated under the assumption that the $(2n-1)$-dimensional homology of the manifold is trivial and that the intersection of the set of n polar hypersurfaces of the integrating form ω is discrete. In terms of the corresponding long Mayer–Vietoris sequence, we have obtained (Theorem 3.1) a necessary and sufficient condition under which any separating cycle is represented in terms of local cycles (and therefore the integral is calculated in terms of the residues). However, this is only a reformulation of the problem in the language of homological algebra. Theorem 3.2 gives a more practical sufficient condition (in terms of homology of complements of polar hypersurfaces of the form ω) under which any separating cycle is represented in terms of local cycles. This condition, in particular, is satisfied for Stein manifolds and arbitrary set of n polar hypersurfaces, which allows us to obtain another proof of the Tsikh theorem on separating cycles in Stein manifolds (Theorem 3.3). Therefore, the condition from Theorem 3.2 gives the desired relaxation of the condition of Steinnes of the manifold.

1. Homology of the union of open subspaces

Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of topological space X, where I is an ordered index set. We will use the standard notation $S_q(X)$ for the group of singular chains of dimension q (with coefficients in $\mathbb{C}$) in X. Also, by $S_* = S_*(X)$ we will denote the corresponding chain complex with the boundary operator ∂.

Definition 1.1. *A $\mathfrak{U}$-chain in X of multiplicity p and dimension q is an alternating function σ on I^{p+1} with values*

$$\sigma(i_0, i_1, \ldots, i_p) \in S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}),$$

which vanishes except at a finite number of a points of I^{p+1}.

Note that $\mathfrak{U}$-chains can be identified with elements of the bigraded group

$$C_{p,q} = \bigoplus_{i_0<i_1<\ldots<i_p} S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}), \quad p, q = 0, 1, \ldots$$

We denote by $S_q^{\mathfrak{U}} = S_q^{\mathfrak{U}}(X)$ the subgroup in $S_q = S_q(X)$ generated by singular q-simplices Δ, such that $\operatorname{supp}\Delta \subset U_i$ for some $U_i \in \mathfrak{U}$. The natural inclusion $\iota\colon S_*^{\mathfrak{U}} \to S_*(X)$ obviously is a chain map.

For the chain complex C_* with the boundary operator d we use the following standard notations for subgroups of cycles, subgroups of boundaries and homology groups:

$$Z_q(C_*) = \ker(d\colon C_q \to C_{q-1}),$$
$$B_q(C_*) = \operatorname{im}(d\colon C_{q+1} \to C_q),$$
$$H_q(C_*) = Z_q(C_*)/B_q(C_*).$$

The following fact shows that when calculating the homology of the space X it is sufficient to use the complex $S_*^{\mathfrak{U}}$.

Theorem 1.1 (see [8])**.** *The homomorphism* $\iota_*\colon H(S_*^{\mathfrak{U}}) \to H(X)$ *induced by the chain map* $\iota\colon S_*^{\mathfrak{U}} \to S_*(X)$ *is the isomorphism.*

Consider first the case $I = \{1, 2\}$. In this situation, there is the well-known long exact Mayer–Vietoris sequence for homology of the union of two open subsets. This sequence is obtained from short exact sequences

$$S_q^{\mathfrak{U}}(X) \xleftarrow{\varepsilon} S_q(U_1) \oplus S_q(U_2) \xleftarrow{\delta} S_q(U_1 \cap U_2), \tag{1}$$

where $\varepsilon\colon (\sigma_1, \sigma_2) \mapsto \sigma_1 + \sigma_2$ is an epimorphism and $\delta\colon \sigma \mapsto (\sigma, -\sigma)$ is a monomorphism, at that ε and δ are chain maps. Passing to homology, we get sequences

$$H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(U_1) \oplus H_q(U_2) \xleftarrow{\delta_*} H_q(U_1 \cap U_2),$$

in which in general ε_* is not an epimorphism and δ_* is not a monomorphism. The following properties hold: 1) $\operatorname{im}\delta_* = \ker\varepsilon_*$; 2) the *connecting homomorphism* $\varphi\colon H_q(S_*^{\mathfrak{U}}) \to H_{q-1}(U_1 \cap U_2)$ induced by the multivalued map $\delta^{-1}\partial\varepsilon^{-1}$ is correctly defined, at that $\operatorname{im}\varphi = \ker\delta_*$ and $\ker\varphi = \operatorname{im}\varepsilon_*$. We get the required long exact sequence:

$$\ldots \longleftarrow H_{q-1}(U_1 \cap U_2) \xleftarrow{\varphi} H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(U_1) \oplus H_q(U_2) \xleftarrow{\delta_*} H_q(U_1 \cap U_2) \longleftarrow \ldots,$$

in which groups $H_q(S_*^{\mathfrak{U}})$ are replaced by isomorphic groups $H_q(X)$.

Let now $\operatorname{card}(I) \geqslant 2$. We will describe a sequence of chain groups generalizing the short exact sequence (1). Inclusions $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \hookrightarrow U_{i_0} \cap U_{i_1} \cap \ldots [i_k] \ldots \cap U_{i_p}$, $k = 0, \ldots, p$, induce the Čech boundary operator $\delta\colon C_{p,q} \to C_{p-1,q}$ defined by the formula "alternating sum":

$$(\delta\sigma)(i_0, i_1, \ldots, i_{p-1}) = \sum_{i \in I} \sigma(i, i_0, \ldots, i_{p-1}).$$

In turn, the inclusions $U_i \subset X$ induce the operator $\varepsilon\colon C_{0,q} \to S_q^{\mathfrak{U}}$, which acts according to the same "alternating sum" formula as follows:

$$\varepsilon\sigma = \sum_{i \in I} \sigma(i),$$

at that $\varepsilon\delta = 0$. We obtain the following Mayer–Vietoris sequence for the groups of singular chains of the union

$$0 \longleftarrow S_q^{\mathfrak{U}} \xleftarrow{\varepsilon} C_{0,q} \xleftarrow{\delta} C_{1,q} \xleftarrow{\delta} C_{2,q} \xleftarrow{\delta} \ldots. \tag{2}$$

Theorem 1.2 (see [1])**.** *The sequence* (2) *is exact for all* $q = 0, 1, \ldots$.

If $\operatorname{card}(I) = 2$ then this sequence coincides with the sequence (1). In what follows, we will assume that $\operatorname{card}(I) > 2$. Maps $\delta\colon C_{p,*} \to C_{p-1,*}$ and $\varepsilon\colon C_{0,*} \to S^{\mathfrak{U}}_*$ are, as above, chain maps of the corresponding complexes with the boundary operator ∂. In passing to homology, the exact sequence (2) goes over into the sequence

$$0 \longleftarrow H_q(S^{\mathfrak{U}}_*) \xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} H_q(C_{1,*}) \xleftarrow{\delta_*} H_q(C_{2,*}) \xleftarrow{\delta_*} \ldots, \tag{3}$$

about which, in general, we can say that it is only semi-exact. In this case, the measure of "inexactness" is the homology groups of the sequence (3), considered as a chain complex with a boundary operator δ_* (or ε_*).

More abstract point of view on the generalization of the Mayer–Vietoris sequence for homology of the union relates to consideration of two spectral sequences of the double complex $C = (C_{p,q}; \delta, \partial)$ (see [2, 3]). This double complex is a *first quarter complex* ($C_{p,q} = 0$ for $p < 0$ or $q < 0$). By adding to C a column $(S^{\mathfrak{U}}_q; \partial)$ and a chain map $\varepsilon\colon C_{0,*} \to S^{\mathfrak{U}}_*$ we get the *extended double complex*

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 & S^{\mathfrak{U}}_q \arrow[l] \arrow[d, "\partial"'] & C_{0,q} \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] & C_{1,q} \arrow[l, "\delta"'] \arrow[d, "\partial"'] & \ldots \arrow[l, "\delta"'] \\
0 & S^{\mathfrak{U}}_{q-1} \arrow[l] \arrow[d, "\partial"'] & C_{0,q-1} \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] & C_{1,q-1} \arrow[l, "\delta"'] \arrow[d, "\partial"'] & \ldots \arrow[l, "\delta"'] \\
 & \cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] & \\
0 & S^{\mathfrak{U}}_0 \arrow[l] \arrow[d] & C_{0,0} \arrow[l, "\varepsilon"'] \arrow[d] & C_{1,0} \arrow[l, "\delta"'] \arrow[d] & \ldots \arrow[l, "\delta"'] \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4}$$

of singular chains which is dual to the familiar Čech–de Rham double complex for differential forms. Based on a double complex C we build a *total complex* TC, formed by a graded group

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}$$

and a boundary operator $D\colon (TC)_n \to (TC)_{n-1}$ such that $D\big|_{C_{p,q}} = \delta + (-1)^p \partial$.

The first of the spectral sequences $\{(E^r_{p,q}; d^r)\}$ of the complex C corresponds to filtration for TC determined by the formula $F_p(TC)_n = \bigoplus_{i \leqslant p} C_{i,n-i}$. We have $E^0_{p,q} = C_{p,q}$ and $d^0 = \pm\partial$, so $E^1_{p,q} = H_q(C_{p,*})$ (the vertical homology of the complex C) and the differential $d^1\colon E^1_{p,q} \to E^1_{p-1,q}$ coincides with the map induced by the chain map $\delta\colon C_{p,*} \to C_{p-1,*}$, i.e. $d^1 = \delta_*$. Further, the term $E^2_{p,q}$ (the horizontal homology of the vertical homology of the complex C) describes the homology of the sequence (3). Therefore, this spectral sequence (called the Mayer–Vietoris spectral sequence) is a generalization of the long exact Mayer–Vietoris sequence.

The second spectral sequence for C is determined by another filtration of the total complex: $F_p(TC)_n = \bigoplus_{j \leqslant p} C_{n-j,j}$. In this case $E^0_{p,q} = C_{q,p}$, so E^1 is determined by the horizontal homology of the complex C. Since the strings of the extended complex (4) are exact, then $E^1_{p,0} = S^{\mathfrak{U}}_p$ and $E^1_{p,q} = 0$ for $q > 0$. Considering the vertical homology, we obtain $E^2_{p,0} = H_p(X)$ and $E^2_{p,q} = 0$ for $q > 0$. This means that the second spectral sequence "degenerates" and gives the isomorphism $H_q(TC) \cong H_q(S^{\mathfrak{U}}_*)$. Thus, the double complex C "calculates" the homology of the space X.

2. Connecting homomorphism

Pursuing the goal of finding a generalization of the long exact Mayer–Vietoris sequence in the case of any finite open cover of the space X, we construct a homomorphism, which is a connecting homomorphism for semi-exact sequences (3).

We will use the following notation:

$$C^{\partial}_{p,q} = Z_q(C_{p,*}) = \ker(\partial\colon C_{p,q} \to C_{p,q-1}).$$

For the chain complex (2) consider the subcomplex

$$0 \longleftarrow Z_q(S^{\mathfrak{U}}_*) \xleftarrow{\varepsilon} C^{\partial}_{0,q} \xleftarrow{\delta} C^{\partial}_{1,q} \xleftarrow{\delta} C^{\partial}_{2,q} \xleftarrow{\delta} \ldots.$$

In general, there are nontrivial homology groups $H_p(C^{\partial}_{*,q}) = Z_p(C^{\partial}_{*,q})/B_p(C^{\partial}_{*,q})$, $p = 0, 1, \ldots$

Lemma 2.1. *Let ξ be the cycle belongs to $Z_p(C^{\partial}_{*,q})$, where $p \geqslant 0$, $q \geqslant 1$, and $\xi = \delta\xi_{p+1}$ for some $\mathfrak{U}$-chain $\xi_{p+1} \in C_{p+1,q}$. Then $\partial\xi_{p+1}$ belongs to $Z_{p+1}(C^{\partial}_{*,q-1})$ and the homology class $[\partial\xi_{p+1}] \in H_{p+1}(C^{\partial}_{*,q-1})$ depends only on the class $[\xi] \in H_p(C^{\partial}_{*,q})$. The correspondence $[\xi] \to [\partial\xi_{p+1}]$ defines a homomorphism*

$$\varphi_{p+1} = (\partial\delta^{-1})_*\colon H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1}).$$

Proof. Note that the existence of a $\mathfrak{U}$-chain ξ_{p+1} such that $\xi = \delta\xi_{p+1}$ follows from the fact that strings of the complex (4) are exact, due to condition $\delta\xi = 0$ ($\varepsilon\xi = 0$ for $p = 0$). We have $\partial(\partial\xi_{p+1}) = 0$, $\delta(\partial\xi_{p+1}) = \partial(\delta\xi_{p+1}) = \partial\xi = 0$ (see the diagram (5)), so $\partial\xi_{p+1} \in Z_{p+1}(C^{\partial}_{*,q-1})$.

$$\begin{tikzcd} 0 & \xi \arrow[l, "\delta"'] \arrow[d, "\partial"'] \arrow[dr, dashed, "\varphi_{p+1}"] & \xi_{p+1} \arrow[l, "\delta"'] \arrow[d, "\partial"] \\ & 0 & \partial\xi_{p+1} \arrow[l, "\delta"] \end{tikzcd} \qquad \begin{tikzcd} 0 & \xi \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] \arrow[dr, dashed, "\varphi_1"] & \xi_1 \arrow[l, "\delta"'] \arrow[d, "\partial"] \\ & 0 & \partial\xi_1 \arrow[l, "\delta"] \end{tikzcd} \tag{5}$$

If also $\xi = \delta\xi'_{p+1}$, then $\delta(\xi_{p+1} - \xi'_{p+1}) = 0$. So there exists a $\mathfrak{U}$-chain $\tau \in C_{p+2,q}$ such that $\xi_{p+1} - \xi'_{p+1} = \delta\tau$. This implies $\xi'_{p+1} = \xi_{p+1} - \delta\tau$. We have $\partial\xi'_{p+1} = \partial(\xi_{p+1} - \delta\tau) = \partial\xi_{p+1} - \partial(\delta\tau)$, at that $\partial(\delta\tau) \in B_{p+1}(C^{\partial}_{*,q-1})$, because $\partial(\delta\tau) = \delta(\partial\tau)$, where $\partial(\partial\tau) = 0$. Hence it follows $[\partial\xi_{p+1}] = [\partial\xi'_{p+1}] \in H_{p+1}(C^{\partial}_{*,q-1})$.

Finally, let ξ' be an arbitrary representative of the class $[\xi] \in H_p(C^{\partial}_{*,q})$. Then there exist a $\mathfrak{U}$-chain $\sigma \in C_{p+1,q}$, for which $\xi - \xi' = \delta\sigma$ and $\partial\sigma = 0$. If $\delta\xi_{p+1} = \xi$, then $\delta(\xi_{p+1} - \sigma) = \xi - \delta\sigma = \xi'$. The class $[\xi']$ maps to the class of the $\mathfrak{U}$-chain $\partial(\xi_{p+1} - \sigma) = \partial\xi_{p+1} - \partial\sigma = \partial\xi_{p+1}$, which was required.

It remains to note that for the map $[\xi] \mapsto [\partial\xi_{p+1}]$ the image of the sum of classes is obviously equal to the sum of the images, so this map is a homomorphism. $\square$

Remark 2.1. We have $C^{\partial}_{p,0} = C_{p,0}$ for $p \geqslant 0$, so $H_p(C^{\partial}_{*,0}) = H_p(C_{*,0}) = 0$. Hence for $q = 1$ the homomorphism described in the Lemma 2.1 is trivial.

Lemma 2.2. *Let be the cycle belongs to ξ $Z_q(S^{\mathfrak{U}}_*)$, where $q \geqslant 1$, and $\xi = \varepsilon\xi_0$ for some $\mathfrak{U}$-chain $\xi_0 \in C_{0,q}$. Then $\partial\xi_0$ belongs to $Z_0(C^{\partial}_{*,q-1})$ and the homology class $[\partial\xi_0] \in H_0(C^{\partial}_{*,q-1})$ depends only on the class $[\xi] \in H_q(S^{\mathfrak{U}}_*)$. The correspondence $[\xi] \to [\partial\xi_0]$ defines a homomorphism*

$$\varphi_0 = (\partial\varepsilon^{-1})_*\colon H_q(S^{\mathfrak{U}}_*) \to H_0(C^{\partial}_{*,q-1}).$$

Proof. Existence of a $\mathfrak{U}$-chain ξ_0, such that $\xi = \varepsilon\xi_0$, follows from the surjectivity of ε. The fact that $\partial\xi_0 \in Z_0(C^{\partial}_{*,q-1})$ and independence of the class $[\partial\xi_0] \in H_0(C^{\partial}_{*,q-1})$ from choice of ξ_0 can be proved in the same way as in Lemma 2.1.

Let ξ' represents a class $[\xi] \in H_q(S^{\mathfrak{U}}_*)$. So there exists a chain $\sigma \in S^{\mathfrak{U}}_{q+1}$, such that $\xi - \xi' = \partial\sigma$. Since ε is an epimorphism, there exists $\tau \in C_{0,q+1}$, such that $\sigma = \varepsilon\tau$. We have $\varepsilon(\xi_0 - \partial\tau) = \xi - \varepsilon(\partial\tau) = \xi - \partial(\varepsilon\tau) = \xi - \partial\sigma = \xi'$, therefore ∂-cycle ξ' corresponds to the class $[\partial(\xi_0 - \partial\tau)] = [\partial\xi_0 - \partial\partial\tau] = [\partial\xi_0]$. In this way the map $[\xi] \mapsto [\partial\xi_0]$ is correctly defined. Obviously, this is a homomorphism. $\square$

Remark 2.2. For $q = 1$ the homomorphism φ_0 is trivial.

The last two lemmas allow us to write the following "diagonal" sequence of maps:

$$H_r(S^{\mathfrak{U}}_*) \xrightarrow{\varphi_0} H_0(C^{\partial}_{*,r-1}) \xrightarrow{\varphi_1} H_1(C^{\partial}_{*,r-2}) \to \ldots \to H_{r-2}(C^{\partial}_{*,1}) \xrightarrow{\varphi_{r-1}} H_{r-1}(C^{\partial}_{*,0}) \cong 0. \qquad (6)$$

The sequence of homomorphisms (6) is related to the following notion of a *resolution for a cycle* by Gleason (see [4]).

Definition 2.1. *A $\mathfrak{U}$-resolution for the cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$ is a sequence $\{\xi_p\}_{p=0}^{r}$ of $\mathfrak{U}$-chains, $\xi_p \in C_{p,r-p}$, such that:*

1) $\varepsilon\xi_0 = \xi$;

2) $\delta\xi_p = \partial\xi_{p-1}$, $p = 1, \ldots, r$.

Remark 2.3. The existence of a $\mathfrak{U}$-resolution $\{\xi_p\}$ for any cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$ follows from the fact that the strings of the complex (4) are exact. Gleason's definition of a resolution suggests that $\xi \in Z_r(S_*)$. So the condition $\xi \in Z_r(S^{\mathfrak{U}}_*)$ is given as a criterion for the existence of the resolution.

Remark 2.4. The $\mathfrak{U}$-resolution for a cycle ξ (accurate to sign $\pm$ in front of its term) is, in fact, such D-cycle, which represents the image of the class $[\xi]$ under the isomorphism $H_q(S^{\mathfrak{U}}_*) \to H_q(TC)$. It is a "zig-zag" of the double complex C.

Comparing the definitions of homomorphisms φ_{p+1} and φ_0 from Lemmas 2.1, 2.2 with Definition 2.1, we obtain the following statement.

Proposition 2.1. *Let $\{\xi_p\}$ be the $\mathfrak{U}$-resolution of a cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$. Then the sequence of images of the cycle $[\xi] \in H_r(S^{\mathfrak{U}}_*)$ under homomorphisms (6) has the form*

$$[\xi] \longmapsto [\partial\xi_0] \longmapsto [\partial\xi_1] \longmapsto \ldots \longmapsto [\partial\xi_{r-2}] \longmapsto [\partial\xi_{r-1}] = [0].$$

In a similar way, one can consider a part of the sequence of homomorphisms (6) starting from the group $H_p(C^{\partial}_{*,q})$ for $p \geqslant 0$ and $q \geqslant 1$. In this case, we have the following sequence of homomorphisms

$$H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1}) \to \ldots \to H_{p+q-1}(C^{\partial}_{*,1}) \to H_{p+q}(C^{\partial}_{*,0}) \simeq 0,$$

which is naturally leads to following version of the notion of the $\mathfrak{U}$-resolution.

Definition 2.2. *Let ξ by a $\mathfrak{U}$-chain of the multiplicity $p \geqslant 0$ and the dimension $q \geqslant 1$ such that $\partial\xi = 0$ and $\delta\xi = 0$ ($\varepsilon\xi = 0$ for $p = 0$). A sequence $\{\xi_k\}$ of $\mathfrak{U}$-chains $\xi_k \in C_{k,p+q-k+1}$, is said to be the $\mathfrak{U}$-resolution of the ξ, if the following conditions hold:*

1) $\delta\xi_{p+1} = \xi$;

2) $\delta\xi_k = \partial\xi_{k-1}$, $k = p+2, \ldots, p+q+1$.

Remark 2.5. As in the case $\xi \in Z_r(S_*^{\mathfrak{U}})$, the resolution exists for any cycle $\xi \in Z_p(C_{*,q}^{\partial})$.

The existence of the connecting homomorphism assumes that the open cover $\mathfrak{U}$ of topological space X is finite ($\operatorname{card}(I) < \infty$). In what follows, we will assume that this covering consists of m ($m \geqslant 2$) elements.

In this case, for the double complex (4), we have $C_{p,q} \cong 0$ for $p \geqslant m$. Since the strings of the complex are exact, we see that $\delta\colon C_{m-1,*} \to C_{m-2,*}$ is monomorphism. Hence

$$Z_{m-1}(C_{*,q}) \cong 0, \quad H_{m-1}(C_{*,q}^{\partial}) = H_{m-1}(C_{*,q}) \cong 0.$$

For any resolution $\{\xi_k\}$ of a cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$, $r \geqslant m$, we have $\xi_k = 0$ for all $k > m-1$, at that $\partial\xi_{m-1} = 0$. We assume that the ∂-cycle ξ_{m-1} is the end term of this resolution, ignoring the following zero terms. We will proceed similarly with the resolution of cycle $\xi \in Z_p(C_{*,q}^{\partial})$.

The following statement is the last step to the construction of the desired generalization of a connecting homomorphism.

Lemma 2.3. *Let $\xi \in Z_{m-3}(C_{*,q}^{\partial})$, $q \geqslant 1$, and let $\{\xi_{m-2}, \xi_{m-1}\}$ be the resolution of ξ. The correspondence of classes $[\xi] \in H_{m-3}(C_{*,q}^{\partial})$ and $[\xi_{m-1}] \in H_{q-1}(C_{m-1,*})$ defines correctly a homomorphism of homology groups*

$$\psi_{m-1} = (\delta^{-1}\partial\delta^{-1})_*\colon H_{m-3}(C_{*,q}^{\partial}) \to H_{q-1}(C_{m-1,*}).$$

Proof. The action of the homomorphism ψ_{m-1} is illustrated by the following diagram:

$$\begin{tikzcd}
0 & \xi \arrow[l, "\delta"'] \arrow[d, "\partial"'] \arrow[drr, dashed, "\psi_{m-1}"] & \xi_{m-2} \arrow[l, "\delta"'] \arrow[d, "\partial"] & \\
 & 0 & \partial\xi_{m-2} \arrow[l, "\delta"] & \xi_{m-1} \arrow[l, "\delta"]
\end{tikzcd}$$

First, we show that the image $[\xi_{m-1}]$ does not depend on the choice of the resolution. Let $\{\xi_{m-2}, \xi_{m-1}\}$ and $\{\xi'_{m-2}, \xi'_{m-1}\}$ are resolutions of $\xi \in Z_{m-3}(C_{*,q}^{\partial})$. We have $\delta(\xi_{m-2} - \xi'_{m-2}) = = \delta\xi_{m-2} - \delta\xi'_{m-2} = \xi - \xi = 0$, so there is (the only one) $\mathfrak{U}$-chain $\tau \in C_{m-1,q}$ for which $\delta\tau = \xi_{m-2} - \xi'_{m-2}$. Hence $\xi'_{m-2} = \xi_{m-2} - \delta\tau$, and

$$\delta\xi'_{m-1} = \partial\xi'_{m-2} = \partial(\xi_{m-2} - \delta\tau) = \partial\xi_{m-2} - \partial(\delta\tau) = \delta\xi_{m-1} - \delta(\partial\tau) = \delta(\xi_{m-1} - \partial\tau).$$

Since $\delta\colon C_{m-1,*} \to C_{m-2,*}$ is a monomorphism, we get $\xi'_{m-1} = \xi_{m-1} - \partial\tau$. Therefore $[\xi'_{m-1}] = = [\xi_{m-1}]$ in $H_{q-1}(C_{m-1,*})$.

Next, we will show that the image $[\xi_{m-1}]$ also does not depend on the choice of the cycle representing the class in $H_{m-3}(C_{*,q}^{\partial})$. If $[\zeta] = [\xi]$ then $\zeta = \xi + \delta\sigma$, where $\partial\sigma = 0$. We put $\zeta_{m-2} = \xi_{m-2} + \sigma$, $\zeta_{m-1} = \xi_{m-1}$. We have $\delta\zeta_{m-2} = \delta(\xi_{m-2} + \sigma) = \delta\xi + \delta\sigma = \xi + \delta\sigma = \zeta$,

$$\partial\zeta_{m-2} = \partial(\xi_{m-2} + \sigma) = \partial\xi_{m-2} + \partial\sigma = \partial\xi_{m-2} = \delta\xi_{m-1} = \delta\zeta_{m-1}.$$

Hence, $\{\zeta_{m-2}, \zeta_{m-1}\}$ is the resolutions for ζ, at that $\zeta_{m-1} = \xi_{m-1}$.

In order to prove that this correspondence of homology classes is a homomorphism, it suffices to note that a resolution for the sum of $\mathfrak{U}$-chains is the sum (term-by-term) of resolutions. $\square$

Remark 2.6. Let $m = 2$. For the cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$ and its resolution $\{\xi_0, \xi_1\}$ similarly the correspondence $[\xi] \mapsto [\xi_1]$ gives the homomorphism

$$\psi_1 = (\delta^{-1}\partial\varepsilon^{-1})_*\colon H_r(S_*^{\mathfrak{U}}) \to H_{r-1}(C_{1,*}),$$

which is the connecting homomorphism φ for the usual long exact Mayer–Vietoris sequence.

Consider the homomorphisms $\varphi_0, \varphi_1, \ldots, \varphi_{m-3}$ from the sequence (6) and complement them with the homomorphism ψ_{m-1} from Lemma 2.3. We obtain a sequence of homomorphisms whose action, according to Proposition 2.1 and Lemma 2.3, is described in terms of a resolution of the cycle as follows.

Theorem 2.1. *For any $r \geqslant m$ there exists a sequence of homomorphisms*

$$H_r(S_*^{\mathfrak{U}}) \xrightarrow{\varphi_0} H_0(C^{\partial}_{*,r-1}) \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{m-3}} H_{m-3}(C^{\partial}_{*,r-m+2}) \xrightarrow{\psi_{m-1}} H_{r-m+1}(C_{m-1,*}), \tag{7}$$

given by the following sequence of images:

$$[\xi] \longmapsto [\partial\xi_0] \longmapsto \ldots \longmapsto [\partial\xi_{m-3}] \longmapsto [\xi_{m-1}],$$

where $\{\xi_p\}$ is the resolution for the cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$.

Finally, taking the composition of all homomorphisms from (7), we obtain the desired connecting homomorphism $\varphi = \psi_{m-1}\varphi_{m-3}\ldots\varphi_1\varphi_0 = (\delta^{-1}\partial\delta^{-1}\ldots\partial\delta^{-1}\partial\varepsilon^{-1})_*$.

Theorem 2.2. *Let $\mathfrak{U} = \{U_i\}$ be a finite open cover of a topological space X, consisting of $m \geqslant 2$ elements. Then the correspondence of homology classes $[\xi] \to [\xi_{m-1}]$, where $\xi \in Z_r(S_*^{\mathfrak{U}})$ and $\{\xi_p\}$ is arbitrary $\mathfrak{U}$-resolution of cycle ξ, defines a connecting homomorphism*

$$\varphi\colon H_r(S_*^{\mathfrak{U}}) \to H_{r-m+1}(C_{m-1,*}).$$

For $m > 2$ this homomorphism generates a semi-exact long sequence of homology groups

$$\begin{aligned}\ldots \longleftarrow H_{q-m+1}(C_{m-2,*}) \xleftarrow{\delta_*} H_{q-m+1}(C_{m-1,*}) \xleftarrow{\varphi} H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} \ldots \\ \ldots \xleftarrow{\delta_*} H_q(C_{m-1,*}) \xleftarrow{\varphi} H_{q+m-1}(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_{q+m-1}(C_{0,*}) \longleftarrow \ldots\end{aligned} \tag{8}$$

Proof. It remains to show that $\operatorname{im}\varphi \subset \ker\delta_*$ and $\operatorname{im}\varepsilon_* \subset \ker\varphi$. The first inclusion follows from the equality $\delta_*[\xi_{m-1}] = [\delta\xi_{m-1}] = [\partial\xi_{m-2}] = 0$. Let us prove the second inclusion. Let $[\xi] \in \operatorname{im}\varepsilon_*$. Then $\xi = \varepsilon\xi_0$ for some $\mathfrak{U}$-chain $\xi_0 \in C_{0,q}$, and $\partial\xi_0 = 0$. Therefore $\varphi_0[\xi] = [\partial\xi_0] = 0$, and hence $\varphi[\xi] = 0$. □

Remark 2.7. In what follows, we will assume that the codomain of the connecting homomorphism is the subgroup $H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*}) = \ker\delta_* \subset H_{q-m+1}(C_{m-1,*})$. The notation $H^{\mathrm{sep}}_q(C_{m-1,*})$ will be discussed later. In this way,

$$\varphi\colon H_q(S_*^{\mathfrak{U}}) \to H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*}).$$

Similarly, from the previous proof, we can conclude that $\operatorname{im}\psi_{m-1} \subset \ker\delta_*$. Therefore, we further assume that $\psi_{m-1}\colon H_{m-3}(C^{\partial}_{*,q}) \to H^{\mathrm{sep}}_{q-1}(C_{m-1,*})$.

The proofs of the following properties of the homomorphisms φ_0, φ_{p+1} and ψ_{m-1}, from which the connecting homomorphism is "glued", are completely standard. Moreover, conditions on the homology appear as sufficient conditions for inverting the required vertical arrows of the complex (4) at diagrammatic search.

Lemma 2.4. *If $H_{q-1}(C_{0,*}) \cong 0$, then the homomorphism $\varphi_0\colon H_q(S_*^{\mathfrak{U}}) \to H_0(C^{\partial}_{*,q-1})$ is an epimorphism. If $H_q(C_{0,*}) \cong H_{q-1}(C_{1,*}) \cong 0$, then it is a monomorphism.*

Lemma 2.5. *If $H_{q-1}(C_{p+1,*}) \cong 0$, then the homomorphism $\varphi_{p+1}\colon H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1})$ is an epimorphism. If $H_{q-1}(C_{p+2,*}) \cong 0$, then it is a monomorphism.*

Lemma 2.6. *The homomorphism* $\psi_{m-1}\colon H_{m-3}(C^{\partial}_{*,q}) \to H^{sep}_{q-1}(C_{m-1,*})$ *is an isomorphism.*

Remark 2.8. For $m = 2$ the homomorphism $\psi_1\colon H_q(S^{\mathfrak{U}}_*) \to H^{\mathrm{sep}}_{q-1}(C_{1,*})$ is only an epimorphism in general case.

Considering that the composition of epimorphisms is an epimorphism, and the composition of monomorphisms is a monomorphism, we obtain the following property of the connecting homomorphism.

Theorem 2.3. *For the connecting homomorphism*

$$\varphi\colon H_q(S^{\mathfrak{U}}_*) \to H^{sep}_{q-m+1}(C_{m-1,*})$$

to be an epimorphism, it suffices to satisfy the condition

$$H_{q-1}(C_{0,*}) \cong H_{q-2}(C_{1,*}) \cong \ldots \cong H_{q-m+2}(C_{m-3,*}) \cong 0, \tag{9}$$

and for φ *to be a monomorphism, it suffices to satisfy the condition*

$$H_q(C_{0,*}) \cong H_{q-1}(C_{1,*}) \cong \ldots \cong H_{q-m+2}(C_{m-2,*}) \cong 0. \tag{10}$$

Remark 2.9. The homomorphism $\varphi\colon H_q(S^{\mathfrak{U}}_*) \to H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*})$ is surjective if and only if the sequence (8) is exact in the term $H_{q-m+1}(C_{m-1,*})$.

Remark 2.10. The condition (9) can be replaced by the following weaker condition: if $\xi \in C_{p,q-p-1}$ such that $\partial\xi = 0$ and $\delta\xi = 0$ ($\varepsilon\xi = 0$), then $[\xi] = 0$ in $H_{q-p-1}(C_{p,*})$, $p = 0, \ldots, m-3$.

3. Separating cycles

The notion of the *separating cycle* appeared in complex analysis in connection with a property of the Grothendieck residue. Let ω be a meromorphic n-form on an n-dimensional complex-analytic manifold M, and $F_1, \ldots, F_n$ are polar hypersurfaces of ω, $F = F_1 \cup \ldots \cup F_n$. In a sufficiently small neighborhood U_a of an isolated point a of the intersection $Z = F_1 \cap \ldots \cap F_n$ the form ω is given by

$$\omega = \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{f_1(z) \ldots f_n(z)}, \tag{11}$$

where $h, f_1, \ldots f_n$ are holomorphic germs at a, $F_k|_{U_a} = \{f_k = 0\}$. The *grothendieck residue* of the form ω at the point a is represented by the integral

$$\mathrm{res}_a\, \omega = \frac{1}{(2\pi i)^n} \int_{\gamma^{(a)}} \omega, \tag{12}$$

where $\gamma^{(a)}$ is a *local cycle* at a having the form

$$\gamma^{(a)} = \{z \in U_a \colon |f_1(z)| = \varepsilon_1, \ldots, |f_n(z)| = \varepsilon_n\}. \tag{13}$$

The orientation of $\gamma^{(a)}$ is determined by the condition $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) \geqslant 0$. It is not hard to see that $\gamma^{(a)} \in Z_n(M \setminus F)$.

The mentioned property of the residue (12) is as follows: it is zero if h belongs to the ideal generated by $f_1, \ldots, f_n$ in the ring $\mathcal{O}_a$ of germs of holomorphic functions. This property is of a topological nature. Indeed, if $h = h_j f_j$, then ω has poles only on the $n-1$ hypersurfaces $F_k = \{f_k = 0\}$, $k = 1, \ldots [j] \ldots, n$, in the complement of which the n-cycle $\gamma^{(a)}$ becomes homologically trivial. Indeed, $\gamma^{(a)}$ is a boundary of the $(n+1)$-chain $\sigma_j = \{|f_1| = \varepsilon_1, \ldots, |f_j| \leqslant \varepsilon_j, \ldots, |f_n| = \varepsilon_n\}$ taken with a suitable orientation, so $\gamma^{(a)} \sim 0$. Therefore, according to Stokes' formula, the integral (12) for the $h = h_j f_j$ is zero.

Definition 3.1. *A n-dimensional cycle $\Gamma \in Z_n(M \setminus F)$ separates hypersurfaces $F_1, \dots, F_n$, if it satisfies the conditions*

$$\Gamma \sim 0 \text{ in } M \setminus (F_1 \cup \dots [j] \dots \cup F_n) \text{ for all } j = 1, \dots, n.$$

By the above, the local cycle $\gamma^{(a)}$ from the definition of the Grothendieck residue separates the set of polar hypersurfaces of ω.

An important argument for the use of Grothendieck residues of meromorphic forms is their rational computability in terms of a finite number of Taylor coefficients of functions $h, f_1, \dots, f_n$ at the point a. In this connection there is a problem of the representation of the integral of a meromorphic form ω by residues. The topological formulation of this problem is as follows. Let $\mathcal{F} = \{F_1, \dots, F_n\}$ be a set of hypersurfaces in an n-dimensional complex-analytic manifold M. Let us denote by F the union of these hypersurfaces, and by Z_0 the discrete part of their intersection $Z = F_1 \cap \dots \cap F_n$. It is required to find out in which case the given n-cycle Γ in $M \setminus F$ is homologically expressed in terms of local cycles $\gamma^{(a)}$, $a \in Z_0$. In view of the above, for this it is necessary the cycle Γ separates the given set of hypersurfaces $\mathcal{F}$.

We denote by $H_n^{\text{loc}}(M \setminus F)$ the subgroup in $H_n(M \setminus F)$ generated by the classes of all local cycles $\gamma^{(a)}$, $a \in Z_0$. We also denote by $H_n^{\text{sep}}(M \setminus F)$ the subgroup of classes of all cycles separating the set of hypersurfaces $\mathcal{F}$. We have

$$H_n^{\text{loc}}(M \setminus F) \subset H_n^{\text{sep}}(M \setminus F).$$

We are interested in sufficient conditions on the manifold M and the collection of hypersurfaces $\mathcal{F}$ under which $H_n^{\text{sep}}(M \setminus F) = H_n^{\text{loc}}(M \setminus F)$, that is, in which any separating cycle is homologically represented in terms of local cycles.

Consider the space $X = M \setminus Z$ and its cover $\mathfrak{U}$ formed by open sets $U_j = M \setminus F_j$, $j = 1, \dots, n$. We get the corresponding extended double complex (4) and the semi-exact sequence (8) for $q = m = n$. Given the isomorphism of Theorem 1.1, this sequence can be written as

$$\dots \longleftarrow H_n(C_{m-2,*}) \xleftarrow{\delta_*} H_n(M \setminus F) \xleftarrow{\varphi} H_{2n-1}(M \setminus Z) \xleftarrow{\varepsilon_*} H_{2n-1}(C_{0,*}) \longleftarrow \dots . \tag{14}$$

The condition for separating the set $\mathcal{F}$ by a cycle Γ means that $\delta\Gamma$ is an ∂-boundary in the group $C_{n-2,n}$. So $H_n^{\text{sep}}(M \setminus F) = \ker \delta_*$, which explains the previously used (see Remark 2.7) notation for the subgroup $H_{q-m+1}^{\text{sep}}(C_{m-1,*}) = \ker \delta_* \subset H_{q-m+1}(C_{m-1,*})$. Let us show that $H_n^{\text{loc}}(M \setminus F) \subset \operatorname{im} \varphi$.

It suffices to show that each generator $[\gamma^{(a)}]$, $a \in Z_0$, of the group $H_n^{\text{loc}}(M \setminus F)$ have preimage in $H_{2n-1}(M \setminus Z)$. For a fixed point $a \in Z_0$, consider the $(2n-1)$-dimensional sphere S_a centred at the point a of a small radius. The class $[S_a]$ can be represented as a cycle $\partial\Pi_a$, where

$$\Pi_a = \{z \in U_a \colon |f_i(z)| < \varepsilon_i,\ i = 1, \dots, n\}, \tag{15}$$

where the orientation of the *special analytical polyhedron* Π_a is induced by the orientation of the manifold M. Moreover, the boundary $\partial\Pi_a$ of the polyhedron Π_a is the sum of its $(n-1)$-dimensional faces $\tau_j = \{|f_1| \leqslant \varepsilon_1, \dots, |f_j| = \varepsilon_j, \dots, |f_n| \leqslant \varepsilon_n\}$, $j = 1, \dots, n$, taken with suitable orientation, at that $\operatorname{supp} \tau_j \subset U_j$. Therefore, $\partial\Pi_a \in Z_{2n-1}(S_*^{\mathfrak{U}})$, and for the cycle $\partial\Pi_a$ can be built the $\mathfrak{U}$-resolution $\{\xi_p\}$. It is directly verified that terms of the resolution can be taken as follows:

$$\xi_p(i_0, i_1, \dots, i_p) = \pm\tau_{i_0} \cap \tau_{i_1} \cap \dots \cap \tau_{i_p}.$$

Moreover, the final term $\xi_{n-1} = \xi_{n-1}(1, \dots, n)$ of the resolution is the local cycle $\gamma^{(a)}$. So $\varphi[S_a] = \varphi[\Pi_a] = [\gamma^{(a)}]$, as required to prove. It also follows from the last reasoning that if the group $H_{2n-1}(M \backslash Z)$ is generated by the classes of cycles S_a, $a \in Z_0$, in particular if $H_{2n-1}(M) \cong 0$ and $Z = Z_0$, then $H_n^{\mathrm{loc}}(M \setminus F) = \operatorname{im} \varphi$. This fact, with considering Remark 2.10, proves the following theorem.

Theorem 3.1. *Let $H_{2n-1}(M) \cong 0$ and let the intersection $Z = F_1 \cap \ldots \cap F_n$ be discrete. Then the groups $H_n^{sep}(M \setminus F)$ and $H_n^{loc}(M \setminus F)$ are coincide if and only if the semi-exact sequence* (14) *is exact in the term $H_n(M \setminus F)$.*

Remark 3.1. For $n = 2$ the sequence (14) turns into a long exact Mayer–Vietoris sequence. Therefore, under the assumptions made on the manifold and the set of hypersurfaces, the equality $H_2^{\mathrm{loc}}(M \setminus F) = H_2^{\mathrm{sep}}(M \setminus F)$ is always the case.

Consider (see Remark 2.7) the connecting homomorphism φ from the sequence (14) as the homomorphism

$$\varphi \colon H_{2n-1}(M \setminus Z) \to H_n^{\mathrm{sep}}(M \setminus F).$$

Sequence (14) is exact in the term $H_n(M \backslash F)$ if and only if φ is an epimorphism (see Remark 2.9). Considering Theorem 2.3, we obtain the following sufficient condition.

Theorem 3.2. *Let $H_{2n-1}(M) \cong 0$ and let the intersection $Z = F_1 \cap \ldots \cap F_n$ be discrete. Then for the equality of groups $H_n^{sep}(M \setminus F) = H_n^{loc}(M \setminus F)$ it suffices to satisfy the condition*

$$H_{2n-2}(C_{0,*}) \cong H_{2n-3}(C_{1,*}) \cong \ldots \cong H_{n+1}(C_{n-3,*}) \cong 0. \tag{16}$$

Remark 3.2. In last two theorems, instead the triviality of the group $H_{2n-1}(M)$ and the discreteness of the intersection $Z = F_1 \cap \ldots \cap F_n$, we can assume the following weaker condition: the group $H_{2n-1}(M \setminus Z)$ is generated by classes of cycles S_a, $a \in Z_0$. The condition (16) can also be replaced (see Remark 2.10) by the following weaker condition: if $\xi \in C_{p,2n-p-2}$ such that $\partial \xi = 0$ and $\delta \xi = 0$ ($\varepsilon \xi = 0$), then $[\xi] = 0$ in group $H_{2n-p-2}(C_{p,*})$, $p = 0, \dots, n-3$.

As a consequence of Theorem 3.2, it is easy to obtain the following theorem on separating cycles in Stein manifolds which was proved by Tsikh.

Theorem 3.3 (see [5])**.** *Let M be a Stein manifold of dimension n. Then the equality of groups $H_n^{sep}(M \setminus F) = H_n^{loc}(M \setminus F)$ holds for any set $\mathcal{F} = \{F_1, \dots, F_n\}$ of hypersurfaces in M.*

Proof. As it was noted in [5], it suffices to prove the statement of the theorem under the following assumptions: 1) $H_{2n-1}(M) \cong 0$; 2) $M \setminus F_j$, $j = 1, \dots, n$, are the Stein manifolds; 3) the intersection $Z = F_1 \cap \ldots \cap F_n$ is discrete. It remains to note that all possible intersections of the sets $U_j = M \setminus F_j$ are also Stein manifolds. Condition (16) follows from the fact that for an arbitrary Stein manifold X the homology groups (with coefficients in the field) $H_q(X)$ are trivial for $q > \dim X$. □

Remark 3.3. In the case of the Stein manifold M and an arbitrary set of hypersurfaces $\mathcal{F}$ in M the connecting homomorphism $\varphi \colon H_{2n-1}(M \setminus Z) \to H_n^{\mathrm{sep}}(M \setminus F)$ is an isomorphism. The injectivity follows from the fulfillment of conditions of the form (10).

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Связывающий гомоморфизм и разделяющие циклы

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Аннотация. Обсуждается построение длинной полуточной последовательности Майера–Виеториса для гомологий объединения конечного числа открытых подпространств. Эта последовательность применяется для получения топологических условий, при которых интеграл от мероморфной дифференциальной формы в многомерном комплексном многообразии представляется в виде суммы вычетов Гротендика. Для существования такого представления интеграла необходимо, чтобы цикл интегрирования разделял семейство полярных гиперповерхностей формы. Условие разделения в ряде случаев оказывается достаточным условием для представления интеграла в виде суммы вычетов. Ранее при описании таких случаев (в работах А. К. Циха, А. П. Южакова, Р. В. Ульверта и др.) ключевым оказывалось условие штейновости многообразия. Основным результатом данной статьи является ослабление этого условия.

Ключевые слова: последовательность Майера–Виеториса, вычет Гротендика, разделяющий цикл.

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On an Inverse Problem for a Stationary Equation with Boundary Condition of the Third Kind

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Abstract. The identification of an unknown coefficient in the lower term of elliptic second-order differential equation $Mu + ku = f$ with boundary condition of the third kind is considered. The identification of the coefficient is based on integral boundary data. The local existence and uniqueness of the strong solution for the inverse problem is proved.

Keywords: inverse problem for PDE, boundary value problem, second-order elliptic equation, existence and uniqueness theorem.

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Introduction

In this paper an inverse problem for some stationary equation is considered.

Problem. For given functions $f(x), \sigma(x), \beta(x), h(x)$ and constant μ find function $u(x)$ and constant k that satisfy the equation

$$-\mathrm{div}(\mathcal{M}(x)\nabla u) + m(x)u + ku = f, \tag{1}$$

boundary condition

$$\left(\frac{\partial u}{\partial \overline{N}} + \sigma(x)u\right)\Big|_{\partial\Omega} = \beta(x), \tag{2}$$

and the condition of overdetermination

$$\int_{\partial\Omega} uh(x)ds = \mu. \tag{3}$$

Here $\Omega \subset \mathbf{R}^n$ is a bounded domain with boundary $\partial\Omega$, $t \in (0, T)$, $\mathcal{M}(x) \equiv (m_{ij}(x))$ is a matrix of functions $m_{ij}(x)$, $i, j = 1, 2, \ldots, n$; $m(x)$ is a scalar function, $\dfrac{\partial}{\partial \overline{N}} = (\mathcal{M}(x)\nabla, \mathbf{n})$, $\mathbf{n}$ is the unit vector of the outward normal to the boundary $\partial\Omega$.

A main goal of this paper is to establish the existence and uniqueness of the strong solution of inverse problem (1)–(3). The additional integral boundary data similar to condition of overdetermination (3) were considered [1–3]. Following the idea given in [1–3] and using method

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developed in [4], we prove the existence of the solution by reducing the inverse problem to an operator equation of the second kind for the unknown coefficient. Note that the problem for the same equation with Dirichlet boundary conditions was considered [5].

The study of inverse problems for the elliptic equations goes back to fundamental work of M. M. Lavrentiev [6]. Inverse problems for the elliptic equation with special boundary conditions (non-local conditions, non-classical conditions) were considered [7–9].

Such problems arise in determination of unknown physical properties of a medium. In particular, the lowest coefficient k specifies, for instance, the catabolism of contaminants due to chemical reactions [10] or the absorption in diffusion and acoustic problems [11].

1. The preliminaries

The following notations are used $\|\cdot\|_R$, $(\cdot,\cdot)_R$ — the norm and the inner product in $\mathbb{R}^n$; $\|\cdot\|$, $(\cdot,\cdot)$ — the norm and the inner product in $L^2(\Omega)$; $\|\cdot\|_j$, $\langle\cdot,\cdot\rangle_1$ — the norm in $W_2^j(\Omega)$, $j=1,2$, and the duality relation between $\overset{\circ}{W}{}_2^1(\Omega)$ and $W_2^{-1}(\Omega)$, respectively. The linear operator $M: W_2^1(\Omega)\to(W_2^1(\Omega))^*$ of the form

$$M=-\mathrm{div}(\mathcal{M}(x)\nabla)+m(x)I,$$

Is introduced, where I is the identity operator. The notation

$$\langle Mv_1,v_2\rangle_M=\int_\Omega((\mathcal{M}(x)\nabla v_1,\nabla v_2)_R+m(x)v_1v_2)dx$$

is also used for $v_1,v_2\in W_2^1(\Omega)$. The following assumptions hold throughout the paper

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i,j,l=1,2,\ldots,n$, and $m(x)$ are bounded in Ω. Operator M is elliptic, that is, there exist positive constants m_0 and m_1 such that for all $v\in W_2^1(\Omega)$

$$m_0\|v\|_1^2\leqslant\langle Mv,v\rangle_M\leqslant m_1\|v\|_1^2. \tag{4}$$

II. M is self-adjoint, that is, $m_{ij}(x)=m_{ji}(x)$ for $i,j=1,\ldots,n$.

The existence and uniqueness results for problem (1)–(3) is based on two lemmas for direct problem (1)–(2) with known coefficient k.

Lemma 1.1. *Let u be the strong solution of problem (1)–(2). If $f\geqslant 0$, $\beta\geqslant 0$, $\sigma\geqslant 0$, $k>0$ and assumptions I, II are fulfilled, then $u\geqslant 0$ almost everywhere in Ω.*

Proof. Multiplying (1) by $\bar u=\min\{\bar u,0\}$ in terms of the inner product in $L_2(\Omega)$ and integrating by parts in first term, we obtain

$$\langle M\bar u,\bar u\rangle_1+k\|\bar u\|^2+\int_{\partial\Omega}\sigma\bar u^2ds-\int_{\partial\Omega}\beta\bar u ds-(f,\bar u)=0.$$

Taking into account the lemma conditions, the last equality implies that

$$m_1\|\bar u\|_1^2\leqslant 0.$$

So, $\bar u=0$ almost everywhere in Ω. Lemma is proved. □

Lemma 1.2. *Let $u_1, u_2 \in W_2^2(\Omega)$ are the solutions of the problems*

$$Mu_i + k_i u_i = f_i,$$
$$\left(\frac{\partial u_i}{\partial \overline{N}} + \sigma u_i\right)\Big|_{\partial\Omega} = \beta_i,$$

here $i = 1, 2$.

If $0 \leqslant k_1 \leqslant k_2,\ 0 \leqslant \beta_2 \leqslant \beta_1,\ 0 \leqslant f_2 \leqslant f_1$ and $\sigma(x) \geqslant 0$ then $u_1 \geqslant u_2 \geqslant 0$ for almost all $x \in \bar{\Omega}$.

Proof. By Lemma (1.1), $u_i \geqslant 0$, $i = 1, 2$, for almost all $x \in \bar{\Omega}$. The difference $u_1 - u_2$ satisfies equation

$$M(u_1 - u_2) + k_1(u_1 - u_2) = (k_2 - k_1)u_2 + f_1 - f_2, \tag{5}$$

and boundary condition

$$\left(\frac{\partial(u_1 - u_2)}{\partial \overline{N}} + \sigma(x)(u_1 - u_2)\right)\Big|_{\partial\Omega} = \beta_1 - \beta_2.$$

Taking into account the lemma conditions, the right side of (5) is non-negative and $\beta_1 - \beta_2 \geqslant 0$. So, by Lemma (1.1), $u_1 - u_2 \geqslant 0$ for almost all $x \in \bar{\Omega}$. Lemma is proved. □

2. Existence and uniqueness

First of all the solution of the inverse problem should be defined. By the solution of the inverse problem is meant function $u \in W_2^2(\Omega)$ and a positive real number k. They satisfy equation (1) almost everywhere in Ω and conditions (2)–(3) almost everywhere on $\partial\Omega$. Now, to formulate the theorem functions a, a^τ and b are introduced as the solution of the problems

$$Ma = f(x), \qquad \left(\frac{\partial a}{\partial \overline{N}} + \sigma(x)a\right)\Big|_{\partial\Omega} = \beta(x); \tag{6}$$

$$Ma^\tau + \tau a^\tau = f, \qquad \left(\frac{\partial a^\tau}{\partial \overline{N}} + \sigma(x)a^\tau\right)\Big|_{\partial\Omega} = \beta(x); \tag{7}$$

$$Mb = 0, \qquad \left(\frac{\partial b}{\partial \overline{N}} + \sigma(x)b\right)\Big|_{\partial\Omega} = h(x), \tag{8}$$

where $\tau > 0$ is a real number.

Theorem 2.1. *Let $\partial\Omega \in C^2$ and assumptions I, II are fulfilled. Suppose also that*

(i) *$f(x) \in L^2(\Omega)$, $\beta(x), h(x) \in W_2^{3/2}(\partial\Omega)$, $\sigma(x) \in C(\partial\Omega)$;*

(ii) *$f(x) \geqslant 0$ almost everywhere in Ω; $\beta(x) \geqslant 0$, $\sigma(x) \geqslant 0$, $h(x) \geqslant 0$ for almost all $x \in \partial\Omega$ and there is a smooth piece Γ of the boundary $\partial\Omega$ and a constant $\delta > 0$ such that $\beta \geqslant \delta$ and $\omega \geqslant \delta$ almost everywhere on Γ.*

Then problem (1)–(3) has a solution $\{u, k\}$. Moreover, the estimates

$$a^\tau \leqslant u \leqslant a, \quad 0 \leqslant k \leqslant \tau, \quad \|u\|_2 \leqslant C(\tau + 1)\|a\| + \|a\|_2 \tag{9}$$

hold with some $\tau > 0$, and constant C depends on $mes\Omega, \tau, m_0$ and m_1. If

$$0 \leqslant \mu - \Psi \leqslant \frac{m_0(a, b)^2}{\|a\|\|b\|}, \tag{10}$$

where $\Psi = \int\limits_{\partial\Omega} ahds - (f, b)$, then the solution is unique.

Proof. Following the idea given in [4] and the method developed in [1], the original problem is reduced to an equivalent inverse problem with a non-linear operator equation for k. It follows from (1)–(3) that function $w = a - u$ and the constant k satisfy the following relations

$$Mw + kw = ka, \tag{11}$$

$$\left(\frac{\partial w}{\partial \overline{N}} + \sigma(x)w\right)\Big|_{\partial\Omega} = 0, \tag{12}$$

$$\int_{\partial\Omega} whds = \int_{\partial\Omega} ahds - \mu. \tag{13}$$

Taking into account (8), (11) and (12), multiplying (9) by b in terms of the inner product in $L_2(\Omega)$ and integrating by parts twice, we obtain

$$k(u,b) = \int_{\partial\Omega} ahds + (f,b) - \mu = \Psi - \mu.$$

Let operator $A : R_+ \to R$ maps every $y \in R_+$ into the real number Ay by the rule

$$Ay = \frac{\Psi - \mu}{(u_y, b)}, \tag{14}$$

where u_y is the solution of direct problem (1)–(2) with $y = k$. One can show that the original problem is solvable if and only if operator A has a fixed point, i.e., the operator equation $Ak = k$ has a solution.

Now we need to prove that there exists $\tau > 0$ such that operator A defined for all $k \in [0,\tau]$, is continuous on $[0,\tau]$, and maps $[0,\tau]$ into itself. Indeed, Lemma 1.2 implies that for all $0 \leqslant y \leqslant \tau$

$$a^\tau \leqslant u_y \leqslant a. \tag{15}$$

Therefore

$$Ay \geqslant \frac{\Psi - \mu}{(a,b)} \geqslant 0.$$

On the other hand, let us introduce the difference between (6) and (7)

$$M(a - a^\tau) + \tau(a - a^\tau) = \tau a.$$

Then, multiplying the difference by $a - a^\tau$ in terms of the inner product in $L^2(\Omega)$, integrating by parts in the first term and estimating the left-hand side of the result with the help of (4), we obtain

$$m_0\|a - a^\tau\|_1^2 + 2\int_{\partial\Omega} \sigma(a - a\tau)^2 ds + 2\tau\|a - a^\tau\|^2 \leqslant \frac{\tau^2}{m_0}\|a\|^2.$$

This estimate and (15) allows one to obtain the lower bound of (u_y, b) in (14)

$$(u_y, b) \geqslant (a^\tau, b) = (a,b) - (a - a^\tau, b) \geqslant (a,b) - \frac{\tau}{\sqrt{m_0}}\|a\|\|b\| \geqslant 0. \tag{16}$$

Hence

$$0 \leqslant \tau \leqslant \frac{\sqrt{m_0}(a,b)}{\|a\|\|b\|}.$$

In view of (14) and (16)

$$Ay \leqslant \frac{\Psi - \mu}{(a,b) - \frac{\tau}{\sqrt{m_0}}\|a\|\|b\|} \leqslant \tau.$$

Accordingly, the relation $Ay \leqslant \tau$ holds for all $\tau > 0$ such that

$$\frac{\tau^2}{\sqrt{m_0}}\|a\|\|b\| - \tau(a,b) + \Psi - \mu \leqslant 0. \tag{17}$$

The last inequality is possible, because it follows from the theorem conditions that

$$D \equiv (a,b)^2 - \frac{4(\Psi-\mu)\|a\|\|b\|}{\sqrt{m_0}} \geqslant 0.$$

Then (17) is valid for τ that obeys the inequality

$$\frac{\sqrt{m_0}((a,b)-\sqrt{D})}{2\|a\|\|b\|} \leqslant \tau \leqslant \frac{\sqrt{m_0}((a,b)+\sqrt{D})}{2\|a\|\|b\|}.$$

Thus, the operator A maps the segment into itself.

Now one can obtain the estimate of u_y in $W_2^1(\Omega)$ provided that $y \in [0,\tau]$. Let $w_y = a - u_y$. This function satisfies (11)–(13) with $y = k$. Multiplying (11) for $k = y$ by w_y in terms of the inner product in $L_2(\Omega)$ and integrating by parts in the first term, we obtain

$$(Mw_y, w_y) + y\|w_y\|^2 = \langle Mw_y, w_y\rangle_M + y\|w_y\|^2 + \int_{\partial\Omega} \sigma w_y^2 ds = y(a, w_y).$$

In view of (15) and the definition of w_y, we have

$$\left| y\int_\Omega a w_y dx \right| \leqslant \tau\|a\|^2.$$

Taking into account the ellipticity of operator M, the last two relations implies that

$$\|u_y\|_1 \leqslant \sqrt{\frac{\tau}{m_0}}\|a\| + \|a\|_1. \tag{18}$$

In accordance with [12], direct problem (11)–(12) has a unique solution $w_y \in W_2^2(\Omega)$ for all $y \geqslant 0$. Furthermore, (11) is fulfilled almost everywhere in Ω and $Mw_y \in L_2(\Omega)$. Multiplying (11) with $k = y$ by Mw_y in terms of the inner product in $L_2(\Omega)$ and integrating by parts in the second component, one can obtain the equality

$$\|Mw_y\|^2 + y\langle w_y, Mw_y\rangle_M + \int_{\partial\Omega} \sigma w_y^2 ds = y(aM, w_y). \tag{19}$$

In accordance with (4), the second term of (19) is non-negative and

$$y|(a, Mw_y)| \leqslant \tau\|a\|\|Mw_y\| \leqslant \frac{1}{2}\tau^2\|a\|^2 + \frac{1}{2}\|Mw_y\|^2.$$

it follows from the last two relations that

$$\|Mw_y\|^2 \leqslant \tau^2\|a\|^2. \tag{20}$$

In view of the definition of w_y and the inequality [12]

$$\|v\|_2 \leqslant C_M(\|Mv\| + \|v\|),$$

valid for all $v \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$ with the constant C_M depending on M and $mes\Omega$, relations (18), (20) imply the estimate

$$\|u\|_2 \leqslant \|w_y\|_2 + \|a\|_2 \leqslant C_M(\tau + 1)\|a\| + \|a\|_2.$$

Now one can show that operator A is continuous on segment $[0, \tau]$. Let $y_1, y_2 \in [0, \tau]$ and u_{y_1}, u_{y_1} are the solutions of problem (11), (12) with $y_1 = k$ and $y_2 = k$, respectively. By the definition of operator A, (15) and (16)

$$|A_{y_1} - A_{y_2}| \leqslant \frac{\|u_{y_2} - u_{y_1}\|\|b\|(\Psi - \mu)}{(a^\tau, b)^2} \leqslant \frac{\|u_{y_2} - u_{y_1}\|\|b\|(\Psi - \mu)}{((a, b) - \frac{\tau}{\sqrt{m_1}}\|a\|\|b\|)^2}. \tag{21}$$

On the other hand, multiplying the difference of equation (1) for $k = y_1$ and $k = y_2$ by $u_{y_1} - u_{y_2}$ in terms of the inner product in $L_2(\Omega)$ and integrating by parts in the first term of the resulting equality,we obtain

$$\langle M(u_{y_1} - u_{y_2}, u_{y_1}) \rangle_1 + \int_{\partial\Omega} \sigma(u_{y_1} - u_{y_2})^2 ds + y_1\|u_{y_1} - u_{y_2}\|^2 = (y_2 - y_1)(u_{y_2}, u_{y_1} - u_{y_2}). \tag{22}$$

In accordance with (4) and the non-negativity of y_1, the left side of (22) can be estimated as

$$\langle M(u_{y_1} - u_{y_2}), u_{y_1} \rangle_1 + \int_{\partial\Omega} \sigma(u_{y_1} - u_{y_2})^2 ds + y_1\|u_{y_1} - u_{y_2}\|^2 \geqslant m_0\|u_{y_1} - u_{y_2}\|_1^2.$$

The right term of (22) is estimated with the use of (15) as

$$|(y_2 - y_1)(u_{y_2}, u_{y_1} - u_{y_2})| \leqslant \frac{1}{2m_0}|y_2 - y_1|^2\|a\|^2 + \frac{m_0}{2}\|u_{y_1} - u_{y_2}\|_1^2.$$

Hence, we obtain the relation

$$\|u_{y_1} - u_{y_2}\|_1 \leqslant \frac{1}{m_0}\|a\||y_2 - y_1|. \tag{23}$$

Then, joining (21) with

$$\tau = \frac{2\sqrt{m_0}((a, b) - \sqrt{D})}{\|a\|\|b\|} = \tau_0,$$

and (23), we obtain the inequality

$$|Ay_1 - Ay_2| \leqslant \frac{\|a\|\|b\|(\Psi - \mu)}{((a, b) + 2\sqrt{D})^2}|y_1 - y_2| \tag{24}$$

which implies the continuity of operator A. Thus, according to the Brouwer fixed point theorem, operator A has a fixed point $k^* \in [0, \tau_0]$ and the pair $\{u^*, k^*\}$, where function u^* satisfies (1)–(2) with $k = k^*$, gives a solution of problem (1)–(3).

It remains to prove that the solution of problem (1)–(3) is unique under assumption (10). In this case, operator A is a contractor on the segment $[0, \tau_0]$ because A satisfies (24) with

$$q = \frac{\|a\|\|b\|(\Psi - \mu)}{(a, b) + 2\sqrt{D}} < \frac{(a, b)^2}{((a, b) + 2\sqrt{D})^2} < 1.$$

Let us assume that (u', k') and (u'', k'') are two solutions of problem (1)–(2). Then k', k'' are the fixed points of operator A. By (24)

$$|k' - k''| = |Ak' - Ak''| \leqslant q|k' - k''|$$

whence $k' - k'' = 0$. This in turn implies $u' - u'' = 0$ in view of (23). Theorem is proved. $\square$

Under assumption (10) the solution $\{u, k\}$ depends continuously on the input data of original problem.

Remark 1. Condition (4) is valid when $m(x) \geqslant m_0 > 0$ almost everywhere in Ω, or $\sigma(x) \geqslant \sigma_0 > 0$ almost everywhere in $\partial\Omega$, here m_0, σ_0 are some constants. In the last case left inequality holds due to the Friedrichs inequality.

Remark 2. The main theorem is correct for a more general type of operator M:

$$M = -\mathrm{div}(\mathcal{M}(x)\nabla) + (\bar{m}\nabla) + m(x)I,$$

where $\bar{m} \in L_\infty(\Omega)$ is vector of functions $m_i(x),\ i = 1, \dots, n$.

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Об одной обратной задаче для эллиптического уравнения со смешанными граничными условиями третьего рода

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Аннотация. В данной работе рассматривается обратная задача для эллиптического уравнения с граничными условием третьего рода и условием интегрального переопределения. Доказано существование и единственность решения, а также непрерывная зависимость решения от входных данных.

Ключевые слова: обратная задача, краевая задача, эллиптическое уравнение, теорема существования и единственности.

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Satisfiability in Boolean Logic (SAT problem) is Polynomial

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Abstract. We find a polynomial algorithm to solve SAT problem in Boolean Logic.

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Introduction

The satisfiability problem (SAT) in Boolean propositional logic is the question to determine if any given formula F is satisfiable (i.e. if there is a substitution of literals TUE-$FALSE$ instead the propositional letters from given formula F making the formula $TRUE$). Extended SAT problem is to find a such substitution if the one exists. SAT is an NP-complete problem, and is one of the most intensively studied problems.

As well known SAT was the first known NP-complete problem, that was proved by S.Cook at the University of Toronto in 1971 (cf. [1]) and also independently by L.Levin in 1973 (cf. [2]). Remarkably that before these results, the idea, the concept, of an NP-complete problem did not even exist, so was totally out of consideration. .

That generated a very active area in complexity theory; since the SAT problem is NP-complete, and only algorithms with exponential worst-case complexity are until now known for it, better algorithms for SAT where in grate demand. In particular researchers looked for efficient and scalable algorithms for SAT for formulas in restricted form; and during the 2000s algorithms making dramatic advances in our ability to automatically solve problem was developed (cf. [3, 4, 10–12]). In this paper we will prove that SAT may be solved in polynomial time.

1. Proof, deterministic algorithm with random any choice

We first need to restrict the amount of necessary formulas in our considerations. We will do this restriction by Theorem 1 placed below. Actually this result was known for long ago, for example the author introduced reduced normal forms for inference rules in [5, 6] where he solved Friedman problem about recognition inference rules for intuitionistic propositional logic. This technique was efficiently applied in [5–9] for study inference rules and unification. The point here is that the premises of such rules are exactly the normal reduced forms for just formulas. These approach also possibly was observed even earlier when researchers used reduction formulas for 3-Sat problem and relative subsequent research (cf. [3,4,10–12]). I am not sure about history

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and priority not being very expert in the SAT area. Though it turned out that Theorem 1 is also very useful for positive solution SAT problem and we present it now.

Definition. *We say that Boolean formula F has reduced normal form if*

$$F = A_1 \wedge A_2 \wedge \cdots \wedge A_k,$$

where

$$A_j := B_{j_1} \vee B_{j_2} \vee B_{j_3},$$

and $B_{j_i} \in \{p, \neg p\}$, $p \in Prop$ and $Prop$ is a set of letters.

In the sequel we may consider also A_j containing less then 3 disjunct members. Simply for convenience and simplicity in notation we will always refer to 3 disjunct members, thinking that it might be less.

Theorem 1. *There is a polynomial algorithm constructing by any given boolean formula G a formula F in reduced normal form which has the following properties. (1) F has all variables of G and some more in amount not bigger then the length of G. (2) F is equivalent to G w.r.t satisfiability. (3) Any substitution σ for F satisfying G acting at only variables of G satisfies G, (and vice versa any substitution satisfying G may be computably extended on additional variables of F satisfying it).*

Proof. It is a simple statement; as much as I remember I myself proved it first time in my works for constructing reduced normal forms for inference rules in my research to resolve Friedman problem about recognizing admissible rules in intuitionistic logic cf. [6], 1984. A short draft of the proof is as follows. In fact, we simply shall specify the general algorithm described already several times in [5–9] to the language of our logic.

So, let we start. If $\varphi = \alpha \circ \beta$, where $\circ$ is a binary logical operation and both formulas α and β are not simply variables or unary logical operations applied to variables (which both we call final formulas), take two new variables x_α and x_β and the formula

$$\mathbf{f_1} := (x_\alpha \circ x_\beta) \wedge (x_\alpha \equiv \alpha) \wedge (x_\beta \equiv \beta).$$

If one from formulas α or β is final and another one not, we apply this transformation to the non-final formula. It is clear that $\mathbf{f}$ and $\mathbf{f_1}$ are equivalent w.r.t. satisfiability.

If $\varphi = \neg\alpha$ and α is not a variable, take a new variable x_α and the formula

$$\mathbf{f_1} := \neg x_\alpha \wedge (x_\alpha \equiv \alpha).$$

Again $\mathbf{f}$ and $\mathbf{f_1}$ are equivalent w.r.t. satisfiability. We continue this transformation over the resulting formulas

$$\bigwedge_{j \in J_1} \gamma_j \wedge \bigwedge_{i \in I_1} x_{\alpha_i} \equiv \alpha_i$$

until all formulas α_i and γ_j in the formula above will be either atomic formulas, i.e. logical operations applied to variables, or variables.

Evidently this transformation is *polynomial.* Further, we transform the formulas using $\equiv$ in the ones using only disjunctions and conjunctions and negations. After that we obtain formula in form as required for reduced normal forms. The only point is that the conjuncts may continue less that 3 disjunct formulas, we then may double some until 3 members. As the result we get the final formula $\mathbf{f_2}$ which evidently has all required properties. Q.E.D.

Now we turn to SAT problem. In the sequel a literal is either a propositional letter or a letter with applied negation ($\neg p$ for p). Below we will always refer to only 3 disjuncts in

$F = A_1 \wedge A_2 \wedge \cdots \wedge A_k$,, where $A_j := B_{j_1} \vee B_{j_2} \vee B_{j_3}$. But we could admit less disjuncts, simply we keep all 3 for simplicity of notation.

Theorem 2. *If a formula F has reduced normal form then there is a polynomial algorithm verifying its satisfiability and constructing its some unifier if F is satisfiable.*

Proof. Let
$$F = A_1 \wedge A_2 \wedge \cdots \wedge A_k,$$
where
$$A_j := B_{j_1} \vee B_{j_2} \vee B_{j_3},$$
and $B_{j_i} \in \{p, \neg p\}, p \in Prop$ where $Prop$ are propositional letters. Assume F has exactly m letters.

It is evident that F is satisfiable iff there is at least one path from A_1 to A_k passing through each A_j via some unique B_{j_i} (in this A_j) not containing contradictory literals along all path.

We will try to construct such a path now. We do it by induction on j in A_j so we do it by induction on k in $A_1 \wedge A_2 \wedge \cdots \wedge A_k$. Let $k = 1$ then we have the path.

Inductive step. Assume that n steps are already done and (1) all sets $Imp(B_{n_i})$ are constructed and any $Imp(B_{n_i})$ contains absolutely all possible literals g (in given m letters) to which we cannot make step from B_{n_i} that is g is any literal contradicting to some lateral in any possible not contradictory paths leading from any disjunct from A_1 to B_{n_i}. (2) The sets $To(B_{n_i})$ contain all B_{n-1_j} which themselves are reached by non-contradictory (no matter which) paths from A_1 and from which we (non-contradictory) moved in B_{n_i}. Note that we do not store (record or memorize) paths themselves - we just fix (record) their existence by marking all B_{n-1_j} from which we made final steps to (B_{n_i}). (That is why we summarize things (steps, actions) while the procedure but do not multiply them.)

Of course we assume all sets $To(B_{1_i})$ to be $\{B_{1_i}\}$ and for $n = 1$ all is ready, and for all i, $Imp(B_{1_i}) = \{\neg B_{1_i}\}$ if B_{1_i} is a letter and $Imp(B_{1_i}) = \{B_{1_i}\}$ otherwise.

Now we turn to the inductive step itself, we try to move to $(n+1)$-th conjunct from A_n, to make $(n+1)$-th step. Assume that n steps are done and we arrived to B_{n_i} (which informally means by not-contradictory path; which in our formalism only means that $To(B_{n_i}) \neq \emptyset$, or $n = 1$) and all $Imp(B_{n_j})$ and $To(B_{n_j})$ are constructed. Consider any B_{n+1_j}. Define immediately
$$Imp(B_{n+1_j}) := [Imp(B_{n_1}) \cap Imp(B_{n_2}) \cap Imp(B_{n_3})] \cup \{\neg B_{n+1_j}\}$$
and
$$To(B_{n+1_j})$$
are unions of all B_{n_l} which do not contradict B_{n+1_j} and where $To(B_{n,l})$ are not empty (which, by the way, means that $B_{n,l}$ are reached by some non-contradictory paths, and, recall the inductive assumption, — sets $Imp(B_{n_l})$ are already successfully constructed).

It is clear that $Imp(B_{n+1_j})$ is the set containing all literals to which we cannot step form B_{n+1_j} further at all, even, in particular, to literals which occurs A_{n+2}. Consider all B_{n+2_l} from level $n+2$; if any of them occurs in $Imp(B_{n+1_j})$ we cross out such B_{n+1_j} from further consideration. And if that indeed holds for all B_{n+1_l} the procedure stops and formula F is not satisfiable.

If we can continue, recall that earlier we put in $To(B_{n+1_j})$ all B_{n_i} from which we arrived to B_{n+1_j}. The step (n+1) is completed.

If we came to some B_{k_j} the formula F is satisfiable otherwise not. And the sets $To(B_{n+1_j})$ give us a satisfying substitution, if we will move from B_{k_j} back to A_1 via sets $To(B_{k_j})$ subsequently using $To(B_{n_j})$ towards A_1. The trick here is that we do not do any choice at all; during moving we just take any B_{k_j} from $To(B_{n+1_j})$ (and we have at most 3 options for that each choice) and move towards A_1.

The interesting thing here is that we do not do choice at all — we can take subsequently any from any occurring in B_{n_j} moving to the origin A_1. That looks as not deterministic algorithm but in fact it is the one, a good one, since we can take any disjunct from at most 3 possible options and any of them will lead us to success.

The amount of steps in this algorithm is polynomial: the general amount of steps used in our inductive procedure in constructing $Imp(B_{n_j})$ and $To(B_{n+1_j})$ is at most k. And in the inductive step itself from n to $n+1$ for constructing $Imp(B_{n+1_j})$ and $To(B_{n+1_j})$, the amount of steps is at most $3 \times [3 \times (2m)^2]$ (m is the number of letters in formula) for computing intersections $Imp(B_{n_1}) \cap Imp(B_{n_2}) \cap Imp(B_{n_3})]$. Q.E.D.

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Проблема выполнимости формул в булевой логике (SAT) полиномиальна

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Аннотация. Находится полиномиальный алгоритм решающий проблему SAT в Булевой логикуе.

Ключевые слова: булева логика, проблема выполнимости, алгоритм SAT.