ISSN 1997-1397 (Print)
ISSN 2313-6022 (Online)

Журнал Сибирского федерального университета

Математика и физика

Journal of Siberian Federal University

Mathematics & Physics

2022 15 (5)

ISSN 1997-1997-1397 (Print)

ISSN 2313-6022 (Online)

ЖУРНАЛ СИБИРСКОГО ФЕДЕРАЛЬНОГО УНИВЕРСИТЕТА
Математика и Физика

2022 15 (5)

JOURNAL OF SIBERIAN FEDERAL UNIVERSITY
Mathematics & Physics

Издание индексируется Scopus (Elsevier), Emerging Sources Citation Index (WoS, Clarivate Analytics), Российским индексом научного цитирования (НЭБ), представлено в международных и российских информационных базах: Ulrich's periodicals directiory, ProQuest, EBSCO (США), Google Scholar, MathNet.ru, КиберЛенинке.

Включено в список Высшей аттестационной комиссии «Рецензируемые научные издания, входящие в международные реферативные базы данных и системы цитирования».

Все статьи представлены в открытом доступе http://journal.sfu-kras.ru/en/series/mathematics_physics.

Журнал Сибирского федерального университета. Математика и физика.
Journal of Siberian Federal University. Mathematics & Physics.

Учредитель: Федеральное государственное автономное образовательное учреждение высшего образования "Сибирский федеральный университет"(СФУ)
Главный редактор: А.М. Кытманов. Редакторы: В.Е. Зализняк, А.В. Щуплев.
Компьютерная верстка: Г.В. Хрусталева
№ 5. 26.10.2022. Индекс: 42327. Тираж: 1000 экз. Свободная цена
Адрес редакции и издательства: 660041 г. Красноярск, пр. Свободный, 79, оф. 32-03.
Отпечатано в типографии Издательства БИК СФУ
660041 г. Красноярск, пр. Свободный, 82а.
Свидетельство о регистрации СМИ ПИ № ФС 77-28724 от 27.06.2007 г., выданное Федеральной службой по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия
http://journal.sfu-kras.ru
Подписано в печать 15.10.22. Формат 84×108/16. Усл.печ. л. 12,1.
Уч.-изд. л. 11,9. Бумага тип. Печать офсетная.
Тираж 1000 экз. Заказ 15828
Возрастная маркировка в соответствии с Федеральным законом № 436-ФЗ:16+

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Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 555-558

DOI: 10.17516/1997-1397-2022-15-5-555-558
УДК 517.9

Uniqueness of an Interpolating Entire Function with Some Properties

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Received 19.04.2022, received in revised form 23.05.2022, accepted 20.06.2022

Abstract. We consider the problem of power series analytic continuation by coefficients interpolation by entire or meromorphic functions. We prove uniqueness of an interpolating function with some properties. Also, under assumptions of Polya‘s theorem about extendability of the sum of power series to the whole complex plane, except, possibly, some boundary arc, we find at least one singular point location.

Keywords: power series, analytic continuation, indicator function.

Citation: A.J. Mkrtchyan, Uniqueness of an Interpolating Entire Function with Some Properties, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 555–558. DOI: 10.17516/1997-1397-2022-15-5-555-558.

Consider a power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

whose domain of convergence is the unit disk $D_1 := \{z \in \mathbb{C} : |z| < 1\}$. One possible approach to treat analytic continuation problem is to interpolate the coefficients of this power series.

The function φ *interpolates* the coefficients f_n of the power series (1), if

$$\varphi(n) = f_n \ \text{ for all } \ n \in \mathbb{N}. \tag{2}$$

Following this approach many results were obtained in the case when the coefficients of the series are interpolated by an analytic function (see for example [1, 2]).

In particular, Lindelöf and Le Roy gave the conditions for which the series extends analytically into a sectorial domain:

Theorem 1 (Le Roy, Lindelöf [3, 4])**.** *The sum of the series* (1) *extends analytically to the open sector* $\mathbb{C} \setminus \Delta_\sigma$ *if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the condition*

$$h_\varphi(\theta) \leqslant \sigma|\sin\theta| \ \textit{ for } \ |\theta| \leqslant \frac{\pi}{2}.$$

The *indicator* function $h_\varphi(\theta)$ for an entire function φ is defined as the upper limit [5]

$$h_\varphi(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln|\varphi(re^{i\theta})|}{r}, \ \ \theta \in \mathbb{R}.$$

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Δ_σ is the sector $\{z = re^{i\theta} \in \mathbb{C} : |\theta| \leqslant \sigma\}$, $\sigma \in [0, \pi)$.

Pólya found conditions for analytic continuability of a series to the whole complex plane except for some boundary arc :

Theorem 2 (Polya [6]). *The series* (1) *extends analytically to* $\mathbb{C}$, *except possibly the arc* $\partial D_1 \cap \Delta_\sigma$, *if and only if there exists an entire function of exponential type* $\varphi(\zeta)$ *interpolating the coefficients* f_n *such that*

$$h_\varphi(\theta) \leqslant \sigma|\sin\theta| \ \textit{for} \ |\theta| \leqslant \pi.$$

Note that despite the fact that an entire function interpolating the coefficients always exists, it is sometimes easier to construct and work with meromorphic interpolating functions of a special form than with entire functions (see [7]).
Let

$$\psi(\zeta) = \phi(\zeta) \frac{\prod_{j=1}^{p} \Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q} \Gamma(c_k\zeta + d_k)}, \tag{3}$$

where $\phi(\zeta)$ is an entire function, $a_j \geqslant 0$, $j = 1, \ldots, p$, and

$$\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} c_k = 0. \tag{4}$$

$$l = \sum_{k=1}^{q} |c_k| - \sum_{j=1}^{p} a_j.$$

Theorem 3. *The series* (1) *extends analytically to the open sector* $\mathbb{C}\setminus\Delta_\sigma$, *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (3) *interpolating the coefficients* f_n, *such that the indicator of the associated with* $\psi(\zeta)$ *entire function*

$$\varphi(\zeta) := \phi(\zeta) \frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k+1}^{q} |c_k|^{c_k\zeta}}$$

satisfies the conditions:

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\left\{h_\varphi\Big(-\frac{\pi}{2}\Big) + \frac{\pi}{2}l, h_\varphi\Big(\frac{\pi}{2}\Big) + \frac{\pi}{2}l\right\} \leqslant \sigma.$$

Also note that all these results do not say anything about uniqueness of the interpolating function. We can see that if the function $\varphi(z)$ is of exponential type and interpolates the coefficients of a power series, then any function of the form $\varphi(z) + A\sin\pi z$ is also of exponential type and also interpolates the same coefficients. We will show that the properties of the indicator of interpolating functions in these theorems ensure uniqueness of the interpolating functions.

Proposition 1. *In each of Theorems 1, 2, 3 above, the interpolating function with the corresponding property is unique.*

Proof. We shall treat here the case of Theorem 2. (proofs for Theorems 1 and 3 are similar). Let $\varphi(z)$ and $g(z)$ be two entire functions of exponential type that interpolate the coefficients f_n and both satisfing the condition

$$h_g(\theta) \leqslant \sigma|\sin\theta| \ \textit{for} \ |\theta| \leqslant \pi,$$

Consider the function $F(z) = \varphi(z) - g(z)$, which is an analytic function of exponential type for $Re\ z \geqslant 0$ and $F(n) = 0, n = 0, 1, \ldots$ according to the Carlson theorem [8]:

Theorem (Carlson). *If the function is analytic and of finite exponential type σ for $Re\ z \geqslant 0$ and $F(n) = 0, n = 0, 1, \dots,$ then either $F(n) \equiv 0$, or its exponential type $\sigma \geqslant \pi$ for $Re\ z \geqslant 0$.*

So either $F(z) \equiv 0$ from which it follows that $g(z) \equiv \varphi(z)$, or of exponential type $\sigma \geqslant \pi$ for $Re\ z \geqslant 0$, that is, there is $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $h_F(\theta) \geqslant \pi$. According to the property of the indicator

$$h_F(\theta) \leqslant \max\{h_g(\theta), h_\varphi(\theta)\}$$

and we obtain

$$\max\{h_g(\theta), h_\varphi(\theta)\} \geqslant \pi,$$

which contradicts the hypothesis of Theorem 2. Therefore, our assumption about the existence of another interpolating function with properties that satisfy the conditions of the theorem is not true. □

Proposition 2. *Under the assumptions of Theorem 2, at least one of the boundary points $e^{ih_\varphi(\frac{\pi}{2})}$, $e^{-ih_\varphi(-\frac{\pi}{2})}$ is a singular point.*

Proof. Note that Theorem 2 is satisfied when $\sigma \geqslant \max\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\}$. The arc $\partial D_1 \cap \Delta_\sigma$, where the series may not continue, is minimal for $\sigma = \max\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\}$. Assume that both points $e^{ih_\varphi(\frac{\pi}{2})}$ and $e^{-ih_\varphi(-\frac{\pi}{2})}$ are not singular for the sum of the series. This means that, there is an open arc on the boundary $\partial D_1 \setminus \Delta_{\sigma_1}$, where the sum of the series is regular and which includes the arc $\partial D_1 \setminus \Delta_\sigma$. Therefore, according to Theorem 2, there must exist an interpolating function ϕ of exponential type for which the condition

$$h_\phi(\theta) \leqslant \sigma_1 |\sin\theta| \ \ \textit{for} \ \ |\theta| \leqslant \pi$$

is satisfied and, therefore, for which

$$\max\left\{h_\phi(\frac{\pi}{2}), h_\phi(-\frac{\pi}{2})\right\} < \max\left\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\right\}$$

that is, another interpolating function must exist, which contradicts Proposition 1. □

The research is supported by a grant of the Russian Science Foundation (project no. 20-11-20117)

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Единственность интерполирующей целой функции с определенными свойствами

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Аннотация. Рассматривается вопрос аналитической продолжимости степенных рядов путем интерполяции коэффициентов целыми или мероморфными функциями. Доказывается единственность интерполирующей функции с определенными свойствами. Также в теореме Полиа о продолжимости суммы ряда на всю комплексную плоскость кроме быть может некоторой граничной дуги находится местоположение по крайней мере одной особой точки.

Ключевые слова: степенные ряды, аналитичиское продолжение, индикатор функция.

DOI: 10.17516/1997-1397-2022-15-5-559-564
УДК 512.5

Sharply Doubly Transitive Groups with Saturation Conditions

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Received 27.12.2021, received in revised form 14.05.2022, accepted 27.06.2022

Abstract. A number of conditions were found under which a sharply doubly transitive permutation group has an abelian normal divider.

Keywords: exactly doubly transitive group, Frobenius groups, saturation condition, finite and generalized finite the elements.

Citation: A.I. Sozutov, E.B. Durakov, Sharply Doubly Transitive Groups with Saturation Conditions, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 559–564. DOI: 10.17516/1997-1397-2022-15-5-559-564.

Recall that the permutation group G of the set F ($|F| \geqslant k$) is called *exactly k-transitive* on F if for any two ordered sets $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ elements from F such that $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for $i \neq j$, there is exactly one element of the group G taking α_i to β_i ($i = 1, \ldots, k$).

As C. Jordan [1] proved, in a finite sharply doubly transitive group T regular permutations (moving each element of the set F) together with the identity substitution constitute an abelian normal subgroup. Is Jordan's theorem true for infinite groups in the general case is unknown (see, for example, [2, questions 11.52, 12.48]). To particular solutions of this central question in the theory of near-fields and near-domains [3] dozens of works by famous authors are devoted. We especially note that in 2014 in [4,5] exactly doubly transitive groups without abelian normal subgroups were constructed, but only in characteristic 2. [6] and [7] constructed examples of exactly 2-transitive groups and simple exactly 2-transitive groups, respectively, that have characteristic 0 and do not contain non-trivial Abelian normal subgroups. And in 2008 [8] studied the question for exactly 2-transitive groups of characteristic 3. For other characteristics the question remains open. Sharply 2-transitive groups are closely related to algebraic structures such as near-fields, near-areas, KT-fields (Kerby-Tits fields), projective planes, etc. (see [3, Ch. V], [9, chap. 20]).

We continue to investigate infinite exactly doubly transitive groups and related near-domains [10, 11]. In this paper, a number of conditions are found under which a group has an abelian normal divisor (see [2, questions 11.52, 12.48]), and the corresponding near-domain [3] is a near-field. All necessary definitions are given in Section 1. We only recall that the group G is saturated with groups of some set of finite groups $\mathfrak{X}$ if every finite subgroup of G is contained in a subgroup of the group G isomorphic to some group from $\mathfrak{X}$.

Theorem 1. *A sharply doubly transitive group T saturated with finite Frobenius groups of substitutions of the set F of odd characteristic has a regular abelian normal subgroup and the near-domain $F(+,\cdot)$ is a near-field if at least one of the following conditions on the stabilizer T_α of the point $\alpha \in F$ is satisfied:*

1. *T_α is a Shunkov group;*

2. *T_α is a periodic group and T_α does not contain conjugate dense subgroups;*

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3. T_α is (locally) a finite normal subgroup of order greater than 2;

4. T_α contains a finite element a of prime odd order not equal to five.

Theorem 2. *Let in a sharply triply transitive permutation group G odd characteristic the point stabilizer have a regular abelian normal subgroup and each of its finite subgroups is contained in a finite unsolvable subgroup of the group G. Then G is locally finite.*

1. Results and concepts used

Let T be a sharply doubly transitive group of permutations of the set F. According to M. Hall's theorem [3], we assume that F is a near-domain with operations of addition $+$ and multiplication $\cdot$, and T is its affine transformation group $x \to a + bx$ $(b \neq 0)$, and $T_\alpha \simeq F^*(\cdot)$ is the stabilizer of the point $\alpha \in F$. Let T_α be the stabilizer of the point $\alpha \in F$, J denote the set of involutions of the group T. For each involution $k \in J$ by N_k we denote the set $kJ = \{kj \mid j \in J\}$. The result of a substitution action $t \in T$ to an element (point) $\gamma \in F$ is denoted by γ^t. The following statements are true.

1. *(T, T_α) is a Frobenius pair, i.e. $T_\alpha \cap T_\alpha^t = 1$ for any $t \in T \setminus T_\alpha$.*

2. *The group T contains involutions, and all involutions in T are conjugate. The set $J \setminus T_\alpha$ is nonempty and T_α is transitive on $J \setminus T_\alpha$. Product of two different involutions from T is a regular substitution, i.e. acting on F without fixed points.*

3. *If T_α contains an involution j, then it is unique in T_α, $T = T_\alpha N_j$ and T acts by conjugation on the set J of all its involutions exactly twice transitively. All products kv, where $k, v \in J$, $k \neq v$ in the group T are conjugate and have the same either prime odd order p or infinite order. In the first case $\operatorname{Char} T = p$, in the second case $\operatorname{Char} T = 0$ (if there are no involutions in T_α, then $\operatorname{Char} T = 2$ by definition).*

4. *The following result was proved in [10] Let $\operatorname{Char} T = p > 2$, b is strictly real with respect to j is an element from $T \setminus T_\alpha$, $A = C_T(b)$ and $V = N_T(A)$. Then:*

1) the subgroup A is periodic, abelian, and is inverted by the involution j and is strongly isolated in T;

2) the subgroup V acts exactly twice transitively on the orbit $\Delta = \alpha^A$, moreover, A is an elementary abelian regular normal subgroup of V, $H = V \cap T_\alpha$ is a point stabilizer and $V = A \lambda H$. If in addition $|\bigcap_{x \in T_\alpha} H^x| > 2$, then A is a regular abelian normal subgroup of T, and the near-domain $F(+, \cdot)$ is a near-field.

The next proposition shows that the saturation condition by finite Frobenius groups partially holds in an arbitrary exactly doubly transitive group of odd characteristic.

5. *When $\operatorname{Char} T = p > 2$, each dihedral subgroup of T is contained in a finite Frobenius subgroup of T with kernel of order p and a cyclic complement of order $p - 1$.*

Recall that (a, b)-*finiteness condition* in the group G means that a and b are its nontrivial elements, and in the group G all subgroups $\langle a, b^x \rangle$ $(x \in G)$ are finite. The elements a and b are called *generalized finite*, and if $a = b$, then a is called a *finite* element in the group G. The following proposition was proved in [11, Lemmas 1, 2 and Theorem]:

6. *Let a sharply doubly transitive group T of characteristics $\operatorname{Char} T \neq 2$ contain elements a, b with (a, b)-finitness condition. Then*

1. *If for any $x \in T$ the subgroup $\langle a^x \rangle$ and $\langle b \rangle$ are not incident, then at least one of the elements a, b belongs to the stabilizer of some point of the near-domain F;*

2. *If $|a| = |b| > 2$, T has a regular abelian normal subgroup;*

3. *If $|a| \cdot |b| = 2k > 4$, T has a regular abelian normal subgroup.*

7 ([12], Theorem 2)**.** *If the group T has is a finite element of order > 2, then T has a regular abelian normal subgroup and the near-domain F is a near-field.*

8. *If T contains a locally finite subgroup containing a regular substitution and intersecting with T_α by a normal subgroup, consisting of more than two elements, then T has a regular abelian normal subgroup, and the near-domain $F(+, \cdot)$ is a near-field.*

9. *If* Char $T \neq 2$, *T contains a Frobenius group V with involution and the complement $H = V \cap T_\alpha$, where T_α is the stabilizer of the point α, and in H there is a normal in T_α subgroup of order > 2, then the group T has a regular abelian normal subgroup.*

10. *In the complement of a finite Frobenius group, each cyclic subgroup of prime order $q > 5$ is normal.*

2. Proof of Theorem 1

Let the group T and the near-domain F satisfy the conditions of Theorem 1. The set of Frobenius subgroups of the group T containing finite subgroup L we denote by $\mathfrak{X}(L)$. For any Frobenius group $M \leqslant T$, we denote its kernel by F_M, and the appropriate to the context complement by H_M. The notation $L \leqslant H_M$ often used in what follows for the subgroup $L \leqslant M \in \mathfrak{X}(L)$ means that L is contained in some complement H_M group M.

Remark. *It is well known that a near-domain $F(+, \cdot)$ is a near-field if and only if T has a regular abelian normal subgroup. Therefore, to prove Theorem 1 it suffices to prove the existence in the infinite group T of a regular abelian normal subgroup.*

Lemma 1. *We can assume that* Char $T \neq 3$.

Proof. In the case of Char $T = 3$, the group T has a regular abelian normal subgroup and without the additional saturation condition (see, for example, [13, Lemma 2.7]. The lemma is proved. □

Lemma 2. *When T_α has a finite element of prime order $q > 5$, the theorem is true.*

Proof. Let Char $T = p > 3$, j be an involution from T_α and a is a finite in T_α element of prime order $q > 5$. Choose an arbitrary element $t \in T_\alpha$. Due to the finiteness of the element a and item 3, the subgroup $L_t = \langle a, a^t, j \rangle$ is finite, and $L_t \leqslant M \in \mathfrak{X}(K)$. In view of item 1 and properties of finite Frobenius groups $M \cap T_\alpha \leqslant H_M$, $F_M \cap T_\alpha = 1$ and F_M is an elementary abelian p-group, consisting of all products jk, where $k \in J \cap M$. According to the structure of complements in finite Frobenius groups $L_t = \langle a, a^t \rangle = \langle a \rangle$. Since the element $t \in T_\alpha$ is arbitrary, we conclude that the subgroup $\langle a \rangle$ is normal in T_α. Therefore, the subgroup $L = \langle a, j \rangle$ is contained in the complement of a finite Frobenius group $M^x \in \mathfrak{X}(A)$ for any $x \in T_\alpha$, and the set-theoretic union of the kernels of all such groups (for $x \in T_\alpha$), contains the set N_j. From this and the equality $T = T_\alpha N_j$ (item 3) it follows that all subgroups $L_g = \langle a, a^g \rangle$ are finite for any $g \in T$, and by item 6 T possesses a regular abelian normal subgroup. The lemma is proved.

□

Lemma 3. *If T_α is a Shunkov group and* $\mathrm{Char}\,T \neq 2$*, then T_α has a local finite periodic part and the subgroup $\Omega_1(T_\alpha)$ generated by all elements of prime orders from T_α is a group of one of the following types: 1) a (locally) cyclic group; $\Omega_1(T_\alpha) = C \times L$, where C is a (locally) cyclic $\{2,3\}'$-group, and $L \simeq SL_2(3)$; 3) $\Omega_1(T_\alpha) = \times L$, where C is a (locally) cyclic $\{2,3,5\}'$-group, $L \simeq SL_2(5)$. In any case, in T_α is a finite normal subgroup of order greater than 2.*

Proof. By virtue of the saturation condition and the condition $\mathrm{Char}\,T \neq 2$, each finite subgroup K of T_α is contained in the complement of a finite Frobenius group $M \in \mathfrak{X}(K)$, and according to the proposition [14, Proposition 11], its subgroup $\Omega_1(K)$ has the structure indicated in the lemma. By [12, Theorem 1], the same structure has the subgroup $\Omega_1(T_\alpha)$, and T_α possesses a local finite periodic part, we denote it by S. If S is not a 2-group, then $\Omega_1(T_\alpha)$ obviously contains a finite normal subgroup in T_α of order, greater 2. If S is a 2-group, then by [15, Theorem 2] it is either a quasicyclic group, either locally quaternionic and also contains a finite normal in T_α subgroup of order 2^n for any $n > 1$. The lemma is proved. □

Lemma 4. *The theorem is true if T_α contains a finite normal subgroup L, $|L| > 2$.*

Proof. Since the involution of j in T_α is unique, we can assume that $j \in L$. By the saturation condition $M \in \mathfrak{X}(L)$, and the subgroup $M \cap T_\alpha$ is strongly isolated in M. Therefore, the subgroup L is the complement of some finite Frobenius group $M \in \mathfrak{X}(L)$. From the normality of L in T_α it follows that $M^t \in \mathfrak{X}(L)$ and $H_M = L$. Hence it follows that the set-theoretic union of the kernels of all such groups (for $t \in T_\alpha$), contains the set N_j. Let a be an element of order greater than 2 from L (since $|L| > 2$, such an element exists by virtue of item 3). It follows from what has been proved that all subgroups $L_g = \langle a, a^g \rangle$ are finite for any $g \in T$, and by item 6 T possesses a regular abelian normal subgroup. The lemma is proved. □

Lemma 5. *When T_α contains a finite element a of order* 3*, the theorem is also true.*

Proof. As in Lemma 2, we prove that for any $t \in T_\alpha$ the finite subgroup $L_t = \langle a, a^t \rangle$ is contained in the complement of a finite Frobenius group M from $\mathfrak{X}(L)$. As in Lemma 3, we conclude that either $L_t = \langle a \rangle$, or L_t is isomorphic to one of the groups $SL_2(3)$, $SL_2(5)$. By the main theorem in [16], the normal closure $L = \langle a^{T_\alpha} \rangle$ of a in T_α is locally finite. As follows from the proof of Lemma 3 either $L = \langle a \rangle$, or L is isomorphic to one of the groups $SL_2(3)$, $SL_2(5)$. By Lemma 4, the theorem is true. The lemma is proved. □

A proper subgroup H of a group G is called *conjugate dense*, if H has a non-empty intersection with every conjugacy class in G elements.

Lemma 6. *The theorem is true when T is a periodic group and T_α has no conjugate dense subgroups.*

Proof. By item 3, T_α has a unique involution j, therefore, for an arbitrary element a of finite order from T_α the subgroup $L = \langle a, j \rangle$ is finite. By the saturation condition $L \leqslant M \in \mathfrak{X}(L)$, and we can assume that $H_M = L$, $F_M \subseteq N_j$. By item 4, for any non-identity element $b \in N_j$, the subgroup $A = C_T(b)$ is periodic, contained in N_j and strongly isolated in T, in this case, $N_T(A) = A \leftthreetimes H$, where $H = T_\alpha \cap N_T(A)$. Since in view of items 2, 3 T_α acts transitively by conjugation on the set N_j, $F_M^x \leqslant A$ for some $x \in T_\alpha$. This implies that $a^x \in H$ and H is the conjugate dense subgroup of the group T_α. According to the conditions of the lemma, $H = T_\alpha$, by item 3, $A = N_j$, and the lemma is proved. □

We now complete the proof of the theorem. The first statement of the theorem follows from Lemmas 3, 4. Statement 2 is proved in Lemma 6. Statement 3 of Theorem coincides with Lemma 4. Statement 4 follows from Lemmas 2 and 5. The theorem is proved.

3. Proof of Theorem 2

Let G be an infinite sharply triply transitive permutation group of $X = F \cup \{\infty\}$ and satisfy the conditions of Theorem 2. As in [17], by B we denote the stabilizer G_α of the point $\alpha \in X$ and by H the stabilizer $G_{\alpha\beta} = G_\alpha \cap G_\beta$ of two points $\alpha = \infty \in X$, $\beta \in F$. Let also J be the set of involutions of the group G, and J_m be the set of involutions stabilizing exactly m points, $m = 0, 1, 2$.

Proof of the theorem. By Lemma 1 $B = U \leftthreetimes H$ is a Frobenius group, H contains an involution z and $N = N_G(H) = H \leftthreetimes \langle v \rangle$, where $v \in J$. For $b \in U^{\#}$ there is an element $a \in H$ of order $p-1$ such that $\langle a, b\rangle = \langle b \rangle \leftthreetimes \langle a \rangle$ is a sharply twice transitive group of order $p(p-1)$. Let S be an arbitrary finite subgroup from U containing an arbitrary element c from $U \setminus \langle b \rangle$ and $K = \langle b, S, a \rangle$. The subgroup K is obviously finite and by the saturation condition $K \leqslant M$, where M is a finite unsolvable group. Let $L = L(M)$ be the layer of the group M [18, Proposition 1.4, p. 53]. It is clear that $Z(L) = 1$ and since the 2-rank of the group G is 2, then L is a simple group. By virtue of Lemma 6 [17] L is isomorphic to $L_2(p^n)$ and $P = U \cap L$ is a Sylow p-subgroup of the group L. As known, all cyclic subgroups of P are conjugate in the subgroup $N_L(P)$, and $P^{\#}$ splits into two conjugate classes. Since $\langle a, b \rangle \leqslant M$ and $\langle a \rangle$ acts transitively to $\langle b \rangle^{\#}$, obviously $|M : L| = 2$ and $M \simeq PGL_2(p^n)$. The subgroup $N_M(P)$ acts transitively on $P^{\#}$, therefore $c = b^h$ for some $h \in H \cap M$. Since the element $c \in P^{\#}$ we conclude that H is a periodic group. By [19, Theorem 2], the group G is locally finite. The theorem is proved. □

This work was supported by the Russian Foundation for Basic Research (project no. 19-01-00566 A) and Krasnoyarsk mathematical center, funded by the Ministry of Education and Science of the Russian Federation in the framework of activities for the creation and development of regional NMCs (Agreement 075-02-2020-1534 / 1).

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О точно дважды транзитивных группах с условиями насыщенности

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Аннотация. Найден ряд условий, при которых точно дважды транзитивная группа подстановок обладает абелевым нормальным делителем.

Ключевые слова: точно дважды транзитивная группа, группы Фробениуса, условие насыщенности, конечные и обобщенно конечные элементы.

DOI: 10.17516/1997-1397-2022-15-5-565-576
УДК 517.958

Determination of Non-stationary Potential Analytical with Respect to Spatial Variables

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Received 31.10.2021, received in revised form 21.03.2022, accepted 28.06.2022

Abstract. The inverse problem of determining coefficient before the lower term of the hyperbolic equation of the second order is considered. The coefficient depends on time and n spatial variables. It is supposed that this coefficient is continuous with respect to variables t, x and it is analytic in other spatial variables. The problem is reduced to the equivalent integro-differential equations with respect to unknown functions. To solve this equations the scale method of Banach spaces of analytic functions is applied. The local existence and global uniqueness results are proven. The stability estimate is also obtained.

Keywords: inverse problem, Cauchy problem, fundamental solution, local solvability, Banach space.

Citation: D.K. Durdiev, Z.D. Totieva, Determination of Non-stationary Potential Analytical with Respect to Spatial Variables, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 565–576.
DOI: 10.17516/1997-1397-2022-15-5-565-576.

1. Introduction and problem formulation

The inverse problem of determining coefficient $a(t, x, y)$, $t \in \mathbb{R}$, $(x, y) = (x, y_1, \ldots, y_m) \in \mathbb{R}^{1+m}$, before the lower term of the hyperbolic equation is studied in this paper. The problem is considered in the class of coefficients that are continuous with respect to variables t, x and it is analytic in variable y. It is known that such problems are referred to as multidimensional inverse problems. For multidimensional inverse problems there are only special cases for which solvability is established. One of such classes of functions in which local solvability takes place is the class of analytic functions. The technique used here is based on the scale method of Banach spaces of analytic functions developed by L. V. Ovsyannikov [1, 2] and L. Nirenberg [3]. This method was first applied to the problem of solvability of multidimensional inverse problems by V. G. Romanov [4–6].

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This method was used to study multidimensional inverse problems of determining the convolution kernel in parabolic and hyperbolic integro-differential equations of the second order; theorems of local unique solvability of inverse problems in the class of functions with finite smoothness with respect to time variable and analytic with respect to spatial [7–13]. variables. This paper generalizes the results given in [4] (Sec. 3) for the case of non-stationary potential.

Let us consider the problem of determining a pair of functions u and a that satisfy the following equations

$$u_{tt} - u_{xx} - \triangle u - a(t,x,y)u = g(y)\delta(x)\delta'(t-t_0),\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0, \tag{1}$$

$$u\Big|_{t<0} \equiv 0, \tag{2}$$

where $\triangle$ is the Laplace operator with respect to variables $(y_1, \dots, y_m) = y$, $\delta(\cdot)$ is the Dirac delta function, $\delta'(\cdot)$ is the derivative of the Dirac delta function, t_0 is a problem parameter. Therefore $u = u(t,x,y,t_0)$, and $g(y)$ is a given smooth function so that $g(y) \neq 0$ for $y \in \mathbb{R}^m$.

It is required to find potential $a(x,t,y)$ in (1) if the solution of problem (1)–(3) is known for $x = 0$, i.e., the condition

$$u(t,0,y,t_0) = f(t,y,t_0),\ t > 0,\ t_0 > 0 \tag{3}$$

is given.

Following monograph [4, sec. 3], we consider the Banach space $A_s(r)$ $s > 0$ of functions $\varphi(y)$, $y \in \mathbb{R}^m$ which are analytic in the neighbourhood of the origin and they satisfy the following relation

$$\|\varphi\|_s(r) := \sup_{|y|<r} \sum_{|\alpha|=0}^{\infty} \frac{s^{|\alpha|}}{\alpha!} |D^\alpha \varphi(y)| < \infty.$$

Here $r > 0,\ s > 0$ and

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \dots \partial y_m^{\alpha_m}},\ \alpha := (\alpha_1, \dots, \alpha_m),$$

$$|\alpha| := \alpha_1 + \dots + \alpha_m,\ \alpha! := (\alpha_1)! \dots (\alpha_m)!.$$

In what follows, parameter r is fixed while parameter s is variable. Then, it is formed a scale of Banach spaces $A_s(r),\ s > 0$ of analytic functions. The following property is obvious: if $\varphi(y) \in A_s(r)$ then $\varphi(y) \in A_{s'}(r)$ for all $s' \in (0,s)$. Consequently, $A_s(r) \subset A_{s'}(r)$ if $s' \in (0,s)$ and the following inequality is valid

$$\|D^\alpha \varphi\|_{s'} \leqslant c_\alpha \frac{\|\varphi\|_s(r)}{(s-s')^{|\alpha|}}$$

for any α with constant c_α which depends only on α.

Solution of problem (1), (2) is considered in the form

$$u(t,x,y,t_0) = \frac{1}{2} g(y)\delta(t - t_0 - |x|) + v(t,x,y,t_0).$$

Substituting this expression into (1) and taking into account that $(1/2)g(y)\delta(t-t_0-|x|)$ satisfies (in a general meaning) equation $u_{tt} - u_{xx} = g(y)\delta(x)\delta'(t-t_0)$, we obtain the following problem for function v:

$$
\begin{aligned}
v_{tt} - v_{xx} = \triangle v + \frac{1}{2}\triangle g(y)\delta(t - t_0 - |x|)+ \\
+ a(t,x,y)\left[\frac{1}{2}g(y)\delta(t - t_0 - |x|) + v(t,x,y,t_0)\right],\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0,
\end{aligned} \tag{4}
$$

$$
v\mid_{t<0} \equiv 0. \tag{5}
$$

In the next section inverse problem (4), (5) and (2) is replaced with the equivalent integro-differential equations. In what follows, we assume that function a is even in x.

2. Reduction of the problem to integro-differential equations

According to the d'Alembert formula the solution of problem (4)–(5) satisfies the integral equation

$$
\begin{aligned}
v(t,x,y,t_0) = \frac{1}{2}\iint\limits_{\Delta(t,x)}\Big\{\triangle v(\tau,\xi,y,t_0) + \frac{1}{2}\triangle g(y)\delta(\tau - t_0 - |\xi|)+ \\
+ a(\tau,\xi,y)\left[\frac{1}{2}g(y)\delta(\tau - t_0 - |\xi|) + v(\tau,\xi,y,t_0)\right]\Big\}d\xi d\tau,\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0,
\end{aligned} \tag{6}
$$

where

$$
\Delta(t,x) = \{(\tau,\xi)|\ 0 \leqslant \tau \leqslant t - |x - \xi|,\ x - t \leqslant \xi \leqslant x + t\}.
$$

Let

$$
Q_T := \{(t,t_0)|\ 0 \leqslant t_0 \leqslant t \leqslant T\},\ T > 0,
$$

$$
\Omega_T := \{(t,x)|\ 0 \leqslant |x| \leqslant t \leqslant T - |x|\},
$$

$$
\Upsilon_T := \Big\{(t,x,t_0)\Big|\ |x| + t_0 \leqslant t \leqslant T - |x|,\ 0 \leqslant t_0 \leqslant t \leqslant T\Big\}.
$$

Domain Υ_T in the space of variables x, t, t_0 is the pyramid with the base Ω_T and vertex $(0,T,T)$.

It follows from (6) that function $v(t,x,y,t_0)$ satisfies the integral equation

$$
\begin{aligned}
v(t,x,y,t_0) = \frac{\triangle g(y)}{4}(t - t_0) + \frac{g(y)}{4}\int\limits_{\frac{x-(t-t_0)}{2}}^{\frac{x+(t-t_0)}{2}} a(t_0 + |\xi|,\xi,y)d\xi+ \\
+\frac{1}{2}\iint\limits_{\square(t,x,t_0)}\left[\triangle v(\tau,\xi,y,t_0) + a(\tau,\xi,y)v(\tau,\xi,y,t_0)\right]d\tau d\xi,\quad (t,x,t_0) \in \Upsilon_T,\quad y \in \mathbb{R}^m,
\end{aligned} \tag{7}
$$

where $\theta(t) = 1,\ t \geqslant 0,\ \theta(t) = 0,\ t < 0$, and $\square(x,t,t_0)$ is domain in the form of a rectangle in the plane of variables (τ,ξ) for each fixed t_0 formed by characteristics passing through the points $(0,t_0)$ and (x,t) of the differential operator $\partial^2/\partial t^2 - \partial^2/\partial x^2$:

$$
\square(x,t,t_0) := \left\{(\xi,\tau)\Big|\ |\xi| + t_0 < \tau < t - |x - \xi|,\ \frac{x - (t - t_0)}{2} < \xi < \frac{x + t - t_0}{2},\ 0 < t_0 < t\right\}.
$$

Obviously, the equalities $f(t,y,t_0) = u(t,0,y,t_0) = v(t,0,y,t_0)$, $t > t_0$ are true. Besides, $f(t_0 + 0,y,t_0) = v(t,0,y,t_0)\big|_{t=t_0+0} = 0$.

First note that if $a(t,-x,y)=a(t,x,y)$ then $v(t,-x,y,t_0)=v(t,x,y,t_0)$. Taking the derivative with respect to t of the both sides of equation (7), we obtain

$$v_t(t,x,y,t_0)=$$

$$=\frac{\triangle g(y)}{2}+\frac{g(y)}{8}\left[a\left(\frac{x+t+t_0}{2},\frac{x+t-t_0}{2},y\right)+a\left(\frac{-x+t+t_0}{2},\frac{x-t+t_0}{2},y\right)\right]+$$

$$+\frac{1}{2}\int\limits_{\frac{x-(t-t_0)}{2}}^{\frac{x+(t-t_0)}{2}}\left[\triangle v\left(t-|x-\xi|,\xi,y,t_0\right)+a\left(t-|x-\xi|,\xi,y\right)v\left(t-|x-\xi|,\xi,y,t_0\right)\right]d\xi,$$

$$(t,x,t_0)\in\Upsilon_T,\ y\in\mathbb{R}^m.$$

Setting $x=0$ in this relation and using evenness of functions $a(t,x,y),\ v(t,x,y,t_0)$ with respect to x, we obtain the equality

$$f_t(t,y,t_0)=\frac{\triangle g(y)}{2}+\frac{g(y)}{4}a\left(\frac{t-t_0}{2},\frac{t+t_0}{2},y\right)+$$

$$+\int_0^{\frac{t-t_0}{2}}\left[\triangle v\left(t-\xi,\xi,y,t_0\right)+a\left(t-\xi,\xi,y\right)v\left(t-\xi,\xi,y,t_0\right)\right]d\xi,\ (t,t_0)\in Q_T,\ y\in\mathbb{R}^m.$$

Substituting $|x|$ for $(t-t_0)/2$ and t for $(t+t_0)/2$ and solving with respect to $a(t,x,y)$, we rewrite this equation in the form

$$\begin{aligned}a(t,x,y)=\frac{2\triangle g(y)}{g(y)}+\frac{4}{g(y)}f_t\left(t+|x|,y,t-|x|\right)-\frac{4}{g(y)}\int_0^{|x|}\Big[\triangle v\left(t+|x|-\xi,\xi,y,t-|x|\right)+\\+a\left(t+|x|-\xi,\xi,y\right)v\left(t+|x|-\xi,\xi,y,t-|x|\right)\Big]d\xi,\ (t,x)\in\Omega_T,\ y\in\mathbb{R}^m.\end{aligned}\tag{8}$$

Thus, in order to find the value of function a at the point (t,x,y) it is necessary to integrate function $a(t,x,y)$ itself over the segment with boundaries $(t+|x|,0,y,0)$, $(t,|x|,y,\ 0)$ and function $v(t,x,y,t_0)$ over the segment with boundaries $(t+|x|,0,y,\ t-|x|)$, $(\ t,|x|,y,\ t-|x|)$ which belong to domain $\Upsilon_T\times\mathbb{R}^m$.

Note that function v, even with respect to $x=0$, satisfies the condition $\partial v/\partial x|_{x=0}$. Taking into account this fact and considering equations (4), (5), (3) for v in the domain $x>0$, we obtain

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-\triangle v-a(t,x,y)v=0,\ 0<x<t-t_0,\ y\in\mathbb{R}^m,$$

$$v\Big|_{x=0}=f(t,y,t_0),\ \frac{\partial v}{\partial x}\Big|_{x=0}=0,\ 0<t-t_0\leqslant T,\ y\in\mathbb{R}^m$$

. Then in accordance with the d'Alembert formula which gives the Cauchy problem solution with an initial data at $x=0$ we find

$$v(t,x,y,t_0)=v_0(t,x,y,t_0)+\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big[\triangle v\left(\tau,\xi,y,t_0\right)+a\left(\tau,\xi,y\right)v\left(\tau,\xi,y,t_0\right)\Big]d\xi d\tau,\tag{9}$$

where

$$v_0(t,x,y,t_0)=\frac{1}{2}\left[f(t+x,y,t_0)+f(t-x,y,t_0)\right],$$

$$\triangle'(t,x) := \{(\tau,\xi) \mid 0<\xi<x, |\tau-t|<x-\xi\}, \quad 0<x<t-t_0<T-x, \quad y\in\mathbb{R}^m.$$

Considering (8) for $x \geqslant 0$, we have

$$\begin{gathered} a(t,x,y) = a_0(t,x,y) - \\ -\frac{4}{g(y)}\int_0^x \Big[\triangle v\,(t+x-\xi,\xi,y,t-x) + a(t+x-\xi,\xi,y)v(t+x-\xi,\xi,y,t-x)\Big]d\xi, \end{gathered} \tag{10}$$

where

$$a_0(t,x,y) = \frac{4}{g(y)}f_t(t+x,y,t-x) + \frac{2\triangle g(y)}{g(y)}, \quad 0\leqslant x\leqslant t\leqslant T-x, \quad y\in\mathbb{R}^m.$$

The system of equations (9), (10) is a closed integro-differential equations for functions a, v. Note that operator $\triangle$ for function v appears in the system only under the integral sign.

Next we consider system (9), (10) in domain

$$D_T = \Upsilon'_T \times \mathbb{R}^m, \quad \Upsilon'_T = \{(t,x,t_0) |\ 0\leqslant x+t_0\leqslant t\leqslant T-x\}.$$

3. The main results and proofs

Let $C_{(t,x,t_0)}\left(\Upsilon'_T;\ A_{s_0}\right)$ denote the class of functions with values in A_{s_0} ($s_0>0$) which are continuous with respect to variables (t,x,t_0) in domain Υ'_T. For fixed (t,x,t_0) the norm of function $v(t,x,y,t_0)$ in A_{s_0} is denoted by $\|v\|_{s_0}(t,x,t_0)$. The norm of function v in $C_{(t,x,t_0)}\left(\Upsilon'_T;\ A_{s_0}\right)$ is defined by the equality

$$\|v\|_{C_{(t,x,t_0)}\left(\Upsilon'_T;\,A_{s_0}\right)} = \sup_{(t,x,t_0)\in\Upsilon'_T}\|v\|_{s_0}(t,x,t_0).$$

Let $C_{(t,x)}\left(G_T;\ A_{s_0}\right)$ be a class of functions with values in A_{s_0} which are continuous with respect to variables (t,x) in domain $G_T=\{(t,x)|\ 0\leqslant x\leqslant t\leqslant T-x\}$. For fixed (t,x) the norm of function $a(t,x,y)$ in A_{s_0} is denoted by $\|a\|_{s_0}(t,x)$. The norm of function a in $C_{(t,x)}\left(G_T;\ A_{s_0}\right)$ is defined as

$$\|a\|_{C_{(t,x)}\left(G_T;\,A_{s_0}\right)} = \sup_{(t,x)\in G_T}\|a\|_{s_0}(t,x).$$

Let us also denote the class of functions with values in A_{s_0} which are continuous with respect to t, t_0 in domain Q_T by $C\left(Q_T;A_{s_0}\right)$.

Theorem 3.1. *Let $f\left(+t_0,y,t_0\right)=0$, $|g(y)|\geqslant g_0>0$, g_0 is a known number and*

$$\left\{\frac{1}{g(y)}, \frac{\triangle g(y)}{g(y)}\right\}\in A_{s_0};\ \{f(t,y,t_0),\ f_t(t,y,t_0)\}\in C\left(Q_T;A_{s_0}\right),$$

in addition, the relations

$$\max\left\{2\left\|\frac{\triangle g(y)}{g(y)}\right\|_{s_0}, \max_{(t,t_0)\in Q_T}\|f(t,y,t_0)\|_{s_0}, \max_{(t,t_0)\in Q_T}\left\|\frac{4f_t(t,y,t_0)}{g(y)}\right\|_{s_0}\right\}\leqslant\frac{R}{2}$$

are valid for some fixed $s_0>0$, R. Then there is such a number $b\in\left(0,T/(2s_0)\right)$, $b=b\left(s_0,R,T\right)$ that for each $s\in(0,s_0)$ in domain $D_T\cap\{(t,x,y,t_0):0\leqslant x+t_0\leqslant b(s_0-s)\}$ there exists the unique solution of equations (9), (10) *and $v(t,x,y,t_0)\in C_{(t,x,t_0)}\left(P_{sT};\ A_{s_0}\right)$,*

$a(t,x,y) \in C_{(t,x)}\left(K_{sT};\ A_{s_0}\right)$, *where* $P_{sT} = \Upsilon'_T \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\}$, $K_{sT} = G_T \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\}$, *moreover*

$$\|v - v_0\|_s (t,x,t_0) \leqslant R,\ (t,x,t_0) \in P_{sT},$$

$$\|a - a_0\|_s (t,x) \leqslant \frac{R}{s_0 - s},\ (t,x) \in K_{sT}.$$

Proof. Under the conditions of Theorem 1 we have

$$v_0 \in C_{(t,x,t_0)}\left(\Upsilon'_T;\ A_{s_0}\right),\quad a_0 \in C_{(t,x)}\left(G_T;\ A_{s_0}\right),$$

$$\|v_0\|_s(t,x,t_0) \leqslant R,\ (t,x,t_0) \in \Upsilon'_T,\quad \|a_0\|_s(t,x) \leqslant R,\quad (t,x) \in G_T,\quad 0 < s < s_0.$$

Let b_n be the member of the monotone decreasing sequence that is defined by the equalities

$$b_{n+1} = \frac{b_n}{1 + 1/(n+1)^2},\ n = 0,\ 1,\ 2,\ \ldots .$$

Let

$$b = \lim_{n\to\infty} b_n = b_0 \prod_{n=0}^{\infty} \left(1 + 1/(n+1)^2\right)^{-1}.$$

The number $b_0 \in (0, T/(2s_0))$ is chosen in an appropriate way. For the system of equations (9), (10) the process of successive approximations is constructed according to the following scheme

$$v_{n+1}(t,x,y,t_0) = v_0(t,x,y,t_0) +$$
$$+ \frac{1}{2} \iint\limits_{\triangle'(t,x)} \Big[\triangle v_n(\tau,\xi,y,t_0) + a_n(\tau,\xi,y)\, v_n(\tau,\xi,y,t_0)\Big] d\tau d\xi,\ 0 \leqslant x \leqslant t - t_0 \leqslant T - x,$$

$$a_{n+1}(t,x,y) = a_0(t,x,y) -$$
$$- \frac{4}{g(y)} \int_0^x \Big[\triangle v_n(t+x-\xi,\xi,y,t-x) + a_n(t+x-\xi,\xi,y) v_n(t+x-\xi,\xi,y,t-x)\Big] d\xi,$$
$$0 \leqslant x \leqslant t \leqslant T - x.$$

Function $s'_n(x)$ is defined by the formula

$$s'_n(x) = \frac{s + \nu^n(x)}{2},\quad \nu^n(x) = s_0 - \frac{x}{b_n}. \tag{11}$$

Let us introduce the following notations: $p_n = v_{n+1} - v_n$, $q_n = a_{n+1} - a_n$, $n = 0,\ 1,\ 2, \ldots$. Then $p_n,\ q_n$ satisfy the relations

$$p_0(t,x,y,t_0) = \frac{1}{2} \iint\limits_{\triangle'(t,x)} \Big[\triangle v_0(\tau,\xi,y,t_0) + a_0(\tau,\xi,y)\, v_0(\tau,\xi,y,t_0)\Big] d\tau d\xi,\quad (t,x,y,t_0) \in D_T,$$

$$q_0(t,x,y) = -\frac{4}{g(y)} \int_0^x \Big\{\triangle v_0(t+x-\xi,\xi,y,t-x) +$$
$$+ a_0(t+x-\xi,\xi,y) v_0(t+x-\xi,\xi,y,t-x)\Big\} d\xi,\quad (t,x,y) \in G_T \times \mathbb{R}^m;$$

$$p_{n+1}(t,x,y,t_0)=\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big\{\triangle p_n(\tau,\xi,y,t_0)+$$
$$+\,q_n\,(\tau,\xi,y)\,v_{n+1}(\tau,\xi,y,t_0)+a_n\,(\tau,\xi,y)\,p_n\,(\tau,\xi,y,t_0)\Big\}\,d\tau d\xi,\quad (t,x,y,t_0)\in D_T,$$

$$q_{n+1}(t,x,y)=-\frac{4}{g(y)}\int_0^x\Big\{\triangle p_n(t+x-\xi,\xi,y,t-x)+$$
$$+\,q_n(t+x-\xi,\xi,y)v_{n+1}(t+x-\xi,\xi,y,t-x)+a_n(t+x-\xi,\xi,y)p_n(t+x-\xi,\xi,y,t-x)\Big\}d\xi,$$
$$(t,x,y)\in G_T\times\mathbb{R}^m.$$

Let us show that $b_0\in\left(0,\dfrac{T}{2s_0}\right)$ can be chosen so that the following inequalities be valid for all $n=0,\,1,\,2,\cdots$:

$$\lambda_n=\max\left\{\sup_{(t,x,s)\in\hat F_n}\left[\|p_n\|_s(t,x,t_0)\frac{\nu^n(x)-s}{x}\right],\sup_{(t,x,s)\in F_n}\left[\|q_n\|_s(t,x)\frac{(\nu^n(x)-s)^2}{x}\right]\right\}<\infty,\quad(12)$$

$$\|\tilde v_{n+1}-v_0\|_s(t,x,t_0)\leqslant R,\quad \|a_{n+1}-a_0\|_s(t,x)\leqslant\frac{R}{s_0-s},\qquad(13)$$

where

$$\hat F_n=\{(t,x,t_0,s)|\ \ (t,x,t_0)\in\Upsilon'_T,\ \ 0\leqslant x+t_0<b_n(s_0-s),\ \ 0<s<s_0\}\,,$$
$$F_n=\{(t,x,s)|\ (t,x)\in G_T,\ 0\leqslant x<b_n(s_0-s),\ 0<s<s_0\}\,.$$

Indeed, using the relations for p_n, q_n, one can find

$$\|p_0\|_s(t,x,t_0)\leqslant\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big[\|\triangle v_0\|_s\,(\tau,\xi,t_0)+\|a_0\|_s\,(\tau,\xi)\,\|v_0\|_s\,(\tau,\xi,t_0)\Big]d\tau d\xi\leqslant$$
$$\leqslant\frac{1}{2}\iint\limits_{\triangle'(t,x)}\left[\frac{Rc_0}{(s'_0(\xi)-s)^2}+R^2\right]d\tau d\xi.$$

Here c_0 is a positive constant such that

$$\|\triangle v_0\|_s\leqslant c_0\frac{\|v_0\|_{s'_n}}{(s'_n-s)^2},\,s'_n>s>0,\ n=0,1,2,\ldots$$

. It is easy to check that $c_0=4m$.

Taking function $s'_n(\xi)$ from (11) for $n=0$, we have

$$\|p_0\|_s(t,x,t_0)\leqslant\frac{1}{2}\int_0^x(x-\xi)\left[\frac{4Rc_0}{(\nu^0(\xi)-s)^2}+R^2\right]d\xi\leqslant$$
$$\leqslant\frac{1}{2}R\left[4c_0+s_0^2R\right]\int_0^x\frac{(x-\xi)d\xi}{(\nu^0(\xi)-s)^2}\leqslant$$
$$\leqslant\frac{1}{2}b_0R\left[4c_0+s_0^2R\right]\frac{x}{\nu^0(x)-s},\ (t,x,s)\in\hat F_0.$$

In a similar way we obtain

$$\|q_0\|_s(t,x) \leqslant 4g_0 \int_0^x \left[\frac{4Rc_0}{(\nu^0(\xi)-s)^2} + R^2\right] d\xi \leqslant$$
$$\leqslant 4g_0 R\left(4c_0 + s_0^2 R\right)\frac{x}{(\nu^0(x)-s)^2},\ (t,x,s)\in F_0.$$

These estimates imply that inequality (12) is valid for $n=0$. Moreover,we find

$$\|\tilde v_1 - \tilde v_0\|_s(t,x,t_0) = \|p_0\|_s(t,x,t_0) \leqslant \frac{\lambda_0 x}{\nu^0(x)-s} \leqslant \frac{\lambda_0 b_1}{1-b_1/b_0} = b_0\lambda_0,\ (t,x,t_0,s)\in \hat F_1,$$

$$\|a_1 - a_0\|_s(t,x) = \|a_0\|_s(t,x) \leqslant \frac{\lambda_0 x}{(\nu^0(x)-s)^2} \leqslant \frac{4b_0\lambda_0}{s_0-s},\ (t,x,s)\in F_1.$$

Choosing b_0 so that $4b_0\lambda_0 \leqslant R$, one can conclude that inequalities (13) are satisfied for $n=0$.

By way of induction, one can show that inequalities (12) and (13) are also valid for other values of n if b_0 is chosen suitably. Let us assume that inequalities (12) and (13) hold for $n = 0, 1, 2, \ldots, i$. Then $(t,x,t_0,s)\in \hat F_{i+1}$ and we have

$$\|p_{i+1}\|_s(t,x,t_0) \leqslant \frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big\{\|\triangle p_i\|_s(\tau,\xi,t_0) +$$
$$+\|q_i\|_s(\tau,\xi)\,\|v_{i+1}\|_s(\tau,\xi,t_0) + \|a_i\|_s(\tau,\xi)\,\|p_i\|_s(\tau,\xi,t_0)\Big\} d\tau d\xi \leqslant$$
$$\leqslant \frac{1}{2}\iint\limits_{\Delta(t,x)}\left[\frac{c_0\lambda_i\xi}{(s_i'(\xi)-s)^2(\nu^i(\xi)-s)} + \frac{2R\lambda_i\xi}{(\nu^i(\xi)-s)^2} + \frac{\lambda_i\xi}{(\nu^i(\xi)-s)}\frac{R(1+s_0)}{(s_0-s)}\right] d\tau d\xi \leqslant$$
$$\leqslant \frac{\lambda_i}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right)\int_0^x \frac{(x-\xi)\xi d\xi}{(\nu^{i+1}(\xi)-s)^3} \leqslant$$
$$\leqslant \frac{\lambda_i}{2} b_0^2\left(4c_0 + 3Rs_0 + Rs_0^2\right)\frac{x}{\nu^{i+1}(x)-s}.$$

Here function s_i' is defined by equality (11) with $n=i$ and the inequalities

$$\|v_i\|_s(t,x,t_0) \leqslant 2R,\quad \|a_i\|_s(t,x) \leqslant R\frac{1+s_0}{s_0-s}$$

are used. The latter is valid by the induction hypothesis together with the obvious inequalities $b_i \leqslant b_0$ and $\nu^{i+1}(x) \leqslant \nu^i(x)$. Similar arguments for q_{i+1} lead to inequalities

$$\|q_{i+1}\|_s(t,x) \leqslant 4g_0\int_0^x \left\{\frac{c_0\lambda_i\xi}{(s_i'(\xi)-s)^2(\nu^i(\xi)-s)} + \frac{2R\lambda_i\xi}{\nu^i(\xi)-s} + \frac{\lambda_i R(1+s_0)\xi}{(\nu^i(\xi)-s)^2}\right\} d\xi \leqslant$$
$$\leqslant 4\lambda_i g_0\left[4c_0 + 3Rs_0^2 + Rs_0\right]\int_0^x \frac{\xi d\xi}{(\nu^{i+1}(\xi)-s)^3} \leqslant$$
$$\leqslant 4\lambda_i g_0 b_0\left[4c_0 + 3Rs_0^2 + Rs_0\right]\frac{x}{(\nu^{i+1}(x)-s)^2},\ (t,x,s)\in F_{i+1}.$$

The obtained estimates yield

$$\lambda_{i+1} \leqslant \lambda_i\rho,\quad \lambda_{i+1} < \infty,$$

$$\rho = b_0 \max\left[\frac{1}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right); 4g_0\left(4c_0 + 3Rs_0^2 + Rs_0\right)\right].$$

Moreover, we have

$$\|\tilde{v}_{i+2} - v_0\|_s(t, x, t_0) \leqslant \sum_{n=0}^{i+1} \|p_n\|_s(t, x, t_0) \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n x}{\nu^n(x) - s} \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n b_{i+2}}{1 - b_{i+2}/b_n} \leqslant$$
$$\leqslant \sum_{n=0}^{i+1} \lambda_n b_n (n+1)^2 \leqslant \lambda_0 b_0 \sum_{n=0}^{i+1} \rho^n (n+1)^2, \ (t, x, t_0, s) \in \hat{F}_{i+2},$$

$$\|a_{i+2} - a_0\|_s(t, x) \leqslant \sum_{n=0}^{i+1} \|q_n\|_s(t, x) \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n x}{(\nu^n(x) - s)^2} \leqslant \frac{1}{s_0 - s} \sum_{n=0}^{i+1} \frac{\lambda_n b_{i+2}}{(1 - b_{i+2}/b_n)^2} \leqslant$$
$$\leqslant \frac{\lambda_0 b_0}{s_0 - s} \sum_{n=0}^{i+1} \rho^n (n+1)^4, \ (t, x, s) \in F_{i+2}.$$

Now we choose $b_0 \in \left(0, \frac{T}{2s_0}\right)$ so as to obtain

$$\rho < 1, \ \lambda_0 b_0 \sum_{n=0}^{\infty} \rho^n (n+1)^4 \leqslant R.$$

Then

$$\|v_{i+2} - v_0\|_s(t, x, t_0) \leqslant R, \quad (t, x, t_0, s) \in \hat{F}_{i+2},$$
$$\|a_{i+2} - a_0\|_s(t, x) \leqslant \frac{R}{s_0 - s}, \ (t, x, s) \in F_{i+2}.$$

Since the choice of b_0 is independent of the number of approximations, all successive approximations v_n, a_n belong to

$$C_{(t,x,t_0)}\left(\hat{F}; A_s\right), \ \hat{F} = \bigcap_{n=0}^{\infty} \hat{F}_n$$

and

$$C_{(t,x)}\left(F; A_s\right), \ F = \bigcap_{n=0}^{\infty} F_n,$$

respectively. Moreover,

$$\|v_n - v_0\|_s(t, x, t_0) \leqslant R, \ (t, x, t_0, s) \in \hat{F},$$
$$\|a_n - a_0\|_s(t, x) \leqslant \frac{R}{s_0 - s}, \ (t, x, s) \in F.$$

For $s \in (0, s_0)$ the series

$$\sum_{n=0}^{\infty}(v_n - v_{n-1}), \quad \sum_{n=0}^{\infty}(a_n - a_{n-1})$$

converge uniformly in the norm of the spaces

$$C_{(t,x,t_0)}\left(P_{sT}; A_s\right), \ P_{sT} = \Upsilon'_T \cap \left\{(t, x, t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\right\},$$
$$C_{(t,x)}\left(K_{sT}; A_s\right), \ K_{sT} = G_T \cap \left\{(t, x, t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\right\}.$$

Therefore $v_n \to v$, $a_n \to a$ and the limit functions v, a are elements of $C_{(t,x,t_0)}(P_{sT}; A_s)$, $C_{(t,x)}(K_{sT}; A_s)$ respectively and they satisfy equations (9), (10).

Now we prove that this solution is unique. Let us assume that $(v,\, a)$ and $(\widehat{v},\, \widehat{a})$ are any two solutions that satisfy the inequalities

$$\|v - v_0\|_s(t,x,t_0) \leqslant R,\ (t,x,t_0,s) \in \hat{F},$$

$$\|a - a_0\|_s(t,x) \leqslant \frac{R}{s_0 - s},\ (t,x,s) \in F.$$

Let us denote $\tilde{p} = v - \widehat{v}$, $\tilde{q} = a - \widehat{a}$,

$$\lambda := \max\left\{ \sup_{(t,x,t_0,s)\in\hat{F}} \left[\|\tilde{p}\|_s(t,x,t_0)\frac{\nu(x)-s}{x}\right],\quad \sup_{(t,x,s)\in F}\left[\|\tilde{q}\|_s(t,x)\frac{(\nu(x)-s)^2}{x}\right]\right\} < \infty$$

, where $\nu(x) = s_0 - x/b$, $b = b_0 \prod\limits_{n=0}^{\infty} \left(1 + 1/(n+1)^2\right)^{-1}$. Then the following relations can be obtained for functions $\tilde{p}$, $\tilde{q}$

$$\tilde{p}(t,x,y,t_0) = \frac{1}{2} \iint\limits_{\triangle'(t,x)} \Big\{ \triangle\tilde{p}\,(\tau,\xi,y,t_0) + \tilde{q}\,(\tau,\xi,y)\,\widehat{v}(\tau,\xi,y,t_0) + a\,(\tau,\xi,y)\,\tilde{p}\,(\tau,\xi,y,t_0)\Big\}\, d\tau d\xi,$$

$$(t,x,y,t_0) \in D_T,$$

$$\tilde{q}(t,x,y) = -\frac{4}{g(y)}\int_0^x \Big\{ \triangle\tilde{p}\,(t+x-\xi,\xi,y,t-x) +$$

$$+\tilde{q}(t+x-\xi,\xi,y)\widehat{v}(t+x-\xi,\xi,y,t-x) + a(t+x-\xi,\xi,y)\tilde{p}(t+x-\xi,\xi,y,t-x)\Big\} d\xi,$$

$$G_T \times \mathbb{R}^m.$$

Let us show that $b_0 \in \left(0, \dfrac{T}{2s_0}\right)$ can be chosen so that the following inequalities are valid for all $n = 0,\, 1,\, 2, \ldots$. Applying the estimates given above to these equations, we find the inequality

$$\lambda \leqslant \lambda\rho',$$

$$\rho' := b \max\left[\frac{1}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right); \frac{4}{\|g\|_s}\left(4c_0 + 3Rs_0^2 + Rs_0\right)\right] < \rho < 1.$$

Consequently $\lambda = 0$. Therefore $v = \widehat{v}$, $a = \widehat{a}$. Theorem 1 is proved. □

Let us consider the set Γ of functions $f(t,y,t_0)$ representing the elements of $C(Q_T;\, A_{s_0})$, $s_0 > 0$ for which conditions of Theorem 1 are valid with R, T, s_0. Then we have the stability theorem

Theorem 3.2. *Let f, $\bar{f} \in \Gamma$. For the corresponding solutions (v,a) and $(\bar{v},\bar{a})$ of (9), (10), we have*

$$\|v - \bar{v}\|_s \leqslant cM,\ (t,x,t_0) \in P_{sT},\quad \|a - \bar{a}\|_s \leqslant \frac{cM}{s_0 - s},\ (t,x) \in K_{sT},\ 0 < s < s_0, \tag{14}$$

where

$$M = \max\left[\max\|f - \bar{f}\|_{s_0}(t,t_0),\ \max\|f_t - \bar{f}_t\|_{s_0}(t,t_0)\right],\ (t,t_0) \in Q_T,$$

and constant c depends on R, T, s_0.

Proof. Taking into account (9)–(10), we obtain the following equalities for the differences $v-\bar{v}=\tilde{v}$, $a-\bar{a}=\tilde{a}$ and $f-\bar{f}=\tilde{f}$

$$\begin{gathered}\tilde{v}(t,x,y,t_0)=\tilde{v}_0(t,x,y,t_0)+\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big\{\triangle\tilde{v}\,(\tau,\xi,y,t_0)+\\+\tilde{a}\,(\tau,\xi,y)\,v(\tau,\xi,y,t_0)+\bar{a}\,(\tau,\xi,y)\,\tilde{v}\,(\tau,\xi,y,t_0)\Big\}\,d\tau d\xi,\ (t,x,y,t_0)\in D_T,\end{gathered}\tag{15}$$

$$\begin{gathered}\tilde{a}(t,x,y)=\tilde{a}_0(t,x,y)-\frac{4}{g(y)}\int_0^x\Big\{\triangle\tilde{v}\,(t+x-\xi,\xi,y,t-x)+\\+\tilde{a}(t+x-\xi,\xi,y)v(t+x-\xi,\xi,y,t-x)+\bar{a}(t+x-\xi,\xi,y)\tilde{v}(t+x-\xi,\xi,y,t-x)\Big\}d\xi,\\(t,x,y)\in G_T\times\mathbb{R}^m,\end{gathered}\tag{16}$$

where

$$\tilde{v}_0(t,x,y,t_0)=\frac{1}{2}[\tilde{f}(t+x,y,t_0)+\tilde{f}(t-x,y,t_0)],\quad \tilde{a}_0(t,x,y)=\frac{4}{g(y)}\tilde{f}_t(t+x,y,t-x).$$

It is obvious that

$$\begin{gathered}\|\tilde{v}_0\|_{s_0}(t,x,t_0)\leqslant M,\ (t,x,t_0)\in P_{sT},\\ \|\tilde{a}_0\|_{s_0}(t,x)\leqslant\frac{4}{\|g(y)\|_{s_0}}M,\ (t,x)\in K_{sT}.\end{gathered}\tag{17}$$

We have from Theorem 1 that

$$\|v\|_s\leqslant 2R,\ \|a\|_s\leqslant\frac{R(1+s_0)}{s_0-s}.$$

Applying the method of successive approximations used for the proof of Theorem 1 to the system of equations (15)–(16) (it is linear with respect to $\tilde{v}$ and $\tilde{a}$), we find that the following inequalities are valid for solution of (15)–(16)

$$\begin{gathered}\|\tilde{v}-\tilde{v}_0\|_s(t,x,t_0)\leqslant c_1M,\ (t,x,t_0)\in P_{sT},\\ \|\tilde{a}-\tilde{a}_0\|_s(t,x)\leqslant\frac{c_1M}{s_0-s},\ (t,x)\in K_{sT},\ 0<s<s_0,\end{gathered}$$

where c_1 depends on R, T, s_0. Hence, taking into account (17), we find that inequalities (14) are true. Theorem 2 is proved. □

The work was supported by the Ministry of Science and Higher Education of the Russian Federation (Grant/Award Number 075-02-2022-896).

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Определение нестационарного потенциала, аналитического по пространственным переменным

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Аннотация. Изучена обратная задача определения коэффициента зависимости временных и n пространственных переменных для младшего члена гиперболического уравнения второго порядка. Предполагается, что этот коэффициент непрерывен по отношению к переменным t, x и аналитичен по другим пространственным переменным. Задача сводится к эквивалентной системе нелинейных интегро-дифференциальных уравнений относительно неизвестных функций. Для решения этих уравнений применяется метод шкал банаховых пространств аналитических функций. Доказаны теоремы локальной разрешимости и единственности в глобальном смысле. Получена оценка устойчивости обратной задачи.

Ключевые слова: обратная задача, фундаментальное решение, задача Коши, локальная разрешимость, устойчивость.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 577–585

DOI: 10.17516/1997-1397-2022-15-5-577-585
УДК 621.396.945, 621.396.25

Computational Modeling of the Electromagnetic Field Distribution of a Horizontal Grounded Antenna in Rock for TTE Communication

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Received 09.04.2022, received in revised form 14.05.2022, accepted 27.06.2022

Abstract. The article investigates the spatial distribution of the electromagnetic (EM) field of an antenna in the form of a long grounded cable in a rock with the corresponding electrophysical parameters. The frequency dependences of the emitted EM fields (300 Hz – 30 kHz) on the depth of the receiver position are determined. This is of practical importance for the problems of wireless mine communications.

Keywords: wireless communication, mine communication, magnetic antenna, grounded antenna, electromagnetic field, ULF, VLF.

Citation: G.Ya. Shaidurov, E.A. Kokhonkova, Computational Modeling of the Electromagnetic Field Distribution of a Horizontal Grounded Antenna in Rock for TTE Communication, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 577–585. DOI: 10.17516/1997-1397-2022-15-5-577-585.

Wireless transmission of signals through rocks is possible by using low frequency electromagnetic waves. The article investigates the magnetic component of the electromagnetic field in a continuous medium with the properties of mountain ranges for the frequency range 300 Hz – 30 kHz. Signal transmission through the rock is possible using a long grounded cable located on the surface of the mine field or in mine roadway as a transmitting antenna. To receive the signal, it is proposed to use a compact magnetic antenna in the form of a coil with a ferrite core.

1. Theoretical assessment

The currently existing wireless communication systems in underground mine workings can be conditionally divided into low-frequency (VLF–LF) and HF and VHF systems [1–8]. As an additional channel for operational communication with personnel, a system of wireless emergency notification and communication through the Earth (TTE) is used. A separate type of connection is wireless magnetic-inductive or near-field magnetic connection (NFMC) in the frequency range of 30–100 kHz. Such systems use an antenna in the form of a magnetic loop with a radius of 10–200 m to transmit a signal through the Earth to mine workings. Studies are being carried out on the possibility of using long conductors in mines to increase the range of low-frequency (400 Hz – 9 kHz) channels [4, 6, 7].

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The Radius-2 wireless notification and personnel search system for mines, developed by the "Radius" company (Krasnoyarsk). It is based on the transmission of a useful signal through a mountain range using an antenna-feeder device in the form of a long feeder line grounded into the ground on the daylight surface [8].

To analyze the energy potential of a wireless channel, it is necessary to estimate the distribution of the magnetic field emitted by a transmitting grounded antenna in a continuous medium with the physical properties of a rock (resistivity ρ, dielectric permittivity ε and magnetic permeability μ). Each of the two grounds is a point source A (see Fig. 1a), directing the current into the ground.

For the case of a homogeneous medium, the current $\overline{j}$ flows uniformly in all directions of the half-space in the form of direct rays from point A. The hemisphere S with radius r is set at some distance from the source to determine the current density in the half-space around it. This will determine the current density at point M on the surface of the hemisphere:

$$j = \frac{J}{2\pi r^2}, \tag{1}$$

where r is equipotential surface radius; J is total current flowing through equipotential surfaces.

The electric field intensity E at point M is defined as:

$$E = \rho \cdot j = \frac{J \cdot \rho}{2\pi r^2}. \tag{2}$$

In an isotropic medium, the orientation of the current and electric field vectors is the same. The distribution of the electric field and currents between the two grounds of the dipole source is shown in Fig. 1b. The current flow of such a source is closed, and the equipotential surfaces on which the potential U is constant, as in the case of a point source, are perpendicular to the current lines and have a hemispherical shape. For a homogeneous medium, the field E of two point electrodes has an almost constant level (see Fig. 1b).

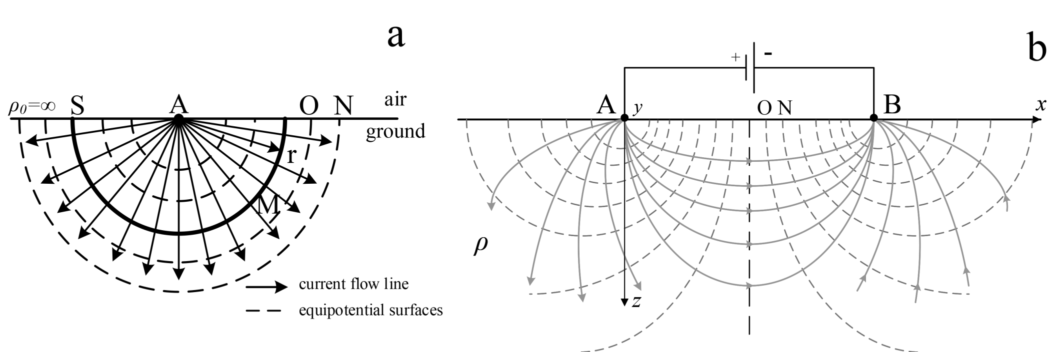


Fig. 1. Scheme of distribution of the field and currents of the current source in a homogeneous half-space: a — point source, b — dipole source

The vector $\overline{E}$ is directed along the radius vector connecting the points AO. In this case, expression (2) takes the form:

$$\overline{E} = \frac{J \cdot \rho}{2\pi r^2} \cdot \frac{\overline{r}}{r}, \tag{3}$$

where $\frac{\overline{r}}{r}$ is unit radius vector.

An electric dipole source is an emitter, because there is a periodic change in the electric moment $\overline{p}$ in the process of oscillation of charges q in the wire:

$$\overline{p} = \overline{p_0}\sin(\omega t), \tag{4}$$

where $\overline{p}_0 = J_0 l\overline{d}$; l is wire length; $\overline{d}$ is unit vector indicating wire orientation.

Consider the electromagnetic field of a straight cable of length l with current J located at the boundary of a conducting half-space with electrical conductivity σ_2. The cable is connected to a power source and grounded at the ends. It is necessary to determine the magnetic field at the depth z1 at the point M, which is formed as the sum of the fields of two opposite charges located at grounding points. It is necessary to determine the magnetic field at the depth z_1 (point M), which is formed as the sum of the fields of two opposite charges located at grounding points. In the work of M. S. Zhdanov [9], a solution was obtained for this problem to determine the current density in a conducting half-space. The current density has three components (j_x, j_y, j_z), let's analyze the distribution of currents in the XZ plane (see Fig. 2):

$$\overline{j}_x = \frac{\overline{J}}{2\pi}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{l-x}{((l-x)^2+y^2+x^2)^{3/2}}\right], \tag{5}$$

$$\overline{j}_z = \frac{\overline{J}}{2\pi}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{z}{((l-x)^2+y^2+x^2)^{3/2}}\right], \tag{6}$$

where $\overline{J}$ is current in the cable.

The magnetic field at point M is represented by the sum of the fields created by the main current of the antenna and the elementary currents flowing in the half-space of rocks. To do this, we divide the entire region $z > 0$ into elementary sections located at a distance Δz from each other, through which the current flows as through an equivalent conductor with electrical conductivity σ_2 and radius a (see Fig. 2).

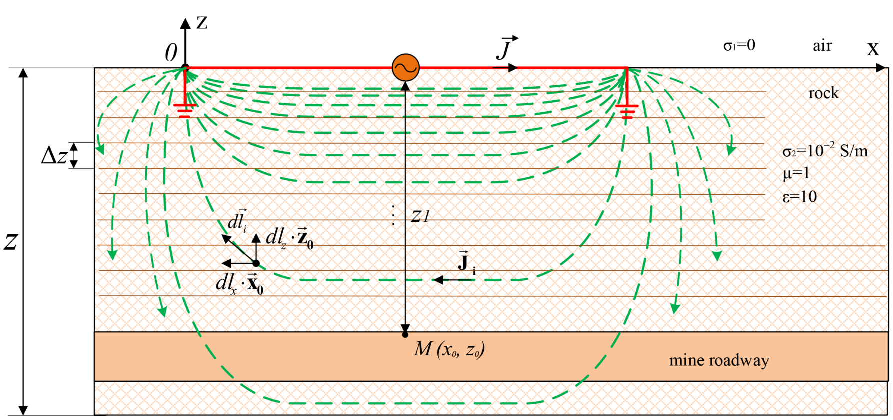


Fig. 2. Scheme of distribution of current density in the rock for a grounded cable with current

The magnetic field at point M is determined by the sum of three components:

$$\overline{H} = \overline{H}_0 + \sum_{i=1}^{k_1} \overline{H}_i + \sum_{i=k_1+1}^{k_2} \overline{H}_i, \tag{7}$$

where $\overline{H}_0$ is the field created by the current $\overline{J}$ flowing in the wire between the grounds on the surface of the half-space.

The second two elements (7) are the sum of the magnetic fields induced by elementary currents located above the observation point M. The total number of such elementary currents is k_1, therefore, $\Delta z \cdot k_1 \leqslant z_1$. The second sum determines the contribution of elementary currents, below the observation point M. The total number of such elementary currents is k_2, which means that $\Delta z \cdot k_2 \leqslant z_1 + z$, where z is the total depth of the studied spaces. The magnetic field of the current element is determined by the Biot-Savart law:

$$\overline{H}_i = \left(\frac{\overline{J}_i}{2\pi} \int\limits_{l_i} \frac{\left[d\bar{l}_i \cdot \bar{l}_r\right]}{r_i^2} \right) \cdot e^{-\alpha r_i}, \tag{8}$$

where $d\bar{l}_i$ is length of the current element $\overline{J}_i$;

$d\bar{l}_r$ is the unit vector of the radius of the vector r_i;

r_i is distance between current element and point M;

l_i is the length of the elementary conductor;

$\alpha = \omega \sqrt{\frac{\varepsilon\mu}{2} \cdot \left[\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1} + 1 \right]}$ is the attenuation coefficient (the real part of the wave number);

$\omega = 2\pi f$ is radian frequency;;

$\varepsilon = \varepsilon_0 \varepsilon_r$ is dielectrical constant of the medium;

$\mu = \mu_0 \mu_r$ is magnetic permeability of the medium;

σ is electrical conductivity.

The current distribution structure is determined by expressions (5), (6). Fig. 3 shows the dependence of j_x and j_z on x for a z value of 500–800 m. The main difference between j_x and j_z current components is that the current density j_z has a value of 0 under the center of the current vector in the cable (see Fig. 3b). This shows a change in the direction of the vector j_z to the opposite and reaches the maximum and minimum extremum under the ground points of the antenna. The current density for the j_x component reaches its maximum under the central part of the antenna (see Fig. 3a). The ratio of these components forms the current flow vector between the grounding points at a sufficiently large depth, while the current density decreases with depth due to the strong divergence of the current vectors over the volume of the half-space.

The magnetic field in the XZ plane is determined based on the ratio:

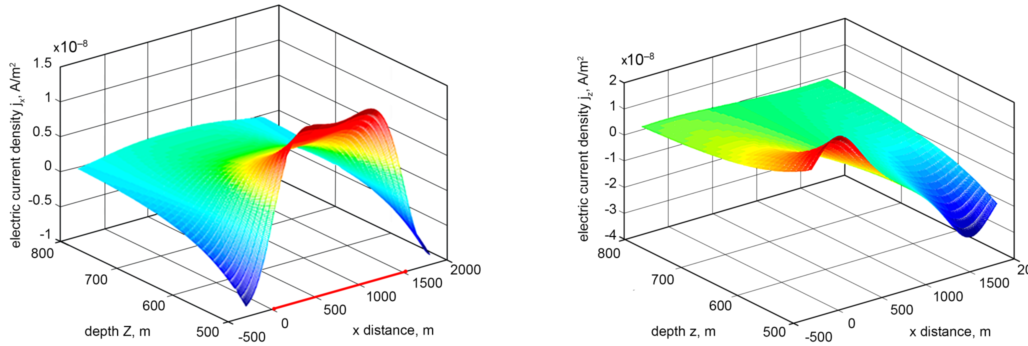
$$[d\bar{l}_i \cdot \bar{l}_r] = [\overline{x}_0 dx \bar{l}_{ir}] + [[\overline{z}_0 dz \bar{l}_{ir}]]. \tag{9}$$

The magnetic field at the point of observation $M(x_0,\, z_0)$ has an orientation along $\overline{y}_0$. The current field J_{xi} of the current component is determined from the equation:

$$\overline{y}_0 \cdot \overline{H}_i^x = \left[\frac{1}{2\pi} \int\limits_a^b J_{xi} \left(-\frac{(z_0 - z)dx}{r^3} \right) \right] \cdot e^{-\alpha r}. \tag{10}$$

where $r = \sqrt{(x_0 - x)^2 + (z_0 - z)^2}$ is distance between element $dl_i = dx$ and observation point M with coordinates x_0, z_0;

J_{xi} is current density at depth z, estimation of relations (5) and (6).



a b

Fig. 3. Current density distribution in the conducting half-space: a — j_x at a depth of 500–800 m; b — j_z at a depth of 500–800 m

The current component J_{zi} creates a magnetic field at the observation point M:

$$\overline{y}_0 \cdot \overline{H}_i^z = \left[\frac{1}{2\pi} \int_a^b J_{zi} \left(-\frac{(x_0 - x)dx}{r^3} \right) \right] \cdot e^{-\alpha r}. \tag{11}$$

The current density J_i at depth Z is determined by expression (5, 6) and the equivalent cross section of the conductor d in the form of a section of rocks (equivalent diameter):

$$J_i = j_{x,z} \cdot \frac{\pi d^2}{4}. \tag{12}$$

where d is cross-sectional diameter of the equivalent wire.

Computational modeling will provide a more accurate solution for analyzing the distribution of magnetic fields. This will make it possible to select the channel parameters depending on the properties of the medium in order to increase the efficiency in terms of the magnetic field level and the coverage area of the rock volume.

2. Computational modeling

Modeling the distribution of currents and magnetic field in an isotropic and anisotropic medium requires extensive computational resources. To analyze these processes, the finite element method (FEM) is proposed as one of the mathematical tools for numerical solution, including physical problems [9, 10]. The method is based on the division of the modeling object into subdomains (finite elements) and the approximation of an unknown function in each element as a combination of basic functions.

For the study, a model with dimensions of 4000×2000×1600 m was created, which includes an air layer (h_1=100 m) and a rock layer (h_2=1500 m). Propagation medium properties: electrical conductivity σ=10^{-3} S/m; dielectric constant ε=10; magnetic permeability μ=1. On the surface of the rock there is a radiating antenna with grounding, formed by a cable with a length l=1400 m and a current of J=5 A. The boundary conditions for the current and magnetic field are determined by the properties of the outer boundaries of the model $n \cdot J = 0$ and $n \times H = 0$ to take into account the absorption of currents and the magnetic field incident on the boundaries. This is done to simulate an infinite space around the model.

To analyze the model by the finite element method, the solution of Maxwell equations and full-current equation is implemented, which show the interaction of currents and electromagnetic fields in an absorbing medium [9, 10]:

$$\operatorname{rot} \overline{H} = \overline{J}, \tag{13}$$

$$\overline{B} = \operatorname{rot} \overline{A}, \tag{14}$$

$$\overline{E} = -j\omega \overline{A}, \tag{15}$$

$$\operatorname{div} \overline{J} = Q_{j,v}, \tag{16}$$

$$\overline{J} = \overline{J}_{cond} + \overline{J}_{EI} + \overline{j}_e, \tag{17}$$

where $Q_{j,v}$ is the density of the bulk current source; j_e is the density of electric current, external source (antenna current); V is the electric potential; σ is electrical conductivity; E is the electric field strength; $D = \varepsilon_0 \varepsilon_r E$ is electrical induction; $J_{EI} = j\omega D$ is electric induction current; $J_{cond} = \sigma E$ is conduction current; H is the magnetic field strength; B is magnetic induction; A is the vector potential; J is the electric current density.

Fig. 4 shows the distribution of the spatial components of the magnetic field (H_x, H_y, H_z) for a grounded antenna 1400 m long. The H_x component is concentrated in the upper part of the model closer to the antenna itself. It has an extremely low level below the center, which makes it the least useful for reception at great depths. The H_z component has a similar depth distribution, but a higher intensity, contributing to the total magnetic field H. The most useful and dominant component in the total field is the H_y component. Possessing the highest tension and coverage area, this component allows you to register it in any area in the emitter area. In the antenna ground region, the Hz and H_y components compensate each other, maintaining an acceptable level of magnetic field strength. An analysis of the distribution of the components shows that the orientation of the receiving magnetic antenna along the Z and Y axes is the most effective for registering a useful signal.

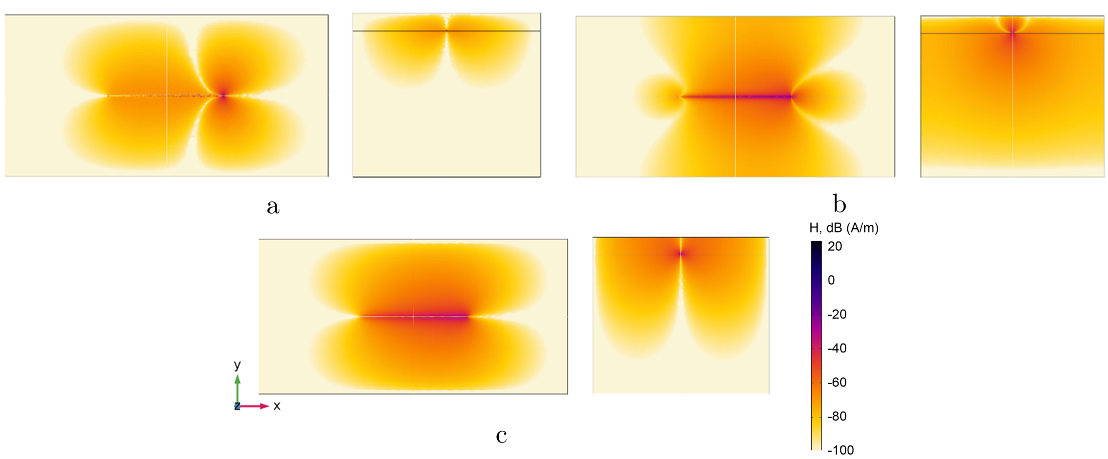


Fig. 4. Distribution of the magnetic field components for a grounded antenna 1400 m long: a — H_x; b — H_y; c – H_z

In practice, signal reception occurs using a ferrite antenna that converts the magnetic field into EMF. In this case, the signal voltage at the receiver input is calculated by the formula [11]:

$$U = \omega \cdot \mu_0 \cdot H \cdot S_{AEA}, \tag{18}$$

where $S_{AEA} = \mu_c ore \cdot S_{core} \cdot n = 1\text{m}^2$ is the antenna-effective area; $n = 1000$ is number of turns; $\mu_{core} = 8000$ is magnetic permeability of the core; $S_{core} = 1.25 \cdot 10^{-7}$ m^2 is the cross-sectional area of the core; $\mu_0 = 4\pi 10^{-7}$ H/m is the magnetic constant; H is the magnetic field strength.

As a result of the simulation, estimates of the level of the magnetic field and the voltage induced on the receiving antenna at a maximum depth of 1000 m were obtained. Analysis in the frequency range of 0.5–30 kHz indicates a general trend of attenuation of the electromagnetic field with increasing frequency due to the absorbing properties of the rock mass, being an electrically conductive medium. For grounded antenna of medium length 1400 m, the simulation shows a level of 1.32 μV at a frequency of 2 kHz (Fig. 5).

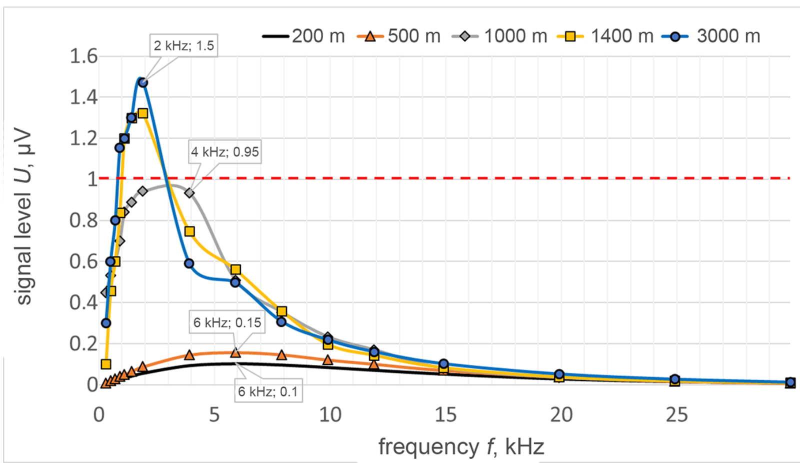


Fig. 5. Dependence of the signal level at the receiver input on the frequency for the depth Z = 1000 m under the center of the antenna, at $\sigma = 10^{-3}$ S/m, and a number of lengths of the transmitting antenna

To analyze the distribution of the EM field under the antenna, simulation data were obtained at a depth of 1000 m in a continuous medium. At the limiting distance at a frequency of 2 kHz, the voltage level above 0.5 μV is maintained in a continuous medium over a length of 1700 m for a grounded current dipole (Fig. 6a). This indicates the advantage of a grounded long antenna for mines located at depths of 200–800 m and having a large extent.

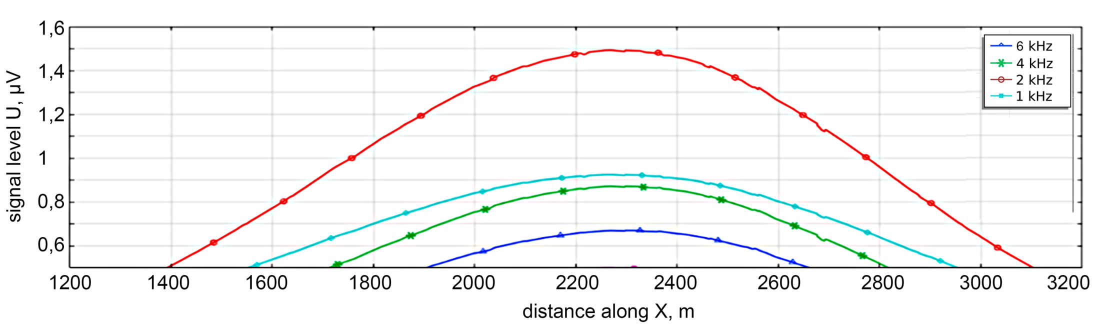


Fig. 6. Distribution of the signal level under the transmitting antenna at a depth of 1000 m

Conclusion

As a result of the study, the magnetic field components that are most suitable for using a receiving magnetic antenna were determined. The components H_y and H_z are predominant at depths greater than 100 m. The optimal operating frequencies of the transmitter are determined depending on the depth of the receiver position. For depths up to 1000 m, with an electrical conductivity of rocks of 10^{-3} S/m, these frequencies are in the range of 1–3 kHz. To increase the coverage area and the maximum depth of signal transmission, it is necessary to increase the length of the transmitting antenna.

The reported study was funded by RFBR, project number 20-07-00267

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Вычислительное моделирование распределения электромагнитного поля горизонтальной заземленной антенны в горной породе

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Аннотация. В статье исследовано пространственное распределение электромагнитного поля антенны в виде длинного заземленного кабеля в горной породе с заданными электрофизическими параметрами. Определены частотные зависимости излучаемых сигналов (300 Гц – 30 кГц) от глубины положения приемника, что имеет большое прикладное значение для задач шахтной радиосвязи.

Ключевые слова: беспроводная связь, шахтная связь, магнитная антенна, заземленная антенна, электромагнитное поле, инфранизкие частоты, очень низкие частоты.

DOI: 10.17516/1997-1397-2022-15-5-586-597
УДК 539.37

Study of a Deformation Localization Direction in Slow Motion of a Granular Medium

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Received 09.11.2021, received in revised form 18.03.2022, accepted 25.06.2022

Abstract. This paper is devoted to the study of the direction of the deformation localization lines in a slow gravity flow of a granular medium in convergent channels with various geometric characteristics. Variational principles of the theory of limiting equilibrium, established within the framework of a special mathematical model of a material that resist tension and compression differently, are used. Assuming a linear deformation localization zone we obtain safety factors and carry out their comparative analysis.

Keywords: variational inequality, materials with different strengths, deformation localization.

Citation: O.I. Kuzovatova, Study of a Deformation Localization Direction in Slow Motion of a Granular Medium, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 586–597.
DOI: 10.17516/1997-1397-2022-15-5-586-597.

Introduction

The theory of materials with different strengths is one of the most interesting and actively developing branch of mechanics. The field of application of this theory is the problems of mechanics of geomaterials. Such materials have significantly different tensile and compressive strength properties. The range of problems related to the mechanics of geomaterials is diverse. In engineering practice, the analysis of the behavior of geomaterials is important in connection with the issues of mechanical treatment of soils, as well as in relation to the issues of mining, construction of engineering structures etc.

The study of the process of localization of deformations in samples made of a material with different strengths is of constant interest. The importance of solving of such problems is dictated by the fact that in practice in narrow zones of localization of tensile deformations where malleability of the material is significantly higher than in the rest of the sample micro-destructions occur. Therefore, when analyzing the structural design for strength, such zones must be determined. At the same time, the possibilities of constructing exact solutions in such problems are limited, thus the development of computational methods is very relevant.

In the branch of geomechanics related to the study of the behavior of granular media, there is an important problem of analyzing movement of granular media in converging channels. Problems of this kind arise when emptying granular media or geomaterials from storage chambers and bunkers, as well as in many mining technologies. The approximate (engineering) solution of the problem and the results of field experiments are presented in works [1,2]. In the work [3] the problem of a flat slow gravity flow of a granular medium in a converging channel was considered.

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For granular sample the safety factor was computed and formulas were obtained for calculating the inclination angle of a narrow linear zone of deformation localization of simple shear deformation with dilatancy. A numerical experiment was also carried out using the finite element method that showed results close to the solution.

The purpose of this work is to construct an approximate solution to the problem of slow gravity flow of a granular medium in converging channels with various geometric structures. During the transition from the static stress-strain state to the movement of a granular medium, the deformation is localized along some surfaces, followed by the movement of the formed blocks. Under assumption of a linear deformation localization zone, it is necessary to calculate the safety factors for various channel samples and conduct the comparative analysis. The solution of the problem will be based on a model that takes into account different strengths of the material [4].

1. Mathematical model

For the description of the stress-strain state of a granular medium as a different strengths material having different tensile and compressive strength limits, we will use a model of a medium with plastic bonds. This model has been developed by V. P. Myasnikov and V. M. Sadovskii in the work [4]. Under compressive or tensile strain lower than the adhesion coefficient (the limit bond strength) such a medium does not deform. As the limit bond strength is approached, the deformation develops according to the theory of linear strain hardening. The rheological scheme of the model is given on Fig. 1 [5]. According to this scheme we have the following

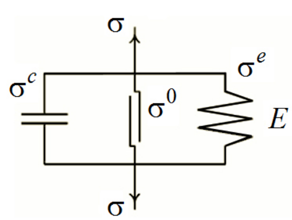

Fig. 1. The rheological scheme

additive representation $\sigma_{ij} = \sigma^c_{ij} + \sigma^0_{ij} + \sigma^e_{ij}$, where σ_{ij} is the total strain tensor, σ^c_{ij} is the rigid contact component, σ^0_{ij} is the cohesion tensor, $\sigma^e_{ij} = E_{ijkl}\varepsilon_{kl}$ is the elastic tensor, $\varepsilon = (\varepsilon_{ij})$ is the deformation tensor, E_{ijkl} is the symmetric positively defined elastic modulus tensor (we assume summing in repeating indices). The tensor σ^c_{ij} satisfies the variational inequality

$$\sigma^c_{ij} \cdot (\tilde{\varepsilon}_{ij} - \varepsilon_{ij}) \leqslant 0, \quad \varepsilon, \tilde{\varepsilon} \in C, \tag{1}$$

where C is the cone of admissible deformations of the form $C = \{\varepsilon \mid \kappa\gamma(\varepsilon) \leqslant \theta(\varepsilon)\}$, κ is the dilatancy parameter, $\gamma(\varepsilon)$ is the intensity of shear, $\theta(\varepsilon)$ is the volume deformation [5].

In this notation, the inequality (1) takes the form

$$\left(E_{ijkl}\varepsilon_{kl} - \sigma_{ij} + \sigma^0_{ij}\right) \cdot (\tilde{\varepsilon}_{ij} - \varepsilon_{ij}) \geqslant 0, \quad \varepsilon, \tilde{\varepsilon} \in C.$$

By definition of a projection, this means that

$$\varepsilon_{ij} = \pi_{ij}\left[E^{-1}_{ijkl}\left(\sigma_{ij} - \sigma^0_{ij}\right)\right],$$

here E^{-1}_{ijkl} are the components of the inverse tensor, π_{ij} the components of the projection of C with respect to the norm $|\varepsilon| = \sqrt{\varepsilon_{ij} E_{ijkl} \varepsilon_{kl}}$.

Consider [3] an element of a construction from a material with different strengths filling a planar domain Ω with the boundary $\partial\Omega = \Gamma$ that consists of two non-intersecting parts Γ_u and Γ_σ. On the first part displacements are absent and on the second part the distributed load p is given. There hold equilibrium equations in variational form and boundary conditions

$$\int\limits_{\Omega} \left(\frac{\partial \sigma_{ij}}{\partial x_j} + f_i \right) (\tilde{u}_i - u_i) \, d\Omega = 0, \tag{2}$$

$$u_i = \tilde{u}_i = 0 \ \text{ on } \ \Gamma_u, \quad \sigma_{ij} \cdot n_j = p_i \ \text{ on } \ \Gamma_\sigma. \tag{3}$$

The problem (2)–(3) reduces to the problem of finding the minimum $\min\limits_{\tilde{u}_i \in U_c} J(\tilde{u}) = J(u)$, where

$$J(u) = \iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma^0_{11} + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma^0_{12} + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma^0_{22} - (f_1 \tilde{u}_1 + f_2 \tilde{u}_2) \right) dx_1 dx_2 - \\ - \int\limits_{\Gamma_\sigma} (p_1 \tilde{u}_1 + p_2 \tilde{u}_2) \, d\Gamma,$$

$$U_C = \left\{ u_i \in H^1(\Omega) \mid u_i|_{\Gamma_u} = 0, \ \varepsilon(u) \in C \right\},$$

for the components of the deformation tensor we have kinematic equations

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

A load (f, p) is called safe if $u \equiv 0$.

Let $p_i = 0$, $f_i = m \cdot f^0_i$, where m is the loading parameter. A load is safe for m varying from zero to the limit value (safety factor)

$$m^* = \min_{\substack{(\tilde{u}_1, \tilde{u}_2) \in U_c \\ (\tilde{u}_1, \tilde{u}_2) \neq 0}} \frac{\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma^0_{11} + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma^0_{12} + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma^0_{22} \right) dx_1 dx_2}{\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2) \, dx_1 dx_2}. \tag{4}$$

This statement is a formulation of a kinematic theorem on limiting equilibrium from plasticity theory [6].

2. Linear deformation localization zone

In the paper [3] we considered a problem of planar gravity flow of a granular medium in a convergent asymmetric channel with sides inclined at angles α and β with the base a, assuming $\alpha > \beta$, $\alpha \in \left(0; \frac{\pi}{2}\right)$ (Fig. 2). The convergent channel fills a planar domain Ω with the boundary $\partial\Omega = \Gamma = \Gamma_u \bigcup \Gamma_\sigma$. On the boundary Γ_u displacements are absent. The vector p of the distributed load on Γ_σ is equal to zero.

The condition $(\tilde{u}_1, \tilde{u}_2) \in U_C$ takes the form

$$\gamma_0 \leqslant \nu \varepsilon_0, \tag{5}$$

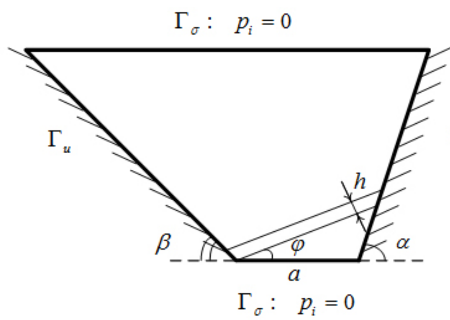

Fig. 2. Direction of a narrow linear localization zone

where $\nu = \sqrt{1/\kappa^2 - 4/3}$, $0 < \kappa < \sqrt{3}/2$.

By (4) the safety factor is m_1:

$$m_1 = \frac{2\tau_s}{\kappa\rho g a}\frac{1}{\sin\alpha}\frac{1}{(\nu\sin\alpha - \cos\alpha)}, \tag{6}$$

here τ_s is the yield point.

In paper [3] we obtained that deformation for a simple shear with dilatancy is localized in a narrow linear zone of thickness h inclined at an angle φ:

$$\varphi = \alpha - \arcsin\frac{1}{\sqrt{\nu^2+1}} \quad \text{or} \quad \varphi = \alpha - \operatorname{arctg}\frac{1}{\nu}. \tag{7}$$

In this case for the angle φ we compute

$$\sin\varphi = \frac{\nu\sin\alpha - \cos\alpha}{\sqrt{\nu^2+1}}, \quad \cos\varphi = \frac{\nu\sin\alpha - \cos\alpha}{\sqrt{\nu^2+1}}. \tag{8}$$

In this paper we shall consider two problems of planar gravity flow of a granular medium in a convergent asymmetric channel. The geometry of a channel in each case will differ from the one on Fig. 2. Under assumption of linearity of the deformation localization zone we compute the safety factors m^* for such channels and compare them. The boundary conditions for the domain Ω are analogous to the conditions of the problem (Fig. 2) considered in [3].

Consider a planar deformed state of a homogeneous sample (Fig. 3). Its geometry differs from that of the one considered in [3] (Fig. 2): the base of the channel is inclined at an angle φ (7).

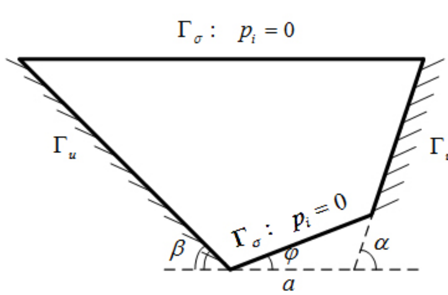

Fig. 3. Cross-section of the sample ($\alpha > \beta$)

Problem 1. Compute the safety factor m^* for the sample (Fig. 3) according to (4).

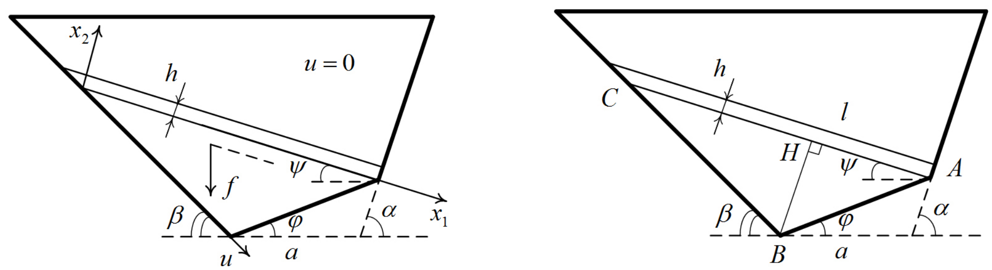

Fig. 4. Geometric constructions

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ be the admissible displacement field describing the deformation localization of simple shear with dilatancy in a narrow linear zone of thickness h inclined at an angle ψ (Fig. 4).

Let us compute some angles. Consider

$$\begin{aligned} \triangle ABH: &\quad \angle BAH = \varphi + \psi, \quad \angle ABH = \frac{\pi}{2} - (\varphi + \psi), \\ \triangle CBH: &\quad \angle CBH = \pi - (\beta + \varphi + \angle ABH) = \frac{\pi}{2} - (\beta - \psi), \\ &\quad \angle BCH = \frac{\pi}{2} - \angle CBH = \beta - \psi. \end{aligned}$$

In the Cartesian coordinates related to the narrow linear zone

$$\begin{cases} \tilde{u}_1 = u_0 \cos(\beta - \psi), \\ \tilde{u}_2 = -u_0 \sin(\beta - \psi), \end{cases} \qquad \begin{cases} \gamma_0 = \dfrac{u_0}{h} \cos(\beta - \psi), \\ \varepsilon_0 = \dfrac{u_0}{h} \sin(\beta - \psi). \end{cases} \tag{9}$$

Then we get

$$\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma_{11}^0 + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma_{12}^0 + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma_{22}^0 \right) dx_1 dx_2 = \varepsilon_0 \, \sigma^0 \cdot S_{\Box}, \tag{10}$$

where

$$\sigma^0 = \frac{\tau_s}{\kappa}, \qquad S_{\Box} = h\, l.$$

The 'separating' triangle domain ABC moves as a solid body, therefore

$$\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2)\, dx_1 dx_2 = f^0 \cdot S_{\triangle}, \tag{11}$$

where

$$f^0 = \rho\, g\, u_0 \sin\beta, \qquad S_{\triangle} = \frac{1}{2} H\, l, \qquad H = L_1 \sin(\varphi + \psi),$$

$$L_1 = AB = a\left(\cos\varphi + \sin\varphi \cdot \operatorname{ctg}(\alpha - \varphi)\right) = a\left(\cos\varphi + \nu \sin\varphi\right),$$

substituting values from (8) we get

$$L_1 = a\sqrt{1 + \nu^2}\, \sin\varphi. \tag{12}$$

Having in mind that (10) and (11), the safety factor m^* is equal to

$$m^* = \frac{\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c \\ (\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\frac{u_0}{h}\,\sin(\beta-\psi)\,h\,l}{u_0\,\sin\beta\cdot\frac{1}{2}\,H\,l} = \frac{2\,\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c \\ (\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\sin(\beta-\psi)}{\sin\beta\cdot L_1\sin(\varphi+\psi)}.$$

Taking into account (5) and (9), we obtain the relation

$$\cos(\beta-\psi) \leqslant \nu\sin(\beta-\psi),$$

$$\frac{\varepsilon_0}{\gamma_0} = \mathrm{tg}(\beta-\psi) \geqslant \frac{1}{\nu} \quad \text{or} \quad \sin(\beta-\psi) \geqslant \frac{1}{\sqrt{\nu^2+1}}.$$

Then

$$\psi \leqslant \beta - \mathrm{arctg}\,\frac{1}{\nu} \quad \text{or} \quad \psi \leqslant \beta - \arcsin\frac{1}{\sqrt{\nu^2+1}}.$$

In this case

$$m^* = \frac{2\,\tau_s}{\kappa\rho g L_1\sin\beta} \min_{\psi} \frac{\sin(\beta-\psi)}{\sin(\varphi+\psi)}. \tag{13}$$

Let us find the minimum of the expression from (13) with respect to ψ

$$\begin{aligned}\frac{d}{d\psi}\left(\frac{\sin(\beta-\psi)}{\sin(\varphi+\psi)}\right) &= \frac{-\cos(\beta-\psi)\sin(\varphi+\psi)-\sin(\beta-\psi)\cos(\varphi+\psi)}{\sin^2(\varphi+\psi)} = \\ &= -\frac{\sin(\varphi+\psi)\cos(\beta-\psi)+\cos(\varphi+\psi)\sin(\beta-\psi)}{\sin^2(\varphi+\psi)} = \\ &= -\frac{\sin(\varphi+\psi+\beta-\psi)}{\sin^2(\varphi+\psi)} = -\frac{\sin(\varphi+\beta)}{\sin^2(\varphi+\psi)} < 0 \qquad \forall\psi,\end{aligned}$$

since $\sin(\varphi+\beta) > 0$ for $\beta,\ \varphi \in \left(0;\ \frac{\pi}{2}\right)$.

Thus, the minimum of the function (13) in ψ is attained at

$$\psi = \beta - \mathrm{arctg}\,\frac{1}{\nu} \quad \text{or} \quad \psi = \beta - \arcsin\frac{1}{\sqrt{\nu^2+1}}. \tag{14}$$

Then formula (13) for the safety factor m^* assumes the form

$$m_2 = \frac{2\,\tau_s}{\kappa\rho g L_1\sin\beta}\,\frac{1}{\sqrt{\nu^2+1}\,\sin(\varphi+\psi)}. \tag{15}$$

From (14) we obtain

$$\sin\psi = \frac{\nu\sin\beta-\cos\beta}{\sqrt{\nu^2+1}}, \qquad \cos\psi = \frac{\nu\cos\beta+\sin\beta}{\sqrt{\nu^2+1}}. \tag{16}$$

Using geometric constructions (Fig. 4) we find the value of $l = L_2$, which is needed further below.

We have $L_2 = AC = l_1 + l_2$. Consider $\triangle ABC = \triangle ABH + \triangle CBH$. Then

$$\triangle ABH \;\Rightarrow\; \mathrm{tg}(\varphi+\psi) = \frac{H}{l_1}, \quad H = L_1\sin(\varphi+\psi),$$

$$\triangle CBH \;\Rightarrow\; \mathrm{tg}(\beta-\psi) = \frac{H}{l_2} = \frac{1}{\nu}.$$

We get

$$L_2 = l_1 + l_2 = \frac{H}{\mathrm{tg}(\varphi+\psi)} + H \cdot \nu = L_1 \cdot \sin(\varphi+\psi) \cdot \left(\frac{\cos(\varphi+\psi)}{\sin(\varphi+\psi)} + \nu \right)$$

or

$$L_2 = L_1 \left(\cos(\varphi+\psi) + \nu \sin(\varphi+\psi) \right), \tag{17}$$

here L_1 is given by (12).

Consider now a planar deformed state of a homogeneous sample (Fig. 5) assuming that $\alpha \in \left(0; \dfrac{\pi}{2}\right)$ and $\alpha > \beta$. The geometry in Problem 2 differs from the geometry in Problem 1: the base of the channel is inclined at an angle ψ. The values of the angles φ and ψ are computed by formulas (7) and (14), respectively, $\varphi < \alpha$, $\psi < \beta$. The boundary conditions are the same as in Problem 1.

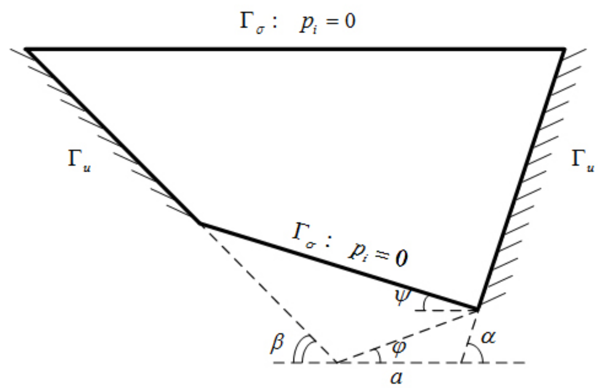

Fig. 5. Cross-section of the sample ($\alpha > \beta$)

Problem 2. Compute the safety factor m^* for the sample (Fig. 5) according to (4).

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ be the admissible displacement field describing the deformation localization of simple shear with dilatancy in a narrow linear zone of thickness h inclined at an angle ϕ (Fig. 6).

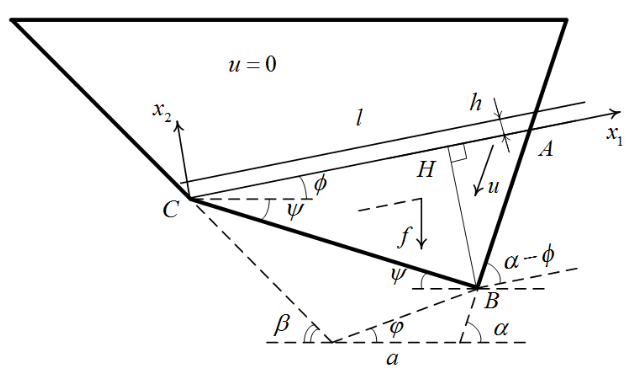

Fig. 6. Geometric constructions

In the Cartesian coordinates related to this zone

$$\begin{cases} \tilde{u}_1 = -u_0 \cos(\alpha - \phi), \\ \tilde{u}_2 = -u_0 \sin(\alpha - \phi), \end{cases} \qquad \begin{cases} \gamma_0 = \dfrac{u_0}{h} \cos(\alpha - \phi), \\ \varepsilon_0 = \dfrac{u_0}{h} \sin(\alpha - \phi). \end{cases} \tag{18}$$

Then we get

$$\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma_{11}^0 + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma_{12}^0 + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma_{22}^0 \right) dx_1 dx_2 = \varepsilon_0 \, \sigma^0 \cdot S_{\square}, \tag{19}$$

where

$$\sigma^0 = \frac{\tau_s}{\kappa}, \quad S_{\square} = h\, l.$$

The 'separating' triangle domain ABC moves as a solid body, hence,

$$\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2)\, dx_1 dx_2 = f^0 \cdot S_{\triangle}, \tag{20}$$

where

$$f^0 = \rho\, g\, u_0 \sin\alpha, \qquad S_{\triangle} = \frac{1}{2} H\, l, \qquad H = L_2 \sin(\psi + \phi),$$

here L_2 is computed by formula (17).

With (19) and (20), the safety factor m^* is

$$m^* = \frac{\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c \\ (\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\frac{u_0}{h} \sin(\alpha - \phi)\, h\, l}{u_0 \sin\alpha \cdot \frac{1}{2} H\, l} = \frac{2\,\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c \\ (\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\sin(\alpha - \phi)}{\sin\alpha \cdot L_2 \cdot \sin(\psi + \phi)}.$$

Taking into account (5) and (18), we get the relation

$$\cos(\alpha - \phi) \leqslant \nu \sin(\alpha - \phi),$$

$$\frac{\varepsilon_0}{\gamma_0} = \operatorname{tg}(\alpha - \phi) \geqslant \frac{1}{\nu} \quad \text{or} \quad \sin(\alpha - \phi) \geqslant \frac{1}{\sqrt{\nu^2 + 1}}.$$

Then

$$\phi \leqslant \alpha - \operatorname{arctg} \frac{1}{\nu} \quad \text{or} \quad \phi \leqslant \alpha - \arcsin \frac{1}{\sqrt{\nu^2 + 1}}.$$

In this case

$$m^* = \frac{2\,\tau_s}{\kappa\rho g L_2 \sin\alpha} \min_{\phi} \frac{\sin(\alpha - \phi)}{\sin(\psi + \phi)}. \tag{21}$$

Let us find the minimum of the expression from (21) with respect to ϕ

$$\frac{d}{d\phi} \left(\frac{\sin(\alpha - \phi)}{\sin(\psi + \phi)} \right) = -\frac{\cos(\alpha - \phi)\sin(\psi + \varphi) + \sin(\alpha - \phi)\cos(\psi + \phi)}{\sin^2(\psi + \phi)} =$$

$$= -\frac{\sin(\alpha - \phi + \psi + \phi)}{\sin^2(\psi + \phi)} = -\frac{\sin(\alpha + \psi)}{\sin^2(\psi + \phi)} < 0 \quad \forall \phi,$$

since $\sin(\alpha + \psi) > 0$ for $\alpha, \psi \in \left(0; \dfrac{\pi}{2}\right)$. Hence, the minimum is attained at

$$\operatorname{tg}(\alpha - \phi) = \frac{1}{\nu} \quad \text{or} \quad \sin(\alpha - \phi) = \frac{1}{\sqrt{\nu^2 + 1}},$$

which means that

$$\phi = \alpha - \operatorname{arctg} \frac{1}{\nu} \quad \text{or} \quad \phi = \alpha - \arcsin \frac{1}{\sqrt{\nu^2 + 1}}. \tag{22}$$

Comparing the expressions (7) and (22) we deduce that the deformation localization zone is inclined at the angle $\phi = \varphi$. Thus, the safety factor (21) takes the form

$$m_3 = \frac{2\,\tau_s}{\kappa\rho g L_2 \sin\alpha} \frac{1}{\sqrt{\nu^2 + 1}\sin(\varphi + \psi)} \tag{23}$$

for the values of L_2 from (17).

3. Safety factors comparison

Let us compare the safety factors m_1, m_2, and m_3 of the form (6), (15), and (23), respectively. For that we consider the quotients m_2/m_1 and m_3/m_2.

By conditions, we know the angles

$$\alpha > \beta, \quad \alpha, \beta \in \left(0; \frac{\pi}{2}\right), \quad \varphi < \alpha, \quad \psi < \beta.$$

Let us carry out auxiliary computations. Using (8) and (16), we find

$$\sin(\varphi + \psi) = \sin\varphi\cos\psi + \cos\varphi\sin\psi =$$

$$= \frac{(\nu\sin\alpha - \cos\alpha)(\nu\cos\beta + \sin\beta) + (\nu\cos\alpha + \sin\alpha)(\nu\sin\beta - \cos\beta)}{\nu^2 + 1}.$$

Modify the numerator of this expression

$$\begin{aligned}
&\nu^2 \sin\alpha\cos\beta + \nu\sin\alpha\sin\beta - \nu\cos\alpha\cos\beta - \cos\alpha\sin\beta + \\
&+ \nu^2\cos\alpha\sin\beta - \nu\cos\alpha\cos\beta + \nu\sin\alpha\sin\beta - \sin\alpha\cos\beta = \\
&= (\nu^2 - 1)(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 2\nu(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = \\
&= (\nu^2 - 1)\sin(\alpha + \beta) - 2\nu\cos(\alpha + \beta).
\end{aligned}$$

Then

$$\sin(\varphi + \psi) = \frac{(\nu^2 - 1)\sin(\alpha + \beta) - 2\nu\cos(\alpha + \beta)}{\nu^2 + 1}. \tag{24}$$

Analogously,

$$\cos(\varphi + \psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi =$$

$$= \frac{(\nu\sin\alpha + \cos\alpha)(\nu\cos\beta + \sin\beta) - (\nu\cos\alpha - \sin\alpha)(\nu\sin\beta - \cos\beta)}{\nu^2 + 1}.$$

After rearranging the numerator

$$\begin{aligned}
&\nu^2 \sin\alpha\cos\beta + \nu\sin\alpha\sin\beta + \nu\cos\alpha\cos\beta + \cos\alpha\sin\beta - \\
&- \nu^2\cos\alpha\sin\beta + \nu\cos\alpha\cos\beta + \nu\sin\alpha\sin\beta - \sin\alpha\cos\beta = \\
&= (\nu^2 - 1)(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\nu(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \\
&= (\nu^2 - 1)\cos(\alpha + \beta) + 2\nu\sin(\alpha + \beta),
\end{aligned}$$

we obtain

$$\cos(\varphi + \psi) = \frac{(\nu^2 - 1)\cos(\alpha + \beta) + 2\nu\sin(\alpha + \beta)}{\nu^2 + 1}. \tag{25}$$

Problem 3. Find the condition for ν for which

$$\frac{m_2}{m_1} < 1.$$

Taking into account (24), we consider the quotient

$$\frac{m_2}{m_1} = \frac{\dfrac{2\,\tau_s}{\kappa\rho g L_1 \sin\beta}\,\dfrac{1}{\sqrt{\nu^2+1}\,\sin(\varphi+\psi)}}{\dfrac{2\,\tau_s}{\kappa\rho g a}\,\dfrac{1}{\sin\alpha\,(\nu\sin\alpha - \cos\alpha)}} = \frac{a\sin\alpha\,(\nu\sin\alpha - \cos\alpha)}{\sqrt{\nu^2+1}\,L_1\sin\beta\sin(\varphi+\psi)} =$$

$$= \frac{\sin\alpha}{\sin\beta}\cdot\frac{1}{(\nu^2+1)}\cdot\frac{\nu\sin\alpha - \cos\alpha}{\sin\alpha\sin(\varphi+\psi)} = \frac{1}{\sin\beta}\cdot\frac{\nu\sin\alpha - \cos\alpha}{(\nu^2-1)\sin(\alpha+\beta) - 2\nu\cos(\alpha+\beta)}.$$

Then

$$\frac{\nu\sin\alpha-\cos\alpha}{\sin\beta}<(\nu^2-1)\sin(\alpha+\beta)-2\nu\cos(\alpha+\beta)$$

i.e., we get a quadratic inequality in ν:

$$\nu^2\sin(\alpha+\beta)-\nu\left(2\cos(\alpha+\beta)+\frac{\sin\alpha}{\sin\beta}\right)-\sin(\alpha+\beta)+\frac{\cos\alpha}{\sin\beta}>0.$$

Solving it, we find

$$\begin{aligned}
D&=\left(2\cos(\alpha+\beta)+\frac{\sin\alpha}{\sin\beta}\right)^2-4\sin(\alpha+\beta)\left(-\sin(\alpha+\beta)+\frac{\cos\alpha}{\sin\beta}\right)=\\
&=4\cos^2(\alpha+\beta)+4\cos(\alpha+\beta)\cdot\frac{\sin\alpha}{\sin\beta}+\frac{\sin^2\alpha}{\sin^2\beta}+4\sin^2(\alpha+\beta)-4\sin(\alpha+\beta)\cdot\frac{\cos\alpha}{\sin\beta}=\\
&=4+\frac{\sin^2\alpha}{\sin^2\beta}-\frac{4}{\sin\beta}\left(\sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha\right)=\\
&=4+\frac{\sin^2\alpha}{\sin^2\beta}-\frac{4}{\sin\beta}\sin(\alpha+\beta-\alpha)=4+\frac{\sin^2\alpha}{\sin^2\beta}-4=\frac{\sin^2\alpha}{\sin^2\beta},
\end{aligned}$$

and

$$\nu_{1,2}=\frac{2\cos(\alpha+\beta)+\dfrac{\sin\alpha}{\sin\beta}\pm\dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)}.$$

Let $\nu_1<\nu_2$, namely,

$$\nu_1=\frac{2\cos(\alpha+\beta)+\dfrac{\sin\alpha}{\sin\beta}-\dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)}=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\operatorname{ctg}(\alpha+\beta),\tag{26}$$

$$\nu_2=\frac{2\cos(\alpha+\beta)+\dfrac{\sin\alpha}{\sin\beta}+\dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)}=\frac{\cos(\alpha+\beta)+\dfrac{\sin\alpha}{\sin\beta}}{\sin(\alpha+\beta)}.\tag{27}$$

Thus, the inequality $\dfrac{m_2}{m_1}<1$ holds for $\nu<\nu_1$ and $\nu>\nu_2$. If $\nu_1<\nu<\nu_2$ then $m_2>m_1$ and the second fragment does not move.

Problem 4. Find the condition for ν for which

$$\frac{m_3}{m_2}<1.$$

Consider the relation

$$\frac{m_3}{m_2}=\frac{\dfrac{2\,\tau_s}{\kappa\rho gL_2\sin\alpha}\dfrac{1}{\sqrt{\nu^2+1}\sin(\varphi+\psi)}}{\dfrac{2\,\tau_s}{\kappa\rho gL_1\sin\beta}\dfrac{1}{\sqrt{\nu^2+1}\sin(\varphi+\psi)}}=\frac{L_1}{L_2}\frac{\sin\beta}{\sin\alpha}=\frac{1}{\cos(\varphi+\psi)+\nu\sin(\varphi+\psi)}\frac{\sin\beta}{\sin\alpha}.$$

Using (24) and (25), we obtain the value of the expression

$$\begin{aligned}
&\cos(\varphi+\psi)+\nu\sin(\varphi+\psi)=\\
&\quad=\frac{\left(\nu^2-1\right)\cos(\alpha+\beta)+2\nu\sin(\alpha+\beta)}{\nu^2+1}+\nu\,\frac{\left(\nu^2-1\right)\sin(\alpha+\beta)-2\nu\cos(\alpha+\beta)}{\nu^2+1}=\\
&\quad=\frac{\left(\nu^2-1-2\nu^2\right)\cos(\alpha+\beta)+\left(2\nu+\nu^3-\nu\right)\sin(\alpha+\beta)}{\nu^2+1}=\\
&\quad=\frac{-\left(\nu^2+1\right)\cos(\alpha+\beta)+\nu\left(\nu^2+1\right)\sin(\alpha+\beta)}{\nu^2+1}=-\cos(\alpha+\beta)+\nu\sin(\alpha+\beta).
\end{aligned}$$

Then

$$\frac{\sin\beta}{\sin\alpha}<\nu\sin(\alpha+\beta)-\cos(\alpha+\beta)$$

or

$$\nu>\nu_0=\frac{\cos(\alpha+\beta)+\dfrac{\sin\beta}{\sin\alpha}}{\sin(\alpha+\beta)}. \tag{28}$$

Thus, the inequality $\dfrac{m_3}{m_2}<1$ holds for $\nu>\nu_0$, if $\nu<\nu_0$ then $m_3>m_2$, and the third fragment does not move.

Let us compare the obtained values $\nu_1<\nu_2$ and ν_0 from (26), (27), and (28).

We have $\sin\alpha>\sin\beta$, or $\dfrac{\sin\alpha}{\sin\beta}>1$, since $\alpha>\beta$ and $\alpha,\ \beta\in\left(0;\dfrac{\pi}{2}\right)$. Therefore $\nu_1<\nu_0<\nu_2$.

Consequently, for $\nu>\nu_2$ of the form (27) the inequalities $\dfrac{m_2}{m_1}<1$ and $\dfrac{m_3}{m_2}<1$ hold simultaneously.

Thus, depending on the value of the coefficient ν that characterize the dilatancy of the medium the deformation zones are localized differently.

Conclusion

In this paper we use the model of a granular medium with different strengths by V. P. Myasnikov and V. M. Sadovskii to study a slow gravity flow of a granular medium in convergent channels. Assuming a linear deformation localization zone we obtain an approximate value of the safety factor and formulas for the slope of a narrow linear zone of the deformation localization for a simple shear with dilatancy. A comparative analysis of the obtained factors is carried out.

This work is supported by Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement no. 075-02-2022-873).

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Исследование направления локализации деформации при медленном движении сыпучей среды

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Аннотация. В статье исследуется направление линий локализации деформации при медленном движении сыпучей среды под действием собственного веса в сходящемся канале с разной геометрической структурой. Используются вариационные принципы теории предельного равновесия, установленные на основе специальной математической модели материала, по-разному сопротивляющегося растяжению и сжатию. В рамках предположения о линейной зоне локализации деформации вычислены коэффициенты безопасности и проведен их сравнительный анализ.

Ключевые слова: вариационное неравенство, разнопрочная среда, локализация деформации.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 598–609

DOI: 10.17516/1997-1397-2022-15-5-598-609
УДК 512

Tutorial on Rational Rotation C^*-Algebras

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Received 10.11.2021, received in revised form 23.04.2022, accepted 30.06.2022

Abstract. The rotation algebra $\mathcal{A}_\theta$ is the universal C^*-algebra generated by unitary operators U, V satisfying the commutation relation $UV = \omega VU$ where $\omega = e^{2\pi i\theta}$. They are rational if $\theta = p/q$ with $1 \leqslant p \leqslant q-1$, othewise irrational. Operators in these algebras relate to the quantum Hall effect [2,26,30], kicked quantum systems [22,34], and the spectacular solution of the Ten Martini problem [1]. Brabanter [4] and Yin [38] classified rational rotation C^*-algebras up to $*$-isomorphism. Stacey [31] constructed their automorphism groups. They used methods known to experts: cocycles, crossed products, Dixmier-Douady classes, ergodic actions, K–theory, and Morita equivalence. This expository paper defines $\mathcal{A}_{p/q}$ as a C^*-algebra generated by two operators on a Hilbert space and uses linear algebra, Fourier series and the Gelfand–Naimark–Segal construction [16] to prove its universality. It then represents it as the algebra of sections of a matrix algebra bundle over a torus to compute its isomorphism class. The remarks section relates these concepts to general operator algebra theory. We write for mathematicians who are not C^*-algebra experts.

Keywords: bundle topology, Gelfand–Naimark–Segal construction, irreducible representation, spectral decomposition.

Citation: W.M. Lawton1, Tutorial on Rational Rotation C^*–Algebras, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 598–609. DOI: 10.17516/1997-1397-2022-15-5-598-609.

1. Uniqueness of universal rational rotation C^*-algebras

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T} \subset \mathbb{C}$ denote the sets of positive integer, integer, rational, real, complex and unit circle numbers. For a Hilbert space H let $\mathcal{B}(H)$ be the C^*-algebra of bounded operators on H. All homomorphisms are assumed to be continuous. We assume famliarity with the material in Section 4.

Fix $p, q \in \mathbb{N}$ with $p \leqslant q-1$ and $\gcd(p,q) = 1$, define $\sigma := e^{2\pi i/q}$ and $\omega := \sigma^p$, and let $\mathfrak{C}_{p/q}$ be the set of all C^*-algebras generated by a set $\{U, V\} \subset \mathcal{B}(H)$ satisfying $UV = \omega VU$. Since $\{U, V\} = \{V, U\}$, $\mathfrak{C}_{(q-p)/q} = \mathfrak{C}_{(q-p)/q}$. M_q and the circle subalgebra of $L^2(\mathbb{T})$ generated by $(Uf)(z) := zf(z)$ and $(Vf)(z) := f(\omega z)$ belong to $\mathfrak{C}_{(q-p)/q}$. The circle algebra is isomorpic to the tensor product $C(\mathbb{T}) \otimes M_q$.

Definition 1. *$\mathcal{A} \in \mathfrak{C}_{p/q}$ generated by $\{U, V\} \subset \mathcal{B}(H)$ satisfying $UV = \omega VU$ is called universal if for every $\mathcal{A}_1 \in \mathfrak{C}_{p/q}$ generated by $\{U_1, V_1\} \subset \mathcal{B}(H_1)$ satisfying $U_1V_1 = \omega V_1U_1$, there exists a $*$-homomorphism $\Psi : \mathcal{A} \mapsto \mathcal{A}_1$ satisfying $\Psi(U) = U_1$ and $\Psi(V) = V_1$.*

Lemma 1. *If $\mathcal{A}, \mathcal{A}_1 \in \mathfrak{C}_{p/q}$ are both universal, then they are isomorphic.*

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Proof. Let U, V, U_1, V_1 be as in Definition 1. There exists $*$-homomorphisms $\Psi : \mathcal{A} \mapsto \mathcal{A}_1$ and $\Psi_1 : \mathcal{A}_1 \mapsto \mathcal{A}$ with $\Psi_1 \circ \Psi(U) = U,\ \Psi_1 \circ \Psi(V) = V,\ \Psi \circ \Psi_1(U_1) = U_1,\ \Psi \circ \Psi_1(V_1) = V_1$. Since $\{U, V\}$ generates $\mathcal{A}$, $\Psi_1 \circ \Psi$ is the identity map on $\mathcal{A}$. Similarly, $\Psi \circ \Psi_1$ is the identity map on $\mathcal{A}_1$. Therefore Ψ is a $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$ and A is $*$-isomorphic to A_1. □

2. Construction of universal rational rotation C^*-algebras

Define the Hilbert space $H_q := L^2(\mathbb{R}^2, \mathbb{C}^q)$ consisting of Lebesgue measurable $v : \mathbb{R}^2 \mapsto \mathbb{C}^q$ satisfying $\int_{\mathbb{R}^2} v^* v < \infty$, equipped with the scalar product $< v, w > := \int_{\mathbb{R}^2} w^* v$. Define $\mathcal{P}_q$ to be the subset of continuous $a : \mathbb{R}^2 \mapsto \mathcal{M}_q$ satisfying

$$a(x_1, x_2) = a(x_1 + q, x_2) = a(x_1, x_2 + q),\ (x_1, x_2) \in \mathbb{R}^2 \tag{1}$$

and regarded as a C^*-subalgebra of $\mathcal{B}(H_q)$ acting by $(av)(x) := a(x)v(x),\quad a \in \mathcal{P}_q,\ v \in H_q$. The operator norm of $a \in \mathcal{P}_q$ satisfies

$$||a|| = \max_{x \in [0,q]^2} ||a(x)||. \tag{2}$$

Define $U, V \in \mathcal{P}_q$ by

$$U(x_1, x_2) := e^{2\pi i x_1/q} U_0, \quad V(x_1, x_2) := e^{2\pi i x_2/q} V_0, \tag{3}$$

where $U_0, V_0 \in \mathcal{M}_q$ are defined by (7), and define $\mathcal{A}_{p/q}$ to be the C^*-subalgebra of $\mathcal{P}_q$ generated by $\{U, V\}$. Choose $r \in \{1, \ldots, q-1\}$ such that $pr = 1 \mod q$. Then r is unique, $\gcd(r, q) = 1$. Define $\sigma := e^{2\pi i/q}$ and $\omega := \omega^p$. Then $\omega^r = \sigma$.

Theorem 1. *If* $a \in \mathcal{A}_{p/q}$ *then*

$$a(x_1 + 1, x_2) = V_0^{-r} a(x_1, x_2) V_0^r \ \text{ and } \ a(x_1, x_2 + 1) = U_0^r a(x_1, x_2) U_0^{-1}. \tag{4}$$

Conversely, if $a \in \mathcal{P}_q$ *satisfies (4), then* $a \in \mathcal{A}_{p/q}$.

Proof. (3) and (8) give $V^{-r} U V^r = \sigma U$ and $U^r V U^{-r} = \sigma V$. If $a = U^m V^n$, then

$$a(x_1 + 1, x_2) = \sigma^m a(x_1, x_2) = V_0^{-r} a(x_1, x_2) V_0^r;\ a(x_1, x_2 + 1) = \sigma^n a(x_1, x_2) = U_0^r a(x_1, x_2) U_0^{-r}.$$

The first assertion follows since $\operatorname{span}\{U^m V^n : m, n \in \mathbb{Z}\}$ is dense in $\mathcal{A}_{p/q}$. Conversely, if $a \in \mathcal{P}_q$, then (1), Lemma 3, and Weierstrass' approximation theorem implies that there exist unique $c(m, n, j, k) \in \mathbb{C}$ with

$$a(x_1, x_2) \sim \sum_{(m,n) \in \mathbb{Z}^2} \sum_{j,k=0}^{q-1} c(m, n, j, k)\, e^{2\pi i (m x_1 + n x_2)/q}\, U_0^j V_0^k$$

where $\sim$ denotes Fourier series. Then (4) gives $c\,(m, n, j, k)\,\sigma^m = c\,(m, n, j, k)\,\sigma^j$ and $c(m, n, j, k)\sigma^n = c(m, n, j, k)\sigma^k$. Since $\sigma^q = 1$, $c(m, n, j, k) = 0$ unless $j = m \mod q$and $k = n \mod q$. Define $c(m, n) := c(m, n, m \mod q, n \mod q)$. Then $a \in A_{p/q}$ since

$$a \sim \sum_{(m,n) \in \mathbb{Z}^2} c(m, n) U^m V^n.$$

Representations $\rho_1, \rho_1 : \mathcal{A} \mapsto \mathcal{B}(H)$ of a C^*-algebra $\mathcal{A}$ are unitarily equivalent if there exists $U \in \mathcal{U}(H)$ such that $\rho_2(a) = U \rho_1(a) U^{-1},\ a \in \mathcal{A}$. □

Theorem 2. *If $\mathcal{A} \in \mathfrak{C}_{p/q}$ is generated by $\{U, V\}$ with $UV = \omega VU$ and $\rho : \mathcal{A} \to \mathcal{B}(H)$ is an irreducible representation then:*

1. *dim $H = q$ so $B(H) = \mathcal{M}_q$,*
2. *there exist $z_1, z_2 \in \mathbb{T}$ such that $\rho = \rho_{z_1,z_2}$ where $\rho_{z_1,z_2}(U^jV^k) := z_1^j z_2^k U_0^j V_0^k$.*
3. *$\rho_{z_1',z_2'}$ is unitarity equivalent to ρ_{z_1,z_2} iff $(z_1'/z_1)^q = (z_2'/z_2)^q = 1$.*

Proof. Boca gives a proof in ([1], p. 5, Lemma 1.8, p. 7, Theorem 1.9). We give a proof based on Schur's lemma. Let $\mathcal{C} \subset \mathcal{A}$ be the C^*-subalgebra generated by $\{U^q, V^q\}$. Since ρ is irreducible and $\rho(\mathcal{C})$ commmutes with $\rho(\mathcal{A})$, there exists a $*$-homomorphism $\gamma : \mathcal{C} \mapsto \mathbb{C}$ such that $\rho(c) = \gamma(c)I$, $c \in \mathcal{C}$. Choose $h \in H\backslash\{0\}$ and define $H_1 := \text{span}\,\{\rho(U^jV^k)h; 0 \leqslant j,k \leqslant q-1\}$. Since H_1 is closed, ρ-invariant, $H_1 \neq \{0\}$, and ρ is irreducible, $H = H_1$. Since $\dim H \leqslant q^2$, $\rho(V)$ has an eigenvector b with eigenvalue $\lambda \in \mathbb{T}$ and $||b|| = 1$. Define $z_2 := \lambda\omega$. Choose $z_1 \in \mathbb{T}$ so $z_1^q = \gamma(U^q)$ and define $b_j := z_1^j \rho(U^{-j})b$, $1 \leqslant j \leqslant q$. Then $\rho(V)b_j = z_2\omega^{j-1}b_j$, $j = 1,\ldots,q$, and $\rho(U)b_1 = z_1 b_q$, and $\rho(U)b_j = z_1 b_{j-1}$, $2 \leqslant j \leqslant q$. Therefore $\{b_1,\ldots,b_q\}$ is a basis for H, and (7) implies that $\rho(U) = z_1U_0$, and $\rho(V) = z_2V_0$ with respect to this basis. This proves assertions 1 and 2. Assertion 3 follows since the set of eigenvalues of $\rho(U)$ is $\{z_1\omega^j, 0 \leqslant j \leqslant q-1\}$, the set of eigenvalues of $\rho(V)$ is $\{z_2\omega^j, 0 \leqslant j \leqslant q-1\}$, and the set of eigenvalues determines unitary equivalence. □

Theorem 3. *$\mathcal{A}_{p/q} \subset \mathcal{B}(H)$ is the universal C^*-algebra in $\mathfrak{C}_{p/q}$.*

Proof. Assume that $\mathcal{B} \in \mathfrak{C}_{p/q}$. Then there exists a Hilbert space H_1 and $U_1, V_1 \in \mathcal{B}(H_1)$ with $U_1V_1 = \omega V_1U_1$ and $\mathcal{B}$ is generated by $\{U_1, V_1\}$. It suffices to construct a continuous $*$-homomorphism $\varphi : \mathcal{A}_{p/q} \mapsto \mathcal{B}$ satisfying $\varphi(U) = U_1$ and $\varphi(V) = V_1$. Define dense $*$-subalgebras

$$\widetilde{\mathcal{A}_{p/q}} := \text{ span } \{U^jV^k : j,k \in \mathbb{Z}\} \subset \mathcal{A}_{p/q}, \quad \widetilde{\mathcal{B}} := \text{ span } \{U_1^jV_1^k : j,k \in \mathbb{Z}\} \subset \mathcal{B},$$

and a $*$-homomorphism $\widetilde{\varphi} : \widetilde{\mathcal{A}_{p/q}} \mapsto \widetilde{\mathcal{B}}$ by $\widetilde{\varphi}(U^jV^k) := U_1^jV_1^k$. To extend $\widetilde{\varphi}$ to $*$-homomorphism $\varphi : \mathcal{A}_{p/q} \mapsto \mathcal{B}$ it suffices to show that for every Laurent polynomial of two variables $p(u,v)$ the following inequality is satisfied $||p(U_1,V_1)|| \leqslant ||p(U,V)||$ since $p(U_1V_1) = \widetilde{\varphi}(p(U,V))$. Then ([13], Corollary I.9.11), which follows directly from the Gelfand-Naimark-Segal construction, implies that there exists an irreducible representation $\rho_1 : \mathcal{B} \mapsto \mathcal{M}_q$ and $v \in H_1$ with $||v|| = 1$ such that $||p(U_1,V_1)|| = ||\rho_1(p(U_1,V_1))v||$. Theorem 2 implies that $\rho_1(U_1) = z_1U_0$ and $\rho_1(V_1) = z_2V_0$ for some $z_1, z_2 \in \mathbb{T}$. Let $\rho : \mathcal{A}_{p/q} \mapsto \mathcal{M}_q$ be the irreducible representation defined by Theorem 2 so $\rho(U) = z_1U_0$ and $\rho(V) = z_2V_0$. Since $\rho_1 \circ \widetilde{\varphi} =$ the restriction of ρ to $\widetilde{\mathcal{A}_{p/q}}$, (2) and (3) imply that

$$||p(U_1,V_1)|| = ||\rho_1(p(U_1,V_1))v|| \leqslant ||\rho(p(U,V))|| \leqslant ||p(U,V)||$$

which concludes the proof. □

3. Bundle topology and isomorphism classes

Define $\mathbb{E}_1$ to be the Cartesian product $[0,1]^2 \times \mathcal{M}_q$ with the identification

$$(1, x_2, M) = (0, x_2, V_0^{-r}MV_0^r), \quad x_2 \in [0,1], \; M \in \mathcal{M}_q$$

and

$$(x_1, 1, M) = (x_1, 0, U_0^rMU_0^{-r}), \quad x_1 \in [0,1], \; M \in \mathcal{M}_q$$

and define the algebra bundle $\pi_1 : \mathbb{E}_1 \mapsto \mathbb{T}^2$ by

$$\pi_1(x_1, x_2, M) = (e^{2\pi i x_1}, e^{2\pi i x_2}), \quad (x_1, x_2, M) \in \mathbb{E}_1.$$

A map $s : \mathbb{T}^2 \mapsto \mathbb{E}_1$ is called a section if it is continuous and $\pi_1 \circ s = I$ where I denotes the identity map on $\mathbb{T}^2$. Since for every $p \in \mathbb{T}^2$, the fiber $\pi_1^{-1}(p) = \mathcal{M}_q$, the set of sections under pointwise operations is a C^*-algebra. The theorems above show that this algebra is isomorphic to $\mathcal{A}_{p/q}$. Furthermore, since points in $\mathbb{T}^2$ correspond to unitary equivalence classes of irreducible representations, isomorphism of algebras induces homeomorphisms of $\mathbb{T}^2$. In order to compute isomorphism classes of universal rational rotation C^*-algebras it is convenient to use a slightly different bundle representation of $\mathcal{A}_{p/q}$. Define $W \in \mathcal{P}_q$

$$W(x_1, x_2) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i x_1/q} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{2\pi i (q-1) x_1/q} \end{bmatrix}$$

and $\mathcal{A}'_{p/q} := W \mathcal{A}_{p/q} W^{-1}$, which is $*$-isomorphic to $\mathcal{A}_{p/q}$. Then $\mathcal{A}'_{p/q}$ is represented as the algebra of sections of the algebra bundle $\pi_2 : \mathbb{E}_2 \mapsto \mathbb{T}^2$ where $\mathbb{E}_2$ is the Cartesian product $\mathbb{T} \times [0,1] \times \mathcal{M}_q$ with the identification

$$(z_1, 1, M) = (z_1, 0, G^r M G^{-r}), \quad z_1 \in \mathbb{T}, \ M \in \mathcal{M}_q$$

and

$$G(z_1) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z_1 \end{bmatrix} U_0.$$

G^r is the clutching function of the bundle. Let $G_{in} : \mathbb{T} \mapsto Aut_q^*$ be the map defined by conjugation by G. Using the arguments for vector bundles in [18], it can be shown that the isomorphism classe of $\mathcal{A}_{p/q}$ is determined the homotopy class of $G_{in}^r : \mathbb{T} \mapsto Aut_q^*$. Since $\pi_1(G_{in}) = -1$, $\pi_1(G_{in}^r) = -r$ which gives:

Theorem 4. *$\mathcal{A}_{p/q}$ is isomorphic to $\mathcal{A}_{p'/q'}$ iff $q' = q$ and either $p' = p$ or $p' = q - p$.*

4. Requisite results

4.1. Hilbert spaces and adjoints

H is a Hilbert space with inner product $< \cdot, \cdot >: H \times H \mapsto \mathbb{C}$, norm $||v|| := \sqrt{< v, v >}$, and metric $d : H \times H \to [0, \infty)$ defined by $d(v, w) := ||v - w||$. $\mathcal{B}(H)$ is the Banach algebra of bounded operators on H (continuous linear maps from H to H) with operator norm

$$||a|| := \sup\{ \, ||av|| \, : \, v \in H, \, ||v|| = 1 \, \}.$$

The dual space H^* is the set of continuous linear functions $L : H \mapsto \mathbb{C}$. For $w \in H$ define $L_w \in H^*$ by $L_w v := < v, w >, \quad v \in H$.

Lemma 2. *If $L \in H^*$ then there exists a unique $w \in H$ such that $L = L_w$.*

Proof. Rudin gives a direct proof ([28], Theorem 4.12). If $\mathfrak{B}$ is an orthonormal basis for H and $w := \sum_{b\in\mathfrak{B}} \overline{Lb}\, b$, then since for every $v \in H$, $v = \sum_{b\in\mathfrak{B}} < v, b > b$, it follows that

$$Lv = \sum_{b\in\mathfrak{B}} < v, b > \ Lb = \left\langle v, \sum_{b\in\mathfrak{B}} \overline{Lb}\ b \right\rangle = < v, w > = L_w\, v.$$

Lemma 2 ensures the existence of adjoints. For $a \in \mathcal{B}(H)$ define its adjoint $a^* \in \mathcal{B}(H)$ by $L_{a^*w} := L_w \circ a, \ \ w \in H$ where $\circ$ denotes composition of functions. Therefore

$$< a\ v, w > = < v, a^* w >, \quad v, w \in H.$$

Clearly $a^{**} = a$, $(ab)^* = b^*a^*$, and the Cauchy–Schwarz inequality gives

$$||a^*|| = \sup\{\, | < a^*v, w > | : v, w \in H,\ ||v|| = ||w|| = 1\} =$$

$$= \sup\{\, | < v, aw > | : v, w \in H,\ ||v|| = ||w|| = 1\} = ||a||$$

and

$$\begin{aligned} ||a^*a|| &= \sup\{\, | < a^*av, w > | : v, w \in H,\ ||v|| = ||w|| = 1\} = \\ &= \sup\{\, | < av, aw > | : v, w \in H,\ ||v|| = ||w|| = 1\} = ||a||^2. \end{aligned} \tag{5}$$

(5) is called the C^*-identity. It makes $\mathcal{B}(H)$ equipped with the adjoint a C^*-algebra. The identity operator $I \in \mathcal{B}(H)$ is defined by $Iv := v$ for all $v \in H$.

$$\mathcal{U}(H) := \{U \in \mathcal{B}(H) : UU^* = U^*U = I\},$$

the set of unitary operators, is a group under multiplication. A subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is a C^*-algebra if it is closed in the metric space topology on $\mathcal{B}(H)$ and $a^* \in \mathcal{A}$ whenever $a \in \mathcal{A}$. The intersection of any nonempty collection of C^*-subalegras of $\mathbb{B}(H)$ is a C^*-algebra. If $S \subset \mathcal{B}(H)$ the intersection of all C^*-subalgebras of $\mathcal{B}(H)$ that contain S is the C^*-algebra generated by S. $\square$

4.2. Matrix algebras

For $m, n \in \mathbb{N}$, $\mathbb{C}^{m\times n}$ denotes the set of m by n matrices with complex entries and $\mathbb{C}^n := \mathbb{C}^{n\times 1}$. The adjoint of $a \in \mathbb{C}^{m\times n}$ is the matrix $a^* \in \mathbb{C}^{n\times m}$ defined by $a^*_{j,k} := \overline{a_{k,j}}$. $\mathbb{C}^n$ is a Hilbert space with scalar product $< v, w > := \ w^*v, \ \ v, w \in \mathbb{C}^n$. Clearly

$$\mathcal{B}(\mathbb{C}^n) = \mathcal{M}_n$$

where for $a \in \mathcal{M}_n$ the adjoint of a as an operator corresponds to the adjoint of a as a matrix. I_n denotes the n by n identity matrix whose diagonal entries equal 1 and other entries equal 0. The operator norm of $a \in \mathcal{M}_n$ is $||a|| = \sqrt{\text{spectral radius } a^*a}$ where the spectral radius is the largest moduli of the eigenvalues of a matrix. Thus $\mathcal{M}_n$ is a C^*-algebra. It is also a Hilbert space a Hilbert space of dimension n^2 with inner product

$$< a, b > := \text{Trace } b^*a \tag{6}$$

and orthonormal basis $e_{j,k}$:= matrix with 1 in row j and column j with all other enties = 0. Fix $p, q \in \mathbb{N}$ with $p \leqslant q-1$ and $\gcd(p,q) = 1$. Define $U_0, V_0 \in \mathcal{M}_q$ by

$$U_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad V_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega^{q-1} \end{bmatrix}, \tag{7}$$

Lemma 3. *$\{(1/\sqrt{q})U_0^j V_0^k : 0 \leqslant j, k \leqslant q-1\}$ is an orthonormal basis for $\mathcal{M}_q$ with the scalar product defined by (6). Furthermore,*

$$U_0 V_o = \omega V_0 U_0. \tag{8}$$

Proof. (8) is obvious. The first assertion follows since

$$< U_0^j V_0^k, U_0^m V_0^n > \text{ Trace } V_0^{-n} U_0^{-m} U_0^j V_0^k = \text{Trace } U_0^{j-m} V_0^{k-n} = \begin{cases} q \text{ if } j = m \text{ and } k = n, \\ 0 \text{ otherwise.} \end{cases}$$

Define the groups of unitary matrices $\mathcal{U}_n := \mathcal{U}(\mathbb{C}^n)$ and special unitary matrices $\mathcal{S}_n := \{a \in \mathcal{U}_n : \det a = 1\}$. Clearly U_0 and V_0 are unitary. Since $\det U_0 = \det V_0 = (-1)^{q-1}$, they are special unitary iff q is odd. A map $\psi : \mathcal{M}_n \mapsto \mathcal{M}_n$ is a homomorphism if it is linear and satisfies $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathcal{M}_n$ and an automorphism if is also a bijective. An automorphism ψ is a $*$-automorpism if $\psi(b^*) = \psi(b)^*$ for all $b \in \mathcal{M}_n$. Aut_n, Aut_n^* denote the group of all automorpisms, $*$-automorphisms of $\mathcal{M}_n$. $\psi \in Aut_n$ is called inner if there exists an invertible $a \in \mathcal{M}_n$ such that $\psi(b) = aba^{-1}$ for every $b \in \mathcal{M}_n$.

Theorem 5 (Skolem–Noether)**.** *Every $\psi \in Aut_n$ is inner.*

Proof. The algebra $\mathcal{M}_n$ is simple, meaning it has no two-sided ideals othe that itself ([29], 11.41), so the result follows from the classic Skolem–Noether theorem. An elementary constructive proof is given in [32]. □

Theorem 6. *If $\psi \in Aut_n^*$ then there exists $a \in \mathcal{U}_n$ such that $\psi(b) = aba^*$ for every $b \in \mathcal{M}_n$.*

Proof. Every $\psi \in Aut_n^*$ induces an irreducible representation $\psi : \mathcal{M}_n \to \mathcal{B}(\mathbb{C}^n)$ so Theorem 2 implies that there exists a basis $\{b_1, \ldots, b_n\}$ with respect to which $\psi(U_0)$ has the matrix representation $z_1 U_0$ and $\psi(V_0)$ has the representation $z_2 V_0$. Since $U_0^n = V_0^n = I$, $z_1^n = z_2^n = 1$ so without loss of generality this basis can be chosen to make $z_1 = z_2 = 1$ and then $\psi(a) = aba^{-1}{-1}$ where $ae_j = b_j$ and $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. This theorem can also be derived as a corollary of of Theorem 5. Clearly $\psi(I_n) = I_n$. Theorem 5 implies that there exists an invertible $a \in \mathcal{M}_n$ such that $\psi(b) = aba^{-1}$ for all $b \in \mathcal{M}_n$. Since ψ is a $*$-homomorphism $ab^*a^{-1} = (aba^{-1})^* = (a^{-1})^* b^* a^*$ hence $a^*ab^* = b^*a^*a$ for every $b \in \mathcal{M}_n$ which implies that $a^*a = c\, I_n$ for some $c > 0$. Replacing a by $a/\sqrt{c}$ gives the conclusion.

Corollary 1. *Let $\mathbb{T}_n \subset \mathbb{T}$ be the subgroup of n-th roots of unity. $\mathbb{T}_n\, I_n \subset \mathcal{S}_n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Aut_n^* is isomorphic to the quotient group $\mathcal{S}_n / \mathbb{T}_n\, I_n$. The fundamental group $\pi_1(Aut_n^*)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Proof. Assertion one is obvious. Define $\zeta : \mathcal{U}_n \mapsto Aut_n^*$ by $\zeta(a)(b) := aba^*$. ζ is a $*$-homomorphism, kernel $\zeta = \mathbb{T}\, I_n$, and Corollary 1 implies that ζ is onto. The first homomorphism theorem of group theory ([33], 7.2) implies that Aut_n^* is isomoprhic to $\mathcal{U}_n / \mathbb{T}\, I_n$. Since

$\mathcal{S}_n = (\mathbb{T}I_n)((\mathbb{T}_n I_n)$ and $\mathbb{T}_n\, I_n = \mathcal{S}_n \cap (\mathbb{T}I_n)$, the second isomorphism theorem of group theory ([33], 7.3) implies that Aut_n^* is isomporphic to $\mathcal{S}_n/\mathbb{T}_n I_n$. $\mathcal{S}_n$ is simply connected ([17], Proposition 13.11) hence since $\mathbb{T}_n I_n$ is discrete $\mathcal{S}_n$ is the univeral cover of $\mathcal{S}_n \cap (\mathbb{T}I_n)$ hence the discussion in ([18], 1.3) implies the last assertion.

4.3. Spectral Decomposition Theorem for Unitary Operators

$E \in \mathcal{B}(H)$ is called a projection if $E^* = E$ and $E^2 = E$. Then $P : H \mapsto PH$ is orthogonal projection. A collection of projections $\{E_\varphi : \varphi \in [0, 2\pi]$ is called a spectral family if $E_{\varphi_1} E_{\varphi_2} = E_{\varphi_2} E_{\varphi_1} = E_{\varphi_1}$ whenver $\varphi_1 \leqslant E_{\varphi 2}$.

Let $A, P, N \in \mathcal{B}(H)$. A is self–adjoint if $A^* = A$. $P \in \mathcal{B}(H)$ is positive if $< Pv, v > \geqslant 0$ for all $v \in H$. $N \in \mathcal{B}(H)$ is called normal if $AA^* = A^*A$. Clearly self-adjoint and unitary operators (or transformations) are normal. Furthermore eigenvalues of self–adjoint operators are real and eigenvalues of unitary operators have modulus 1. If dim $H < \infty$, then H admits an orthonormal basis of eigenvectors ([29], Theorem 9.33). Therefore every unitary matrix in $\mathcal{M}_n$ can be diagonalized and its diagonal entries have modulus 1. The following result, copied verbatim from the classic textbook by F. Riesz and B. Sz.-Nagy ([27], p. 281), extends this diagonalization to unitary operators on arbitrary Hilbert spaces.

Theorem 7. *Every unitary transformation U has a spectral decomposition*

$$U = \int_{-0}^{2\pi} e^{i\varphi} dE_\varphi,$$

where $\{E_\varphi\}$ is a spectral family over the segmen $0 \leqslant \varphi \leqslant 2\pi$. We can require that E_φ be continuous at the point $\varphi = 0$, that is, $E_0 = 0$; $\{E_\varphi\}$ will then be determined uniquely by U. Moreover, E_φ is the limit of a sequence of polynomials in U and U^{-1}.

Proof. The authors of [27] reference 1929 papers by von Neumann [25] and Wintner [37], 1935 papers by Friedricks and Wecken, and a 1932 book by Stone. They observe that the theorem can be deduced from the one on symmetric transformation ([27], p. 280) (since $U = A + iB$ where $A := (U + U^*)/2$ and $B := -i(U - U^*)/2$ are symmetric) or from the theorem on trigonometric moments ([27], Section 53), but they give a direct three page proof. We sketch their proof. For every trigomometric polynomial $p(e^{i\varphi}) = \sum_{-n}^{n} c_k e^{ik\varphi}$ we associate the transformation $p(U) := \sum_{-n}^{n} c_k U^k$. This gives a $*$-homomorphism of the algebra of trigonometric polynomials (where $*$ means complex conjugation) into the subalgebra of $\mathcal{B}(H)$ generated by U and $U^* = U^{-1}$. Clearly if $p(e^{i\varphi})$ is real–valued then $p(U)$ is self-adjoint. If $p(e^{i\varphi}) \geqslant 0$ the Riesz–Fejer factorization Lemma ([27], Section 53) implies that there exists a trigonometric polynomial $q(e^{i\varphi})$ with $p(e^{i\varphi}) = q(e^{i\varphi})\overline{q(e^{i\varphi})}$ hence $p(U) = q(U)q(U)^*$. Therefore $< p(U)v, v >=< q(U)v, q(U) > \geqslant 0, \quad v \in H$, hence $p(U)$ is a positive operator. For $0 \leqslant \psi \leqslant 2\pi$ let e_ψ be the characteristic function of $(0, \psi]$ extended to a 2π periodic function on $\mathbb{R}$. Let p_n be a monotonically sequence of positive trigonometric functions with $\lim_{n\to\infty} p_n(U)v = E_\psi v, \quad v \in H$ ($p_n(U)$ converges to $E_\psi \in \mathbb{B}(H)$ in the strong operator topology). E_ψ is a projection since $E_\psi^* = E_\psi$ and $E_\psi^2 = E_\psi$, so , and the set $\{E_\varphi : \varphi \in [0, 2\pi]$ is a spectral family. Since the functions e_ψ are upper semi–continuous $\lim_{\chi\to\psi,\chi>\psi} E_\chi = E_\psi$.

Given $\epsilon > 0$ choose $0 < \psi_0 < \psi_1 < \cdots < \psi_n = 2\pi$ with max $(\psi_{k+1} - \psi_k) \leqslant \epsilon$ and choose $\varphi_k \in [\psi_{k-1}, \psi_k]$, $k = 1, \ldots, n$. Then for $\varphi \in (\psi_{k-1}, \psi_k]$

$$\left| e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k} \left[e_{\psi_k} - e_{\psi_{k-1}} \right] \right| = |e^{i\varphi} - e^{i\varphi_k})| \leqslant |\varphi - \varphi_k| \leqslant \epsilon$$

with a similar inequality for $\varphi = 0$. Since this inequality holds for all $\varphi \in [0, 2\pi]$

$$\left\| U - \sum_{k=1}^{n} e^{i\varphi_k} \left(E_{\psi_k} - E_{\psi_{k-1}} \right) \right\| \leqslant \epsilon$$

A subspace $H_1 \subset H$ is called proper if $H_1 \neq \{0\}$ and $H_1 \neq H$. The following is an immediate consequence of Theorem 7

Corollary 2. *If $U \in \mathcal{U}(H)$ then either $U = \gamma I$ for some $\gamma \in \mathbb{T}$ or there exists a projection operator $E : H \mapsto H$ satisfying*

1. *E is the limit in the strong operator topology on $\mathcal{B}(H)$ of polynomials $p(U, U^{-1})$.*
2. *EH is a proper closed U-invariant subspace of H.*

4.4. Irreducible representations and Schur's lemma

A representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is a *-homomorphism $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$. A subspace $H_1 \subset H$ is called ρ-invariant if $\rho(a)H_1 \subset H_1$ for every $a \in \mathcal{A}$. ρ is irreducible iff it H has no closed proper ρ-invariant subspaces. The following result extends Shur's lemma for finite dimensional representations ([29], 11.33) for unitary operators.

Theorem 8 (Schur's Lemma)**.** *If $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$ is an irreducible representation and $U \in \mathcal{U}(H)$ commutes with $\rho(a)$ for every $a \in \mathcal{A}$, then there exists $\gamma \in \mathbb{T}$ with $U = \gamma I$.*

Proof. If the conclusion does not hold then Corollary 2 implies that there exists a projection E satisfying conditions (1) and (2). Condition (1) implies that $U\rho(a) = \rho(a)U$ for every $a \in \mathcal{A}$. The $EH \subset H$ is closed and ρ-invariant since for every $a \in \mathcal{A}$, $\rho(a)EH = E\rho(a)H$. Condition (2) asserts that EH is a proper subspace thus contradicting the hypothesis that ρ is irreducible, and concluding the proof.

Corollary 3. *If $\mathcal{A}$ is an commutative C^*-algebra generated by a set of unitary operators and $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$ is irreducible then dim $H = 1$ and there exists a $*$-homomorphism $\Gamma : \mathcal{A} \to \mathcal{C}$ with $\rho(a) = \Gamma(a)I$ for every $a \in \mathcal{A}$.*

Proof. Follows from from Theorem 8 since if $u \in \mathcal{A}$ is unitary the $\rho(u) \in \mathcal{U}(H)$ and $\rho(u)$ commutes with $\rho(a)$ for every $a \in \mathcal{A}$.

5. Remarks

We relate concepts introduced to explain rational rotation algebras to general C^*-algebra theory, especially two breakthrough results obtained by teams of computer scientists.

Remark 1. Launs [14] initiated a program to represent C^*-algebras by continuous sections over bundles over their primitive ideal spaces (kernels of irreducible representations equipped with the hull-kernel topology). The primitive ideal space of rational rotation algebras is homemorpic to the torus $\mathbb{T}^2$.

Remark 2. Bratteli, Elliot, Evans, and Kishimoto [5] represent fixed point C^*-subalgebras of $\mathcal{A}_{p/q}$ by algebras of sections of M_q-algebra bundles over the sphere S^2, which is the space of orbits of $\mathbb{T}^2$ under the map $g \mapsto g^{-1}$.

Remark 3. Elliot and Evans [15] derived the structure of irrational rotation algebras. They proved that if $p/q < \theta < p'/q'$, then $\mathcal{A}_\theta$ can be approximated by a C^*-subalgebra isomorphic to $C(\mathbb{T}) \otimes M_q \oplus C(\mathbb{T}) \otimes M_{q'}$. This approximation, combined with the continued fraction expansion of θ, represents $\mathcal{A}_\theta$ as an inductive limit of these subalgebras.

Remark 4. Williams [36] gives an extensive explanation of crossed product C^*-algebras, which include rotation algebras.

Remark 5. Kadison and Singer [21] formulated a problem about extending pure states. Such an extension is used in the Gelfand–Naimark–Segal construction which we used to prove Theorem 3. This problem was shown to be equivalent to numerous problems in functional analysis and signal processing [6], dynamical systems [23, 24], and other fields [3]. Weaver [35] gave a discrepancy–theoretic formulation that was proved in a seminal paper by three computer scientists: Marcus, Spielman, and Shrivastava [19].

Remark 6. Courtney [8, 9] proved that the class of residually finite dimensional C^*-algebras, those whose structure can be recovered from their finite dimensional representations, coincides with the class of algebras containing a dense set of elements that attain their norm under a finite dimensional representation, this set is the full algebra iff every irreducible representation is finite dimensional (as for rational rotation algebras), and related these concepts to Conne's embedding conjecture [7]. Her publications [10–12] cite many references that discuss equivalent formulations of this conjecture.

Remark 7. In January 2020 five computer scientists: Ji, Natarajan, Vidick, Wright and Yuen submitted a proof that the Conne's embedding conjecture is false. As of November 2021 their paper is still under peer review. However, the editors of the ACM decided, based on the enormous interest that their paper attracted, to publish it [20].

Acknowledgments. We thank Paolo Bertozzini, Kristin Courtney, Sϕren Eilers, Vern Paulsen, Beth Ruskai, David Yost, and Saeid Zahmatkesh for sharing their insights.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of Regional Centers for Mathematics Research and Education (Agreement no. 075-02-2020-1534/1).

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Учебник по рациональному вращению C^*-Алгебры

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Аннотация. Алгебра вращений $\mathcal{A}_\theta$ — это универсальная C^*-алгебра, порожденная унитарными операторами U, V, удовлетворяющими коммутационному соотношению $UV = \omega VU$, где $\omega = e^{2\pi i\theta}$. Они рациональны, если $\theta = p/q$ с $1 \leqslant p \leqslant q-1$, в противном случае иррациональны. Операторы в этих алгебрах связаны с квантовым эффектом Холла [2, 26, 30], квантовыми системами [22, 34] и эффектным решением проблемы Тена Мартини [1]. Брабантер [4] и Инь [38] классифицировали C^*-алгебры рационального вращения с точностью до $*$-изоморфизма. Стейси [31] построила свои группы автоморфизмов. Они использовали известные специалистам методы: коциклы, скрещенные произведения, классы Диксмье-Дуади, эргодические действия, К-теорию и эквивалентность Мориты. Эта пояснительная статья определяет $\mathcal{A}_{p/q}$ как C^*-алгебру, порожденную двумя операторами в гильбертовом пространстве, и использует линейную алгебру, ряды Фурье и конструкцию Гельфанда–Наймарка–Сигала [16] для доказательства его универсальности. Затем он представляет его как алгебру сечений расслоения матричной алгебры над тором для вычисления его класса изоморфизма. Раздел примечаний связывает эти концепции с общей теорией операторной алгебры. Мы пишем для математиков, не являющихся экспертами в C^*-алгебре.

Ключевые слова: топология расслоения, конструкция Гельфанда–Наймарка–Сигала, неприводимое представление, спектральное разложение.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 610–614

DOI: 10.17516/1997-1397-2022-15-5-610-614
УДК 512.54

Irreducible Carpets of Additive Subgroups of Type G_2 Over a Field of Characteristic 0

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Received 08.02.2022, received in revised form 23.04.2022, accepted 27.06.2022

Abstract. It is proved that any irreducible carpet of type G_2 over a field F of characteristic 0, at least one additive subgroup of which is an R-module, where F is an algebraic extension of the field R, up to conjugation by a diagonal element defines a Chevalley group of type G_2 over an intermediate subfield between R and F.

Keywords: Chevalley group, carpet of additive subgroups, carpet subgroup.

Citation: Ya.N. Nuzhin, E.N. Troyanskaya, Irreducible Carpets of Additive Subgroups of Type G_2 Over a Field of Characteristic 0, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 610–614.
DOI: 10.17516/1997-1397-2022-15-5-610-614.

1. Introduction

Let Φ be a reduced indecomposable root system, $\Phi(F)$ be a Chevalley group of type Φ over the field F generated by the root subgroups

$$x_r(F) = \{x_r(t) \mid t \in F\}, \quad r \in \Phi.$$

We call *a carpet of type Φ of rank l over F* a collection of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of the field F with the condition

$$C_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js}, \quad r, s, ir + js \in \Phi,\ i, j > 0, \tag{1}$$

where $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$, and constants $C_{ij,rs}$ are equal to ±1, ±2 or ±3. Inclusions (1) come from the Chevalley commutator formula

$$\big[x_s(u), x_r(t)\big] = \prod_{i,j>0} x_{ir+js}\big(C_{ij,rs}(-t)^i u^j\big), \quad r, s, ir + js \in \Phi. \tag{2}$$

Every carpet $\mathfrak{A}$ defines a *carpet subgroup* $\Phi(\mathfrak{A})$ generated by the subgroups $x_r(\mathfrak{A}_r)$, $r \in \Phi$. A carpet $\mathfrak{A}$ is called *closed* if its carpet subgroup $\Phi(\mathfrak{A})$ has no new root elements, i.e., if

$$\Phi(\mathfrak{A}) \cap x_r(F) = x_r(\mathfrak{A}_r).$$

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The definition of a carpet used here was given by V. M. Levchuk [1] (see also [2, question 7.28]), and in [3] he described irreducible carpets of rank greater than 1 over field F, at least one additive subgroup of which is an R-module, where F is an algebraic extension of the field R, under the assumption that the characteristic of the field F is different from 0 and 2 for types B_l, C_l, F_4, and for the type G_2 is different from 0, 2 and 3. It turned out that, up to conjugation by a diagonal element, all additive subgroups of the carpet coincide with one intermediate subfield between R and F. We call such carpets *constant*. A similar problem for carpets of type G_2 over a field of characteristic 2 and 3 was considered by S. K. Franchuk and she established that non-constant carpets appear in characteristic 3 [4]. We have proved that in the remaining case of characteristic 0 for the type G_2 only constant carpets are possible.

Theorem 1. *Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be an irreducible carpet of type G_2 over a field F of characteristic 0, with at least one additive subgroup $\mathfrak{A}_r$ which is an R-module, where F is an algebraic extension of the field R. Then, up to conjugation by a diagonal element, all additive subgroups $\mathfrak{A}_r$ coincide with some intermediate subfield P between the fields R and F.*

2. Preliminary results

The group $\Phi(F)$ increasing to the extended Chevalley group $\widehat{\Phi}(F)$ by all diagonal elements $h(\chi)$, where χ is a F-character integral root lattice $\mathbb{Z}\Phi$, that is, a homomorphism of the additive group $\mathbb{Z}\Phi$ into the multiplicative group F^* of the field F [5, Sec. 7.1]. Any F-character χ is uniquely defined by the values at the fundamental roots, so for any $r \in \Phi$ and $t \in F$

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t). \tag{3}$$

The next lemma states that the equality (3) fits naturally with the definition of carpet.

Lemma 1 ([6], Lemma 1)**.** *Conjugating the carpet subgroup $\Phi(\mathfrak{A})$ with the diagonal element $h(\chi)$, we obtain the carpet subgroup*

$$h(\chi)\Phi(\mathfrak{A})h(\chi)^{-1} = \Phi(\mathfrak{A}'),$$

defined by the carpet

$$\mathfrak{A}' = \{\mathfrak{A}'_r \mid r \in \Phi\}, \text{ where } \mathfrak{A}'_r = \chi(r)\mathfrak{A}_r.$$

It is natural to call the carpet $\mathfrak{A}'$ from Lemma 1 *conjugate* to the original carpet $\mathfrak{A}$, and we can talk about conjugate carpets without relating them to carpet subgroups. Therefore, such statements are permissible. "Up to conjugation by a diagonal element, the carpet $\mathfrak{A}$ coincides with the carpet $\mathfrak{A}'$."

For a root system of type A_2 (see Fig. 1), there is one kind of commutator formula

$$[x_a(t), x_b(u)] = x_{a+b}(\pm tu).$$

Therefore, the carpet conditions have only one form $\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}$.

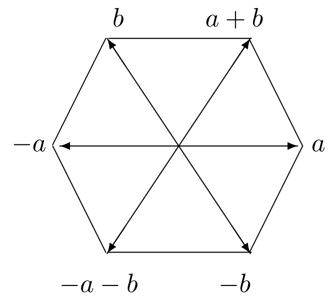

Fig. 1

For a root system of type G_2 (see Fig. 2), there are four kinds of commutator formulas

$$[x_a(t), x_b(u)] = x_{a+b}(\pm tu)x_{2a+b}(\pm t^2u)x_{3a+b}(\pm t^3u)x_{3a+2b}(\pm t^3u^2), \tag{4}$$

$$[x_a(t), x_{a+b}(u)] = x_{2a+b}(\pm 2tu)x_{3a+b}(\pm 3t^2u)x_{3a+2b}(\pm 3tu^2), \tag{5}$$

$$[x_a(t), x_{2a+b}(u)] = x_{3a+b}(\pm 3tu), \tag{6}$$

$$[x_b(t), x_{3a+b}(u)] = x_{3a+2b}(\pm tu). \tag{7}$$

So that, in this case, the carpet conditions look more impressive than for other types of root systems, and the formulas (4), (5), (6), (7) provide, respectively, the following forms

$$\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}, \quad \mathfrak{A}_a^2\mathfrak{A}_b \subseteq \mathfrak{A}_{2a+b}, \quad \mathfrak{A}_a^3\mathfrak{A}_b \subseteq \mathfrak{A}_{3a+b}, \quad \mathfrak{A}_a^3\mathfrak{A}_b^2 \subseteq \mathfrak{A}_{3a+2b},$$

$$2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}, \quad 3\mathfrak{A}_a^2\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{3a+b}, \quad 3\mathfrak{A}_a\mathfrak{A}_{a+b}^2 \subseteq \mathfrak{A}_{3a+2b},$$

$$3\mathfrak{A}_a\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{3a+b},$$

$$\mathfrak{A}_b\mathfrak{A}_{3a+b} \subseteq \mathfrak{A}_{3a+2b}.$$

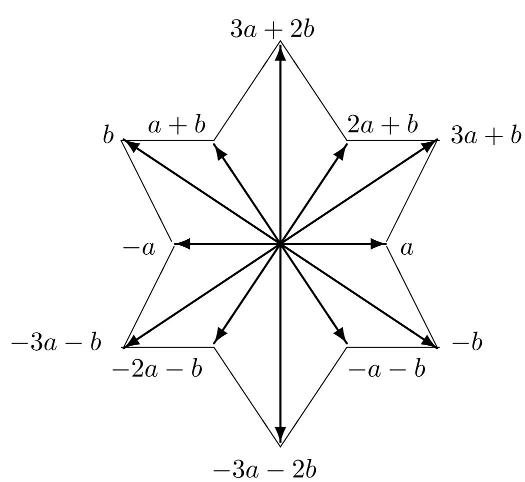

Fig. 2

The proof of the following lemma is elementary, so we omit it.

Lemma 2. *Let F be an algebraic extension of the field R and A is a subring of the field F which is an R-module. Then A is the field between R and F.*

Lemma 3. *Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be an irreducible carpet of type A_2 over a field F, $\{a, b\}$ is the fundamental system for Φ and let $1 \in \mathfrak{A}_{-a} \cap \mathfrak{A}_{-b}$ and the additive subgroup $\mathfrak{A}_{a+b}$ is an R-module, where F is an algebraic extension of the field R. Then all $\mathfrak{A}_r$ coincide with some fixed subfield of the field F.*

Proof. By [3, Lemma 3] all $\mathfrak{A}_r$ coincide with some fixed subring of the field F, and by Lemma 2 this subring is a field. The lemma is proved. □

3. Proof of Theorem 1

Up to conjugation, diagonal elements can be assumed to be $1 \in \mathfrak{A}_{-a} \cap \mathfrak{A}_{-b}$. Then, by virtue of the carpet conditions, from the commutator formula (4) we obtain $1 \in \mathfrak{A}_r$ for all $r \in \Phi^-$. Without loss of generality, we can assume that $\mathfrak{A}_{2a+b}$ or $\mathfrak{A}_{3a+2b}$ is an R-module. Since the field R has characteristic 0, then for any non-zero integer n we use the equality $n\mathfrak{A}_r = \mathfrak{A}_r$ without mentioning in case when the additive subgroup $\mathfrak{A}_r$ is an R-module.

Let $\mathfrak{A}_{2a+b}$ be an R-module. Due to the carpet conditions $2\mathfrak{A}_{-a-b}\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_a$ and $2\mathfrak{A}_{-a}\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{a+b}$ we get the inclusions $\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_a$ and $\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{a+b}$ respectively. Hence, due to the carpet condition $2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}$ it follows that $\mathfrak{A}_{2a+b}$ is a ring, and by virtue of Lemma 2 it is a field. In particular, $1 \in \mathfrak{A}_{2a+b}$. Therefore, due to the carpet conditions from the commutator formula (4), replacing the pair of roots (a, b) with the pairs $(2a+b, -3a-b)$ and $(2a+b, -3a-2b)$ we obtain $1 \in \mathfrak{A}_r$ for all $r \in \Phi$. Let $\mathfrak{A}_{2a+b} = P$. From the six carpet conditions of type $2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}$ we obtain the equalities $\mathfrak{A}_r = P$ for all short roots of r. By Lemma 3, all additive subgroups $\mathfrak{A}_r$ indexed by long roots r coincide with some fixed field Q. Now, from the carpet conditions $\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}$ and $\mathfrak{A}_a\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{3a+b}$ we obtain the inclusions $Q \subseteq P$ and $P \subseteq Q$ respectively. Thus, in this case we have established that all additive subgroups of the carpet coincide with the field P.

Let $\mathfrak{A}_{3a+2b}$ be an R-module. By Lemma 3, all additive subgroups $\mathfrak{A}_r$ indexed by long roots r coincide with some fixed field P. In particular, $1 \in \mathfrak{A}_{3a+2b}$. Therefore, due to the carpet conditions from the commutator formula (4), when the pair of roots (a, b) is replaced by the pairs $(-2a-b, 3a+2b)$ and $(-a-b, 3a+2b)$ we get $1 \in \mathfrak{A}_r$ for all $r \in \Phi$. Further, just as in the previous case, we obtain that all additive subgroups of the carpet coincide with the field P.

The theorem is proved.

This work is supported by Russian Science Foundation, project 22-21-00733.

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Неприводимые ковры аддитивных подгрупп типа G_2 над полем характеристики 0

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Аннотация. Доказано, что любой неприводимый ковер типа G_2 над полем F характеристики 0, хотя бы одна аддитивная подгруппа которого является R-модулем, где F — алгебраическое расширение поля R, с точностью до сопряжения диагональным элементом определяет группу Шевалле типа G_2 над промежуточным подполем между R и F.

Ключевые слова: группа Шевалле, ковер аддитивных подгрупп, ковровая подгруппа.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 615–622

DOI: 10.17516/1997-1397-2022-15-5-615-622
УДК 517.9

Existence and Uniqueness of the Solution to a Class of Fractional Boundary Value Problems Using Topological Methods

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Received 10.04.2021, received in revised form 11.03.2022, accepted 06.07.2022

Abstract. This paper investigates the existence and uniqueness of solutions to boundary value problems involving the Caputo fractional derivative in Banach space by topological structures with some appropriate conditions. It is based on the application of topological methods and fixed point theorems. Moreover, some topological properties of the solutions set are considered. Finally, an example is provided to illustrate the main results.

Keywords: fractional derivatives and integrals; topological properties of mappings, fixed point theorems.

Citation: T.A. Faree, S.K. Panchal, Existence and Uniqueness of the Solution to A class of Fractional Boundary Value Problems using Topological Methods, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 615–622. DOI: 10.17516/1997-1397-2022-15-5-615-622.

Introduction

Fractional differential equations have been verified to be effective modeling of many phenomena in several fields of science, for more details see Kilbas et al. [9], Miller and Ross [10], Podlubny [11], Agnieszka and Delfim [3]. Topological methods are one of the important tools that require weakly compact conditions rather than strongly compact conditions. In fact, the use of topological methods closely to study the existence of solutions for fractional differential equations in the last decades, see [6, 7, 13–16]. The fractional differential equations in Banach space are finding increasing consideration by many researchers such as Agarwal et al. [1, 2], Balachandran and Park [4], Benchohra et al. [5] and Zhang [19]. In 2006, Zhang [20], considered the existence of positive solutions to nonlinear fractional boundary value problems by applying the properties of the Green function and fixed point theorem on cones. In 2009, Benchohra et al. [5], examined the existence and uniqueness of solutions to fractional boundary value problems with nonlocal conditions by fixed point theorem. In 2012, Wang et. al [17,18], obtained the necessary and sufficient conditions for the fractional boundary value problems via a coincidence degree for condensing maps in Banach spaces. In 2015, the result was extended to the case for

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solutions to a fractional order multi point boundary value problem by Khan and Shah [8], who intentioned sufficient conditions for the existence of outcomes for a boundary value problem. In 2017, Samina et al. [12], studied the existence to solutions for nonlinear fractional Hybrid differential equations through some results about the existence of solutions and the Kuratowski's measure of non-compactness.

Stimulated by some of the mentioned results, our aim of this paper is to generate some new results about the following boundary value problem (BVP) for fractional differential equations involving Caputo fractional derivative with topological methods and fixed point theorems in Banach space $\mathcal{X}$.

$$\begin{cases} {}^{c}\mathcal{D}^{q}x(t) = \xi(t, x(t)) & t \in \mathcal{J} = [0, \tau],\ 0 < q \leqslant 1, \\ \beta x(0) + \gamma x(\tau) = \mu, \end{cases} \tag{1}$$

where ${}^{c}\mathcal{D}^{q}$ is the Caputo fractional derivative, $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is a continuous function. $\mathcal{C}(\mathcal{J}, \mathcal{X})$ will be a Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$ with the norm $\|x\|_c := sup\{\|x(t)\| : x \in \mathcal{C}(\mathcal{J}, \mathcal{X})\}$ for $t \in \mathcal{J}$. β, γ, μ are real constants satisfy $\beta + \gamma \neq 0$.

1. Preliminaries

In this section, we recall some of the basic definitions, propositions and basic theorem that will be used in this paper.

Definition 1.1 ([10])**.** *The q^{th} fractional order integral of a continuous function ξ on the closed interval $[a, b]$, is defined as*

$$\mathcal{I}_a^q \xi(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \xi(s) ds, \tag{2}$$

where Γ is the gamma function.

Definition 1.2 ([10])**.** *The q^{th} Riemann–Liouville fractional-order derivative of a continuous function ξ on the closed interval $[a, b]$, is defined as*

$$(\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-q-1} \xi(s) ds, \tag{3}$$

where $n = [q] + 1$ and $[q]$ is the integer part of q.

Definition 1.3 ([10])**.** *For a given continuous function ξ on the closed interval $[a, b]$, the Caputo fractional order derivative of ξ, is defined as*

$$({}^{c}\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} \xi^{(n)}(s) ds, \tag{4}$$

where $n = [q] + 1$.

Theorem 1.1 (Banach contraction mapping principle, [21])**.** *Let $\mathcal{X}$ be a complete metric space, and $\psi : \mathcal{X} \to \mathcal{X}$ is a contraction mapping with a contraction constant $\mathcal{K}$, then ψ has a unique fixed point.*

Theorem 1.2 (Schaefer's fixed point theorem, [21])**.** *Let $\mathcal{K}$ be a non-empty convex, closed and bounded subset of a Banach space $\mathcal{X}$. If $\psi : \mathcal{K} \to \mathcal{K}$ is a complete continuous operator such that $\psi(\mathcal{K}) \subset \mathcal{X}$, then ψ has at least one fixed point in $\mathcal{K}$.*

Definition 1.4 ([14, 21]). *Let $\Omega \subset \mathcal{X}$ and $\mathcal{F} : \Omega \to \mathcal{X}$ be a continuous bounded map. One can say that $\mathcal{F}$ is α-Lipschitz if there exists $k \geqslant 0$ such that*

$$\alpha(\mathcal{F}(B)) \leqslant k\alpha(B) \quad (\forall)\ B \subset \Omega \text{ bounded.}$$

In case, $k < 1$, then we call $\mathcal{F}$ is a strict α-contraction. One can say that $\mathcal{F}$ is α-condensing if

$$\alpha(\mathcal{F}(B)) < \alpha(B) \quad (\forall)\ B \subset \Omega \text{ bounded with } \alpha(B) > 0.$$

We recall that $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz if there exists $k > 0$ such that

$$\|\mathcal{F}_x - \mathcal{F}_y\| \leqslant k\|x - y\| \quad (\forall)\ x, y \subset \Omega,$$

and if $k < 1$ then $\mathcal{F}$ is a strict contraction.

Proposition 1.1 ([14, 21]). *If $\mathcal{F}, \mathcal{G} : \Omega \to \mathcal{X}$ are α-Lipschitz maps with the constants k, k' respectively, then $\mathcal{F} + \mathcal{G} : \Omega \to \mathcal{X}$ is α-Lipschitz with constant $k + k'$.*

Proposition 1.2 ([14, 21]). *If $\mathcal{F} : \Omega \to \mathcal{X}$ is compact , then $\mathcal{F}$ is α-Lipschitz with zero constant.*

Proposition 1.3 ([14, 21]). *) If $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz with a constant k, then $\mathcal{F}$ is α-Lipschitz with the same constant k.*

2. Main results

Definition 2.1. *If $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies the equation ${}^c\mathcal{D}^q x(t) = \xi(t, x(t))$ almost everywhere on $\mathcal{J}$, and the condition $\beta x(0) + \gamma x(\tau) = \mu$ then x is said to be a solution of the fractional BVP (1).*

In order to discuss the existence and uniqueness solutions to BVP(1), we require the following assumptions:
$[H1]$ $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is continuous.
$[H2]$ For each $t \in \mathcal{J}$ and all $x, y \in \mathcal{X}$, there exists constant $\delta > 0$ such that

$$\|\xi(t, x) - \xi(t, y)\| \leqslant \delta\|x - y\|$$

and

$$\frac{\delta\tau^q(1 + \frac{|\gamma|}{|\beta+\gamma|})}{\Gamma(q + 1)} < 1.$$

$[H3]$ For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_1, \delta_2 > 0$, $q_1 \in [0, 1)$ such that

$$\|\xi(t, x)\| \leqslant \delta_1\|x\|^{q_1} + \delta_2.$$

Lemma 2.1. *The fractional integral equation*

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}\xi(s, x(s))ds - \frac{1}{\beta + \gamma}[\frac{\gamma}{\Gamma(q)}\int_0^\tau (\tau - s)^{q-1}\xi(s, x(s))ds - \mu],\ t \in \mathcal{J} \quad (5)$$

has a solution $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ if and only if x is a solution of the fractional BVP (1).

Proof. First, suppose that $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies BVP(1), then we have to show that x is also a solution of FIE(5). We have,

$$x(t) - x(0) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}\xi(s, x(s))ds. \quad (6)$$

Then,

$$x(\tau) - x(0) = \frac{1}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds.$$

By the boundary conditions $\beta x(0) + \gamma x(\tau) = \mu$, we get

$$x(0) = \frac{\mu}{(\beta + \gamma)} - \frac{\gamma}{(\beta + \gamma)\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds, \quad \beta + \gamma \neq 0. \tag{7}$$

Replacing in equation(6), we get

$$x(t) = \frac{\mu}{(\beta + \gamma)} - \frac{\gamma}{(\beta + \gamma)\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \xi(s, x(s)) ds.$$

Conversely, suppose $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies FIE(5). If $t = 0$ then $\beta x(0) + \gamma x(\tau) = \mu$. For $t < \tau \in \mathcal{J}$ using the facts that the Caputo fractional derivative ${}^c\mathcal{D}_t^q$ is the left inverse of the fractional integral $\mathcal{I}_t^q$ and the Caputo derivative of the constant is zero, we can get ${}^c\mathcal{D}_t^q x(t) = \xi(t, x(t))$ which completes the proof. □

Lemma 2.2. *The operator $\mathcal{F} : \mathcal{C}(\mathcal{J}, \mathcal{X}) \to \mathcal{C}(\mathcal{J}, \mathcal{X})$ defined by;*

$$(\mathcal{F}x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \xi(s, x(s)) ds - \frac{1}{\beta + \gamma} \left[\frac{\gamma}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds - \mu \right]$$

is continuous and compact.

Proof. In order to prove the continuity and compactness of $\mathcal{F}$. Consider a bounded set $\mathcal{D}_r := \{\|x\| \leqslant r : x \in \mathcal{C}(\mathcal{J}, \mathcal{X})\}$. Let $\{x_n\}$ be a sequence of a bounded set $\mathcal{D}_r \subseteq \mathcal{C}(\mathcal{J}, \mathcal{X})$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. we have to show that $\|\mathcal{F}x_n - \mathcal{F}x\| \to 0$ as $n \to \infty$. It is obvious that $\|\xi(s, x_n(s)) - \xi(s, x(s))\| \to 0$ as $n \to \infty$ due to the continuity of ξ. Using [H3], we get for each $t \in \mathcal{J}$,

$$\|\xi(s, x_n(s)) - \xi(s, x(s))\| \leqslant \|\xi(s, x_n(s))\| + \|\xi(s, x(s))\| \leqslant 2(\delta_1 |r|^{q_1} + \delta_2).$$

As the function $s \to 2(\delta_1 |r|^{q_1} + \delta_2)$ is integrable for $s \in [0, t]$, $t \in \mathcal{J}$, by means of the Lebesgue Dominated Convergence theorem

$$\int_0^t (t - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds \to 0 \text{ as } n \to \infty.$$

Then, for all $t \in \mathcal{J}$

$$\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\| \leqslant \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds +$$
$$+ \frac{|\gamma|}{|\beta + \gamma|\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds \to 0 \text{ as } n \to \infty.$$

Which implies that $\mathcal{F}$ is continuous.

Let $\{x_n\}$ be a sequence on a bounded set $\mathcal{M} \subset \mathcal{D}_r$, for every $x_n \in \mathcal{M}$.

$$\|\mathcal{F}x_n\| \leqslant \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|\xi(s, x_n(s))\| ds +$$
$$+ \frac{|\gamma|}{|\beta + \gamma|\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \|\xi(s, x_n(s))\| ds + \frac{|\mu|}{|\beta + \gamma|}, \quad t < \tau \in \mathcal{J}.$$

Consequently, by assumption [H3] we can deduce that

$$\|\mathcal{F}x_n\| \leqslant \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds+$$

$$+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds+\frac{|\mu|}{|\beta+\gamma|}\leqslant$$

$$\leqslant \frac{[\delta_1 r^{q_1}+\delta_2]}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}ds+\frac{|\gamma|}{|\beta+\gamma|}\int_0^\tau (\tau-s)^{q-1}ds\right]+\frac{|\mu|}{|\beta+\gamma|}, \quad t<\tau\in\mathcal{J}.$$

Thus,

$$\|\mathcal{F}x_n\| \leqslant \left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\tau^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}+\frac{|\mu|}{|\beta+\gamma|}:=\mathcal{K}. \tag{8}$$

Therefore $(\mathcal{F}x_n)$ is uniformly bounded on $\mathcal{M}$. Hence, $\mathcal{F}(\mathcal{M})$ is bounded in $\mathcal{D}_r\subseteq\mathcal{C}(\mathcal{J},\mathcal{X})$. Now, we need to prove that $(\mathcal{F}x_n)$ is equicontinuous. For $t_1,t_2\in\mathcal{J},\epsilon>0$ and $t_1\leqslant t_2$, let $\rho=\rho(\epsilon)>0$ such that $\|t_2-t_1\|<\rho$. Consider

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\|\leqslant\left\|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}\xi(s,x_n(s))ds-\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}\xi(s,x_n(s))ds\right\|\leqslant$$

$$\leqslant\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|\xi(s,x_n(s))\|ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|\xi(s,x_n(s))\|ds.$$

Consequently, by assumption [H3] we get

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\|\leqslant$$

$$\leqslant\frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right](\delta_1\|x_n\|^{q_1}+\delta_2)ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}(\delta_1\|x_n\|^{q_1}+\delta_2)ds\leqslant$$

$$\leqslant\frac{(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q)}\left[\frac{t_1^q}{q}+\frac{2(t_2-t_1)^q}{q}-\frac{t_2^q}{q}\right]\leqslant\frac{2(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}(t_2-t_1)^q<\frac{2\rho^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}\equiv\epsilon.$$

Therefore, $(\mathcal{F}x_n)$ is equicontinuous. Since $\mathcal{F}$ is uniformly bounded and equicontinuous on $\mathcal{C}(\mathcal{J},\mathcal{X})$, then applying the Arzela Ascoli theorem, we get that $\mathcal{F}(\mathcal{M})$ is a relatively compact subset of $\mathcal{C}(\mathcal{J},\mathcal{X})$. Hence, $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. □

Remark 2.1. *As we proved $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is compact in Lemma (2.2). Consequently, by Proposition (1.2) $\mathcal{F}$ is $\alpha-$Lipschitz with zero constant.*

Theorem 2.1. *Assume that $[H1]-[H3]$ hold then the fractional BVP (1) has at least one solution.*

Proof. It is clear that, the fixed points of the operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ are solutions of BVP (1). Since the operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is continuous and completely continuous then we will prove that $\mathcal{S}(\mathcal{F})=\{x\in\mathcal{C}(\mathcal{J},\mathcal{X}):x=k\mathcal{F}x,\ \textit{for some}\ k\in(0,1)\}$ is bounded. For $x\in\mathcal{S}(\mathcal{F})$ and $k\in(0,1)$, we have

$$\|x\|=k\|\mathcal{F}x\|\leqslant\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\tau^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}+\frac{|\mu|}{|\beta+\gamma|}.$$

The above inequality with $q_1<1$ and equation (8), show that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. Thus, by Schaefer's fixed point theorem, we can conclude that $\mathcal{F}$ has at least one fixed point and the set of fixed points of $\mathcal{F}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. □

Remark 2.2. *If* $[H1]-[H3]$ *hold and* $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ *is a linear operator then the set of solutions of the fractional BVP(1) is convex.*

Theorem 2.2. *Assume that* $[H1]-[H3]$ *hold then the fractional BVP(1) has a unique solution* $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$.

Proof. According to theorem (2.1), the fractional BVP (1) has a solution $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$. It is sufficient to show that $\mathcal{F}$ is a contraction mapping on $\mathcal{C}(\mathcal{J},\mathcal{X})$ by [H2] as follows, for $x,y\in\mathcal{C}(\mathcal{J},\mathcal{X})$, we get

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\|\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds+$$

$$+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau(\tau-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds\leqslant$$

$$\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\delta\|x-y\|ds+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau(\tau-s)^{q-1}\delta\|x-y\|ds,\leqslant$$

$$\leqslant\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\delta\tau^q}{\Gamma(q+1)}\|x-y\|,\quad\beta+\gamma\neq0.$$

Thus, $\mathcal{F}$ is the contraction mapping on $\mathcal{C}(\mathcal{J},\mathcal{X})$ with a contraction constant $\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\delta\tau^q}{\Gamma(q+1)}$. By applying Banach's contraction mapping principle we can conclude that the operator $\mathcal{F}$ has a unique fixed point on $\mathcal{C}(\mathcal{J},\mathcal{X})$. Hence, BVP(1) has a unique solution in $\mathcal{C}(\mathcal{J},\mathcal{X})$. □

Example 2.1. *Consider the following fractional BVP*

$$\begin{cases}{}^c\mathcal{D}^{\frac{2}{3}}x(t)=\dfrac{|x(t)|}{(9+e^t)(1+|x(t)|)}\,,\quad t\in[0,1],\\ \qquad x(0)+x(1)=0.\end{cases}\tag{9}$$

Set $q=\dfrac{2}{3}$, for $(t,x)\in[0,1]\times[0,+\infty)$, we can define $\xi(t,x)=\dfrac{x}{(9+e^t)(1+x)}$. Also, for $t\in[0,1]$ we have $x(t)=\dfrac{1}{9+e^t}$. For $x,y\in[0,+\infty)$, then

$$\Big|\xi(t,x)-\xi(t,y)\Big|=\left|\frac{x}{(9+e^t)(1+x)}-\frac{y}{(9+e^t)(1+y)}\right|\leqslant$$

$$\leqslant\frac{1}{10}\left|\frac{x}{(1+x)}-\frac{y}{(1+y)}\right|\leqslant\frac{1}{10}\left|\frac{x-y}{(1+x)(1+y)}\right|\leqslant$$

$$\leqslant\frac{1}{10}|x-y|\Rightarrow\delta=\frac{1}{10},\quad t\in[0,1].$$

If $q=\dfrac{2}{3}$, $\Gamma(q+1)=\Gamma\left(\frac{5}{3}\right)=0.89$, we have

$$\frac{\delta\tau^q\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)}{\Gamma(q+1)}=\frac{0.15}{0.89}<1.$$

Hence, we see that all assumptions in theorem (2.1) are satisfied which means our results can be used to solve BVP (9).

Conclusion

We have confirmed some sufficient conditions for the existence and uniqueness of solutions for BVP(1) based on the fixed Point theorems as well as the topological technique of approximate solutions. In addition, we studied some topological properties of the solutions set. Finally, an example is provided to verify our results.

The authors appreciate the referees' time and suggestions in helping develop this paper.

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Существование и единственность решений краевых задач с дробной производной с помощью топологических структур

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Аннотация. В статье исследуется существование и единственность решений краевых задач с дробной производной Капуто в банаховом пространстве с помощью топологических структур с некоторыми соответствующими условиями. Он основан на применении топологических методов и теорем о неподвижной точке. Кроме того, рассматриваются некоторые топологические свойства множества решений. Приведен пример, иллюстрирующий основные результаты.

Ключевые слова: дробные производные и интегралы; топологические свойства отображений, теоремы о неподвижной точке

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 623–634

DOI: 10.17516/1997-1397-2022-15-5-623-634
УДК 517.9

Unsteady Flow of two Binary Mixtures in a Cylindrical Capillary with Changes in the Internal Energy of the Interface

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Received 10.09.2022, received in revised form 10.11.2022, accepted 20.12.2022

Abstract. The problem of two-dimensional unsteady flow of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces is studied. The mixtures are contacted through a common interface on which the energy condition is taken into account. The temperature and concentration of mixtures are distributed according to the quadratic law. It is in good agreement with the velocity field of the Hiemenz type. The resulting conjugate boundary value problem is a non-linear problem. It is also an inverse problem with respect to the pressure gradient along the axis of the cylindrical tube. To solve the problem the tau-method is used. It was shown that with increasing time the solution of the non-stationary problem tends to a steady state. It was established that the effect of increments of the internal energy of the inter-facial surface significantly affects the dynamics of the flow of mixtures in the layers.

Keywords: non-stationary solution, binary mixture, interface, energy condition, internal energy, inverse problem, pressure gradient, tau-method, thermal diffusion.

Citation: N.L. Sobachkina, Unsteady Flow of two Binary Mixtures in a Cylindrical Capillary with Changes in the Internal Energy of the Interface, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 623–634. DOI: 10.17516/1997-1397-2022-15-5-623-634.

Introduction

The energy exchange between volume phases and a transition layer between them can lead to the inhomogeneous temperature distribution along the inter-facial surface. The mechanism of formation of the Marangoni stresses has been known for quite a long time. It implies that temperature gradient along the interface can arise and be maintained due to local increments of the internal energy of the inter-facial surface [1]. The temperature gradient, in turn, leads to the concentration gradient in the liquid mixture (these are the effects of thermodiffusion) [2].

For many liquids the surface tension is well approximated by a linear function. In this case *energy condition* is simplified to the following form [3,4]:

$$k_2 \frac{\partial \theta_2}{\partial \mathbf{n}} - k_1 \frac{\partial \theta_1}{\partial \mathbf{n}} = æ\theta \nabla_\Gamma \cdot \mathbf{u}, \tag{1}$$

where $æ = -\partial\sigma/\partial\theta$, $\sigma = \sigma(\theta, c)$ is the surface tension coefficient that depends on temperature and concentration; k_1, k_2 are the coefficients of thermal conductivity; $\mathbf{n}$ is a unit vector of the normal to the interface of liquids Γ directed into the second liquid; $\nabla_\Gamma = \nabla - (\mathbf{n} \cdot \nabla)\mathbf{n}$ is a surface gradient; $\theta = \theta_1 = \theta_2$, $\mathbf{u} = \mathbf{u}_1 = \mathbf{u_2}$ are pairwise coincident values of temperatures and velocity vector of both liquids on Γ; $\nabla_\Gamma \cdot \mathbf{u} = \mathrm{div}_\Gamma\, \mathbf{u}$ is a surface divergence of the velocity vector

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u. According to the condition (1), changes in the surface internal energy induce corresponding changes in heat flows through the interface Γ.

For ordinary liquids at room temperature the effect of changes in the internal energy of the inter-facial surface on the formation of heat fluxes, temperature fields and velocities in its vicinity is insignificant in relation to the viscous friction and heat transfer [1]. However, at sufficiently high temperatures when the viscosity and thermal conductivity of ordinary liquids decrease significantly and for liquids with low viscosity increments of the internal energy of the inter-facial surface can have a significant impact on the dynamics of the flow [5]. The influence of changes in the internal energy of the inter-facial surface on the movement of liquids was studied [6–9].

A mathematical model that describes the two-layer unsteady thermodiffusion motion of binary mixtures in a cylindrical capillary in the absence of mass forces is considered in this paper. The mixtures are in contact through a common interface on which the energy condition is taken into account. The mechanism of influence of changes in the surface internal energy on the dynamics of the flow of binary mixtures in layers is studied. A similar geometry in the case of steady motion of mixtures was studied [10]. The non-linear stationary problem was reduced to a system of non-linear algebraic equations which was solved by the Newton method.

1. Statement of the problem

A two-dimensional unsteady axisymmetric flow of two immiscible incompressible binary mixtures in a cylindrical tube of radius R_2 is considered. The temperature of the system is kept constant. Binary mixtures occupy domains $\Omega_1 = \{0 \leqslant r \leqslant R_1, |z| < \infty\}$ and $\Omega_2 = \{R_1 \leqslant r \leqslant R_2, |z| < \infty\}$, where r, z are radial and axial cylindrical coordinates. Here, the common interface of binary mixtures is at $r = R_1 = \text{const}$, and the solid wall is at $r = R_2 = \text{const}$. Values related to the regions Ω_1 and Ω_2 are denoted by the indices 1 and 2, respectively. It is assumed that the characteristic transverse size of domain Ω_2 is small compared to the radius of domain Ω_1 so $R_2 - R_1 \ll R_1$ (Fig. 1).

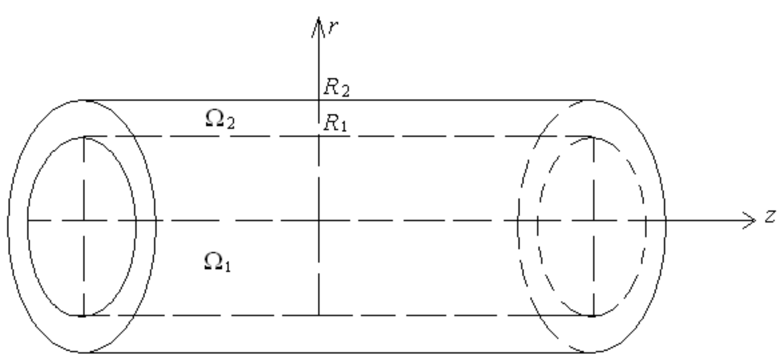

Fig. 1. Schematic of the flow domain

Binary mixture is characterized by constant conductivity k_j, specific heat capacity c_{pj}, dynamic viscosity μ_j, density ρ_j, coefficient of thermal conductivity $\chi_j = k_j/\rho_j c_{pj}$, kinematic viscosity $\nu_j = \mu_j/\rho_j$ (hereinafter, $j = 1, 2$). The influence of gravity is not taken into account. It can be justified for a narrow capillary.

Taking into account the effect of thermal diffusion, the defining equations of motion, heat and mass transfer have the form

$$
\begin{gathered}
u_{jt} + u_j u_{jr} + w_j u_{jz} + \frac{1}{\rho_j}\, p_{jr} = \nu_j \left(\Delta u_j - \frac{u_j}{r^2}\right), \\
w_{jt} + u_j w_{jr} + w_j w_{jz} + \frac{1}{\rho_j}\, p_{jz} = \nu_j \Delta w_j, \\
u_{jr} + \frac{u_j}{r} + w_{jz} = 0, \\
\theta_{jt} + u_j \theta_{jr} + w_j \theta_{jz} = \chi_j \Delta\theta_j, \\
c_{jt} + u_j c_{jr} + w_j c_{jz} = d_j \Delta c_j + \alpha_j d_j \Delta\theta_j,
\end{gathered}
\tag{2}
$$

where u_j, w_j are projections of the velocity vector on r, z axis of the cylindrical coordinate system, respectively; p_j is the pressure in the layers; θ_j, c_j are deviations of temperature and concentration from the average values θ_0, c_0; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the Laplace operator; d_j, α_j are diffusion and thermal diffusion coefficients, respectively. Generally speaking, these coefficients depend on temperature and concentration. However, using assumptions given above, one can consider that they have constant values corresponding to the average values of temperature and concentration. Let us note that the diffusion coefficient is always positive. The thermal diffusion coefficient can be either positive or negative. It depends on the type of a mixture and on the average temperature and average concentration of the selected component [11]. It is assumed that c in system (2) is the concentration of a light component.

It is assumed that inter-facial tension coefficient depends linearly on temperature and concentration:

$$
\sigma(\theta, c) = \sigma_0 - æ_1(\theta - \theta_0) - æ_2(c - c_0). \tag{3}
$$

Here $æ_1 > 0$ is the temperature coefficient, $æ_2$ is the concentration coefficient of the surface tension (usually $æ_2 < 0$, since the surface tension increases with increasing concentration).

The solution of the problem is taken in the special form:

$$
\begin{gathered}
u_j = u_j(r,t), \quad w_j = z v_j(r,t), \quad p_j = p_j(r,z,t), \\
\theta_j = a_j(r,t) z^2 + b_j(r,t), \quad c_j = h_j(r,t) z^2 + g_j(r,t).
\end{gathered}
\tag{4}
$$

In this representation, the velocity field corresponds to a solution of the Hiemenz type [12]. In this case, temperature θ_j takes an extreme value at the point $z = 0$: the maximum value when $a_j(r,t) < 0$ and the minimum value when $a_j(r,t) > 0$. A similar interpretation was obtained for the concentration of c_j. One should only consider function $h_j(r,t)$ instead of $a_j(r,t)$.

Substituting (4) into equations of motion (2) and separating the variable z, one can obtain the following system for unknown functions $u_j(r,t)$, $v_j(r,t)$, $p_j(r,t)$, $a_j(r,t)$, $b_j(r,t)$, $h_j(r,t)$, $g_j(r,t)$

$$
u_{jt} + u_j u_{jr} + \frac{1}{\rho_j}\, p_{jr} = \nu_j (u_{jrr} + \frac{1}{r}\, u_{jr} - \frac{u_j}{r^2}), \tag{5}
$$

$$
z(v_{jt} + u_j v_{jr} + v_j^2) + \frac{1}{\rho_j}\, p_{jz} = z\nu_j (v_{jrr} + \frac{1}{r}\, v_{jr}), \tag{6}
$$

$$
u_{jr} + \frac{u_j}{r} + v_j = 0, \tag{7}
$$

$$
a_{jt} + u_j a_{jr} + 2 v_j a_j = \chi_j (a_{jrr} + \frac{1}{r}\, a_{jr}), \tag{8}
$$

$$
b_{jt} + u_j b_{jr} = \chi_j (b_{jrr} + \frac{1}{r}\, b_{jr} + 2a_j), \tag{9}
$$

$$
h_{jt} + u_j h_{jr} + 2 v_j h_j = d_j (h_{jrr} + \frac{1}{r}\, h_{jr}) + \alpha_j d_j (a_{jrr} + \frac{1}{r}\, a_{jr}), \tag{10}
$$

$$
g_{jt} + u_j g_{jr} = d_j (g_{jrr} + \frac{1}{r}\, g_{jr} + 2h_j) + \alpha_j\, d_j (b_{jrr} + \frac{1}{r}\, b_{jr} + 2a_j). \tag{11}
$$

The pressure gradients (p_{jr}, p_{jz}) can be expressed from equations (5), (6):

$$p_{jr} = \rho_j \nu_j (u_{jrr} + \frac{1}{r}\, u_{jr} - \frac{u_j}{r^2}) - \rho_j\,(u_{jt} + u_j\, u_{jr}), \tag{12}$$

$$p_{jz} = z[\rho_j \nu_j (v_{jrr} + \frac{1}{r}\, v_{jr}) - \rho_j (v_{jt} + u_j\, v_{jr} + v_j^2)]. \tag{13}$$

Conditions for the compatibility of equations (12), (13) are satisfied identically: $p_{jrz} = p_{jzr} = 0$. Hence it follows that pressure in the layers can be restored:

$$p_j = -\rho_j f_j(t) \frac{z^2}{2} + s_j(r,t), \tag{14}$$

where the derivative of function $s_j(r,t)$ with respect to r is exactly the right hand side of equation (12). Integrating this equation, we obtain the following representation of functions $s_j(r,t)$

$$s_j(r,t) = \rho_j \nu_j (u_{jr} + \frac{1}{r}\, u_j) - \rho_j \left(\frac{\partial}{\partial t} \int_0^r u_j dr + \frac{1}{2}\, u_j^2 \right) + s_j(t). \tag{15}$$

Functions $v_j(r,t)$ are determined from the equation:

$$v_{jt} + u_j v_{jr} + v_j^2 = \nu_j (v_{jrr} + \frac{1}{r}\, v_{jr}) + f_j(t). \tag{16}$$

It follows that flow in the layers is induced by longitudinal pressure gradients $f_j(t)$, $j = 1, 2$. These are unknown functions to be defined along with functions v_j, a_j, b_j, h_j, g_j. Therefore, we have an inverse problem.

Conditions on the solid wall $(r = R_2)$ are

$$u_2(R_2,t) = 0, \quad v_2(R_2,t) = 0, \quad a_2(R_2,t) = a_2(t), \quad b_2(R_2,t) = b_2(t); \tag{17}$$

$$h_{2r}(R_2,t) + \alpha_2 a_{2r}(R_2,t) = 0, \quad g_{2r}(R_2,t) + \alpha_2 b_{2r}(R_2,t) = 0, \tag{18}$$

with the specified functions $a_2(t)$, $b_2(t)$.

The inter-facial surface is assumed to be cylindrical and non-deformable. To do this, it is enough to require that Weber number We $\to \infty$. Then taking into account this requirement and relation (3), we have the following boundary conditions at $r = R_1$:

$$u_1(R_1,t) = u_2(R_1,t), \quad v_1(R_1,t) = v_2(R_1,t); \tag{19}$$

$$a_1(R_1,t) = a_2(R_1,t), \quad b_1(R_1,t) = b_2(R_1,t); \tag{20}$$

$$h_1(R_1,t) = h_2(R_1,t), \quad g_1(R_1,t) = g_2(R_1,t), \tag{21}$$

$$\mu_2 v_{2r}(R_1,t) - \mu_1 v_{1r}(R_1,t) = -2[æ_1 a_1(R_1,t) + æ_2 h_1(R_1,t)]; \tag{22}$$

$$d_1[h_{1r}(R_1,t) + \alpha_1 a_{1r}(R_1,t)] = d_2[h_{2r}(R_1,t) + \alpha_2 a_{2r}(R_1,t)]; \tag{23}$$

$$d_1[g_{1r}(R_1,t) + \alpha_1 b_{1r}(R_1,t)] = d_2[g_{2r}(R_1,t) + \alpha_2 b_{2r}(R_1,t)]; \tag{24}$$

$$k_2 a_{2r}(R_1,t) - k_1 a_{1r}(R_1,t) = æ_1 a_1(R_1,t) v_1(R_1,t); \tag{25}$$

$$k_2 b_{2r}(R_1,t) - k_1 b_{1r}(R_1,t) = æ_1 b_1(R_1,t) v_1(R_1,t). \tag{26}$$

Relations (25), (26) are *energy condition* on the interface of two binary mixtures. It can be interpreted as follows: a jump in the heat flow in the direction of the normal to the interface (at $r = R_1$) is compensated by the change in the internal energy of this surface. In turn, this change is associated with both the change in temperature (specific internal energy) and the change in the area of the interface [13].

In addition, it is necessary to require the boundedness of functions on the axis of the cylindrical capillary ($r = 0$):

$$\begin{gathered} |u_1(0,t)| < \infty,\ |v_1(0,t)| < \infty,\ |s_1(0,t)| < \infty, \\ |a_1(0,t)| < \infty,\ |b_1(0,t)| < \infty,\ |h_1(0,t)| < \infty,\ |g_1(0,t)| < \infty. \end{gathered} \tag{27}$$

Initial conditions at $t = 0$ are

$$\begin{gathered} u_j(r,0) = u_{j0}(r), \quad v_j(r,0) = v_{j0}(r), \quad a_j(r,0) = a_{j0}(r), \quad b_j(r,0) = b_{j0}(r); \\ h_j(r,0) = h_{j0}(r), \quad g_j(r,0) = g_{j0}(r), \quad s_j(r,0) = s_{j0}(r), \quad f_j(0) = f_{j0} \equiv \text{const}. \end{gathered} \tag{28}$$

Let us note that functions u_{j0} and v_{j0} should be constrained according to continuity equation (7); functions h_{j0}, a_{j0} should be constrained according to conditions (18), (23); functions g_{j0}, b_{j0} should be constrained according to conditions (18), (24); functions v_{j0}, a_{10}, h_{10} should be constrained according to condition (22); functions v_{10}, a_{j0} should be constrained according to condition (25), and functions v_{10}, b_{j0} should be constrained according to condition (26). Thus, the compatibility conditions are fulfilled.

2. Formulation of the problem in dimensionless variables

It should be noted that equations (7), (8), (10), (16) are independent of the others and they form a closed subsystem for defining functions $v_j(r,t)$, $a_j(r,t)$, $h_j(r,t)$ and $f_j(t)$ $(j = 1,2)$. After solving it, functions $b_j(r,t)$, $g_j(r,t)$ can be determined from equations (9), (11), and functions $s_j(r,t)$ can be uniquely determined from relation (15). Taking into account conditions (27) and adhesion on the solid wall (17), we integrate continuity equation (7) and exclude $u_j(r,t)$ in equations (8), (10), (16). Then one needs to find only functions $v_j(r,t)$, $a_j(r,t)$, $h_j(r,t)$ and $f_j(t)$. We introduce dimensionless variables and functions

$$\begin{gathered} \xi = \frac{r}{R_1}, \quad R = \frac{R_2}{R_1} > 1, \quad \tau = \frac{\nu_1}{R_1^2}, \quad V_j = \frac{R_1^2 v_j}{\mathrm{Ma}\, \nu_1}, \\ A_j = \frac{a_j}{\theta_0}, \quad H_j = \frac{h_j}{c_0}, \quad F_j = \frac{R_1^4 f_j}{\mathrm{Ma}\, \nu_1^2}, \end{gathered} \tag{29}$$

where θ_0, c_0 are the characteristic temperature and concentration.

The main similarity criteria in the problem under consideration are

$$\begin{gathered} \mathrm{Ma} = \frac{æ_1 \theta_0 R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Mc} = \frac{æ_2 c_0 R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Pr}_j = \frac{\nu_j}{\chi_j}, \quad \mathrm{Sc}_j = \frac{\nu_j}{d_j}, \quad \mathrm{Sr}_j = \frac{\alpha_j \theta_0}{d_j c_0}, \\ \mu = \frac{\mu_1}{\mu_2}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad k = \frac{k_1}{k_2}, \quad d = \frac{d_1}{d_2}, \quad \mathrm{M} = \frac{\mathrm{Mc}}{\mathrm{Ma}} = \frac{æ_2 c_0}{æ_1 \theta_0}. \end{gathered} \tag{30}$$

Here Ma is the thermal Marangoni number, Mc is the Marangoni concentration number, Pr_j is the Prandtl number, Sc_j is Schmidt number, Sr_j is the Soret number.

We obtain a nonlinear inverse initial-boundary value problem in the domain of the spatial variable ξ. When $j = 1$ it varies from 0 to 1, and when $j = 2$ it varies from 1 to R.

For $0 < \xi < 1$ we have

$$K_1(V_1, F_1) \equiv V_{1\xi\xi} + \frac{1}{\xi} V_{1\xi} - V_{1\tau} + \frac{\mathrm{Ma}}{\xi} V_{1\xi} \int_0^{\xi} \xi V_1(\xi,\tau)\, d\xi - \mathrm{Ma} V_1^2 + F_1(\tau) = 0, \tag{31}$$

$$S_1(V_1, A_1) \equiv \frac{1}{\mathrm{Pr}_1}\left(A_{1\xi\xi} + \frac{1}{\xi}A_{1\xi}\right) - A_{1\tau} + \frac{\mathrm{Ma}}{\xi}\,A_{1\xi}\int_0^{\xi} \xi V_1(\xi,\tau)\,d\xi - 2\mathrm{Ma}A_1V_1 = 0; \tag{32}$$

$$\begin{gathered} T_1(V_1, A_1, H_1) \equiv \frac{1}{\mathrm{Sc}_1}\left(H_{1\xi\xi} + \frac{1}{\xi}H_{1\xi}\right) + \frac{\mathrm{Sr}_1}{\mathrm{Sc}_1}\left(A_{1\xi\xi} + \frac{1}{\xi}A_{1\xi}\right) - H_{1\tau} + \\ + \frac{\mathrm{Ma}}{\xi}\,H_{1\xi}\int_0^{\xi} \xi V_1(\xi,\tau)\,d\xi - 2\mathrm{Ma}H_1V_1 = 0. \end{gathered} \tag{33}$$

For $1 < \xi < R$ we have

$$K_2(V_2, F_2) \equiv \frac{1}{\nu}\left(V_{2\xi\xi} + \frac{1}{\xi}V_{2\xi}\right) - V_{2\tau} - \frac{\mathrm{Ma}}{\xi}\,V_{2\xi}\int_{\xi}^{R} \xi V_2(\xi,\tau)\,d\xi - \mathrm{Ma}V_2^2 + F_2(\tau) = 0, \tag{34}$$

$$S_2(V_2, A_2) \equiv \frac{1}{\mathrm{Pr}_2\nu}\left(A_{2\xi\xi} + \frac{1}{\xi}A_{2\xi}\right) - A_{2\tau} - \frac{\mathrm{Ma}}{\xi}\,A_{2\xi}\int_{\xi}^{R} \xi V_2(\xi,\tau)\,d\xi - 2\mathrm{Ma}A_2V_2 = 0; \tag{35}$$

$$\begin{gathered} T_2(V_2, A_2, H_2) \equiv \frac{1}{\mathrm{Sc}_2\nu}\left(H_{2\xi\xi} + \frac{1}{\xi}H_{2\xi}\right) + \frac{\mathrm{Sr}_2}{\mathrm{Sc}_2\nu}\left(A_{2\xi\xi} + \frac{1}{\xi}A_{2\xi}\right) - H_{2\tau} - \\ - \frac{\mathrm{Ma}}{\xi}\,H_{2\xi}\int_{\xi}^{R} \xi V_2(\xi,\tau)\,d\xi - 2\mathrm{Ma}H_2V_2 = 0. \end{gathered} \tag{36}$$

Then the following conditions are satisfied on the solid wall ($\xi = R$)

$$V_2(R,\tau) = 0, \quad A_2(R,\tau) = \frac{a_2(\tau)}{\theta_0}, \quad H_{2\xi}(R,\tau) + \mathrm{Sr}_2A_{2\xi}(R,\tau) = 0. \tag{37}$$

On the interface ($\xi = 1$) we have

$$V_1(1,\tau) = V_1(1,\tau), \quad \int_0^1 \xi V_1(\xi,\tau)\,d\xi = 0, \quad \int_1^R \xi V_2(\xi,\tau)\,d\xi = 0, \tag{38}$$

$$A_1(1,\tau) = A_2(1,\tau), \quad H_1(1,\tau) = H_2(1,\tau), \tag{39}$$

$$V_{2\xi}(1,\tau) - \mu V_{1\xi}(1,\tau) = -2[A_1(1,\tau) + \mathrm{M}H_1(1,\tau)], \tag{40}$$

$$d[H_{1\xi}(1,\tau) + \mathrm{Sr}_1A_{1\xi}(1,\tau)] = H_{2\xi}(1,\tau) + \mathrm{Sr}_2A_{2\xi}(1,\tau), \tag{41}$$

$$A_{2\xi}(1,\tau) - kA_{1\xi}(1,\tau) = EA_1(1,\tau)V_1(1,\tau), \tag{42}$$

where parameter $E = æ_1^2\theta_0R_1^2/\mu_2k_2$ characterizes the significance of the process of release or absorption of heat at local increments of the area of the inter-facial surface for the development of convective motion near this surface. The mechanism of local change in the internal energy of the interface should be taken into account for most conventional liquids at elevated temperatures or for liquids with reduced viscosity, for example, for some cryogenic liquids. Calculations carried out for physical parameters of various liquids and phase interfaces showed that $E = O(1)$ is quite realistic [5].

The conditions of boundedness of functions are set on the axis of symmetry:

$$|V_1(0,\tau)| < \infty, \quad |A_1(0,\tau)| < \infty, \quad |H_1(0,\tau)| < \infty. \tag{43}$$

The initial conditions at $\tau = 0$ are

$$\begin{gathered} V_j(\xi,0) = V_{j0}(\xi), \quad A_j(\xi,0) = A_{j0}(\xi), \\ H_j(\xi,0) = H_{j0}(\xi), \quad F_j(0) = F_{j0} \equiv \mathrm{const}. \end{gathered} \tag{44}$$

Note that the integral redefinition conditions in (38) are used as additional ones when solving the inverse problem and they are nothing more than a closed flow condition. They play an important role in finding unknown longitudinal pressure gradients $F_j(\tau)$ in layers of binary mixtures.

Let us consider the creeping unsteady flow of binary mixtures (this is a linear problem).

Let us assume that the thermal Marangoni number $\mathrm{Ma} \ll 1$ (a creeping motion) and $\mathrm{Ma} \sim \mathrm{Mc}$, that is, thermal and concentration effects on the interface $\xi = 1$ are of the same order. Then the equations of momentum, energy and concentration transfer are simplified by discarding convective acceleration. As a result, the conjugate inverse initial-boundary value problem becomes linear. However, such problem cannot be solved consistently because of the non-linearity of energy condition (42).

3. Derivation of a finite-dimensional system of first-order ordinary differential equations

To solve non-linear problem (31)–(44) the tau-method is used. It is a modification of the Galerkin method [14]. For further consideration, it is essential to replace the variables: $\xi' = \xi$ when $j = 1$ and $\xi' = (\xi - R)/(1 - R)$ when $j = 2$ and re-assign $\xi' \leftrightarrow \xi$. Taking into account (43), an approximate solution is taken in the form

$$\begin{aligned} V_j^n(\xi,\tau) &= \sum_{k=0}^{n} V_j^k(\tau) R_k^{(0,1)}(\xi), \\ A_j^n(\xi,\tau) &= \sum_{k=0}^{n} A_j^k(\tau) R_k^{(0,1)}(\xi), \\ H_j^n(\xi,\tau) &= \sum_{k=0}^{n} H_j^k(\tau) R_k^{(0,1)}(\xi), \end{aligned} \tag{45}$$

where $R_k^{(0,1)}(\xi)$ are shifted Jacobi polynomials. In general, they are defined in terms of Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$ as follows $(\alpha > -1, \beta > -1)$ [15]

$$R_k^{(\alpha,\beta)}(y) = P_k^{(\alpha,\beta)}(2y - 1), \quad y \in [0,1]. \tag{46}$$

It is known that shifted Jacobi polynomials $R_k^{(\alpha,\beta)}(y)$ are orthogonal on the segment $[0,1]$ with the weight $(1-y)^\alpha y^\beta$. Then

$$\int_0^1 (1-y)^\alpha y^\beta R_k^{(\alpha,\beta)}(y) R_m^{(\alpha,\beta)}(y)\, dy = \delta_{km} h_m, \tag{47}$$

$$h_m = \frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)}{m!(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}, \quad \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$$

where $\Gamma(x)$ is the Euler gamma function.

In addition, polynomials $R_k^{(\alpha,\beta)}(y)$ form a basis in $L_2(0,1)$ with the weight $(1-y)^\alpha y^\beta$ and they satisfy the following properties [16]

$$R_k^{(\alpha,\beta)}(0) = \frac{(-1)^k(\beta+k)!}{\beta!k!}, \quad R_k^{(\alpha,\beta)}(1) = \frac{(\alpha+k)!}{\alpha!k!}; \tag{48}$$

$$\frac{d^m}{dy^m} R_k^{(\alpha,\beta)}(y) = \frac{\Gamma(k+m+\alpha+\beta+1)}{\Gamma(k+\alpha+\beta+1)} R_{k-m}^{(\alpha+m,\beta+m)}(y). \tag{49}$$

Functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$, $F_j(\tau)$ are determined from the system of Galerkin approximations

$$\int_0^1 K_j(V_j^n, F_j) R_m^{(0,1)}(\xi)\,\xi\,d\xi = 0, \tag{50}$$

$$\int_0^1 S_j(V_j^n, A_j^n) R_m^{(0,1)}(\xi)\,\xi\,d\xi = 0, \tag{51}$$

$$\int_0^1 T_j(V_j^n, A_j^n, H_j^n) R_m^{(0,1)}(\xi)\,\xi\,d\xi = 0, \quad m = 0, \ldots, n-3, \quad j = 1, 2. \tag{52}$$

Taking into account conditions (38) and property (47), we obtain that $V_1^0(\tau) = V_2^0(\tau) = 0$.

Taking into account properties (48) and (49), boundary conditions take the form

$$\sum_{k=0}^{n} (-1)^k (k+1) V_2^k(\tau) = 0, \quad \sum_{k=0}^{n} (-1)^k (k+1) A_2^k(\tau) = \frac{a_2(\tau)}{\theta_0}, \tag{53}$$

$$\sum_{k=1}^{n} (-1)^{k-1} k(k+1)(k+2)[H_2^k(\tau) + \mathrm{Sr}_2 A_2^k(\tau)] = 0. \tag{54}$$

$$\begin{gathered} \sum_{k=0}^{n} V_1^k(\tau) = \sum_{k=0}^{n} V_2^k(\tau), \quad \sum_{k=0}^{n} A_1^k(\tau) = \sum_{k=0}^{n} A_2^k(\tau), \\ \sum_{k=0}^{n} H_1^k(\tau) = \sum_{k=0}^{n} H_2^k(\tau), \end{gathered} \tag{55}$$

$$\frac{1}{1-R} \sum_{k=1}^{n} k(k+2)(V_2^k(\tau) - \mu V_1^k(\tau)) = -2 \sum_{k=0}^{n} (A_1^k(\tau) + \mathrm{M} H_1^k(\tau)). \tag{56}$$

$$d \sum_{k=1}^{n} k(k+2)[H_1^k(\tau) + \mathrm{Sr}_1 A_1^k(\tau)] = \sum_{k=1}^{n} k(k+2)[H_2^k(\tau) + \mathrm{Sr}_2 A_2^k(\tau)], \tag{57}$$

$$\frac{1}{1-R} \sum_{k=1}^{n} k(k+2)(A_2^k(\tau) - k A_1^k(\tau)) = E \sum_{k=0}^{n} A_1^k(\tau) \sum_{k=0}^{n} V_1^k(\tau). \tag{58}$$

The finite-dimensional system of Galerkin approximations for functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$ ($k = 0, \ldots, n$, $j = 1, 2$) and the calculation of the resulting definite integrals over various products of shifted Jacobi polynomials are described in detail in [17].

The system of integro-differential equations can be transformed to a closed system of first-order ordinary differential equations with respect to unknown functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$, $F_j(\tau)$ ($k = 0, \ldots, n-3$, $j = 1, 2$). It involves rather cumbersome treatment and it is not given here. The initial conditions follows from (44) and (45):

$$V_j^k(0) = V_{j0}^k, \quad A_j^k(0) = A_{j0}^k, \quad H_j^k(0) = H_{j0}^k, \quad F_j(0) = F_{j0} \equiv \mathrm{const}, \tag{59}$$

where constants $V_{j0}^k, A_{j0}^k, H_{j0}^k$ are the coefficients of the expansions of functions $V_{j0}(\xi), A_{j0}(\xi), H_{j0}(\xi)$ in terms of the shifted Jacobi polynomials $R_k^{(0,1)}(\xi)$.

4. Numerical solution of the non-linear problem

The system of ordinary differential equations of the first order was integrated numerically using the Runge-Kutta method of the fourth order of accuracy. Note that when using the tau method in order to ensure the exact fulfilment of the boundary conditions it is necessary to use a sufficient number of coefficients in the trial solution. In this case calculations were performed for $n = 10, 11, 12$ in Galerkin approximations. At the same time, with an increase in n a rapid increase in the accuracy of the solution is observed.

Some results of numerical solution are presented for the model system that consists of an aqueous solution of formic acid (mixture 1) – transformer oil (mixture 2). According to tabular data , the dimensionless parameters of the specified system are as follows

$$\mu = 0.0898, \quad \nu = 0.0649, \quad \chi = 1.4, \quad k = 2.41, \quad d = 0.0152,$$

$$\mathrm{Pr}_1 = 13.8, \quad \mathrm{Pr}_2 = 298, \quad \mathrm{Sc}_1 = 963, \quad \mathrm{Sc}_2 = 225,$$

$$\mathrm{Sr}_1 = 6, \quad \mathrm{Sr}_2 = 7, \quad \mathrm{Ma} = 20, \quad \mathrm{Mc} = 15.$$

The following parameter values were also used: $R = 1.5$, $E = 0.7$.

Fig. 2 shows the results of calculations of the velocity field. Function $V_j(\xi, \tau)$ and the radial velocity profile $U_j(\xi, \tau)$ are shown at various points in time. Analysing this result, we came to the conclusion that solution of a non-stationary problem with increasing time tends to the stationary mode obtained by solving the non-linear problem by the Newton method [10]. In turn, the pressure gradients $F_j(\tau)$ in the layers are stabilized with time and they converge to the values $F_1 = -1.78305$, $F_2 = -71.22054$. Calculations show that the pressure gradient in the second layer significantly exceeds the pressure gradient in the first layer in absolute value. This is because transformer oil is more viscous compared to the aqueous solution of formic acid. The greater is viscosity of the liquid the smaller is its mobility. Consequently, much greater pressure forces are required to cause movement in the second layer.

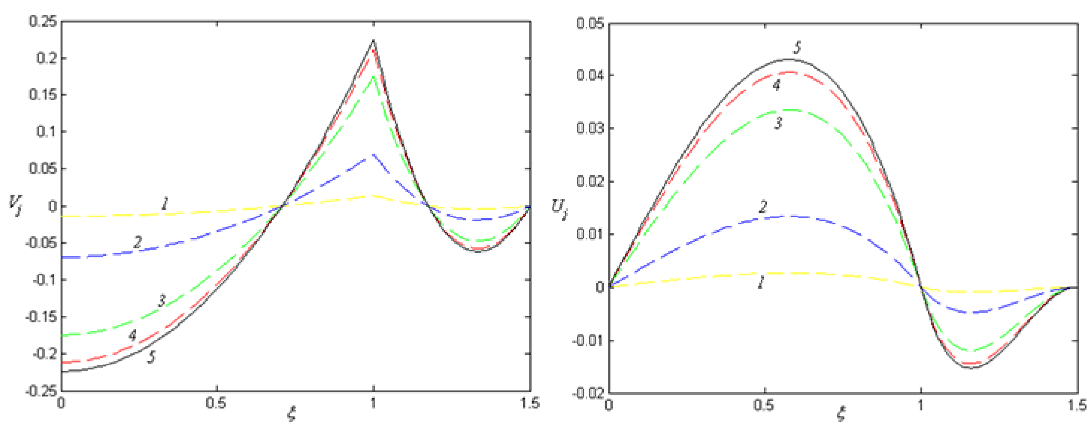

Fig. 2. Plots of functions $V_j(\xi, \tau)$ and $U_j(\xi, \tau)$ at various points in time: $1 - \tau = 0.08$, $2 - \tau = 0.4$, $3 - \tau = 1.21$, $4 - \tau = 1.42$, $5 - \tau = \infty$

It is of interest to consider how the change in the internal energy of the interface affects the flow pattern of binary mixtures. As a result of calculations it was found that with an increase in the energy parameter E at a fixed τ the absolute values of function $V_j(\xi, \tau)$ decrease (see Fig. 3). One can be concluded that the effects associated with the heat of formation of the inter-facial surface contribute to a decrease in the intensity and laminarization of the flow near this surface.

Note that function $V_j(\xi, \tau)$ when passing through zero on the intervals $0 < \xi < 1$ and $1 < \xi < 1.5$ changes sign. This means that flows of binary mixtures change the direction of

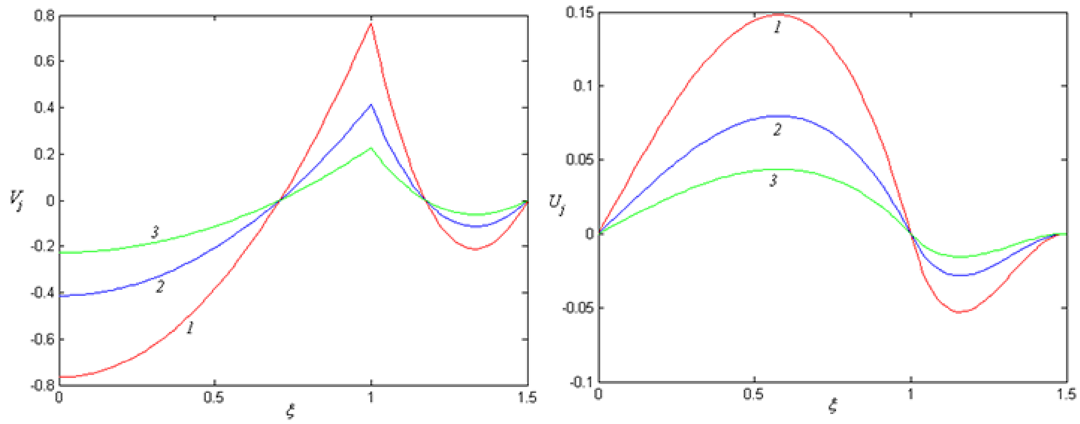

Fig. 3. The relationship between functions $V_j(\xi,\tau)$, $U_j(\xi,\tau)$ and parameter E: 1 – $E = 0.05$, 2 – $E = 0.2$, 3 – $E = 0.7$

movement. Thus, return flow zones appear in liquid layers near the interface. This happens not only due to the gradient of surface tension but also due to the non-stationary pressure drop in the layers that occurs during heating.

Let us consider the obtained numerical results for other functions. Due to the formation of heat function $A_j(\xi,\tau)$ increases in both layers. As for "concentration", function $H_j(\xi,\tau)$ decreases (see Fig. 4). One should take into account the Soret number Sr_j. This dimensionless parameter has a great impact on the concentration distribution in mixtures. Depending on the thermal diffusion coefficient α_j, the Soret number can be either positive or negative. If the Soret number for both mixtures is negative then the directions of the temperature gradient and the diffusion mass flow are the same. As a result, light components move to the more heated area, and the heavy ones stay in areas with reduced temperature. This corresponds to the phenomenon of normal thermal diffusion. For positive Soret numbers, the direction of movement of components changes to the opposite. At the same time, the corresponding effect of thermal diffusion is abnormal. The results of numerical calculation allow one to conclude that abnormal thermal diffusion takes place in this model.

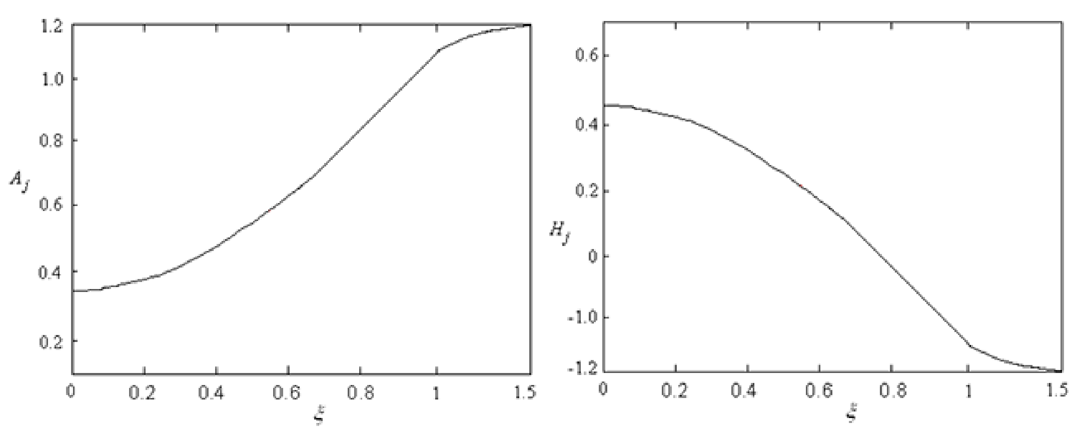

Fig. 4. Plots of functions $A_j(\xi,\tau)$, $H_j(\xi,\tau)$ at fixed τ

Conclusion

A study of the unsteady two-layer flow of binary mixtures in a cylindrical capillary was carried out with consideration for changes in internal energy on the inter-facial surface. The resulting conjugate initial-boundary value problem is non-linear and inverse with respect to pressure gradients along the axis of the cylindrical capillary. To solve the problem the tau method was used. Shifted Jacobi polynomials were taken as basis functions. As a result, the system of integro-differential equations was reduced to a closed system of ordinary differential equations of the first order. To solve the system of equations the Runge-Kutta method of the fourth order was used. It was shown that with increasing time the solution of the non-stationary problem tends to the stationary mode. As a result of calculations for the model problem it was found that when energy parameter increases the characteristic convection velocity changes and intensity decreases. The increase of the energy parameter also promotes laminarization of the flow near the inter-facial surface.

The work was supported by the Russian Foundation for Basic Research (grant no. 20-01-00234).

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Нестационарное течение двух бинарных смесей в цилиндрическом капилляре с учетом изменений внутренней энергии поверхности

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Аннотация. Исследуется задача о двумерном нестационарном течении двух несмешивающихся несжимаемых бинарных смесей в цилиндрическом капилляре в отсутствие массовых сил. Смеси контактируют через общую поверхность раздела, на которой учитывается энергетическое условие. Температура и концентрация в них распределены по квадратичному закону, что хорошо согласуется с полем скоростей типа Хименца. Возникающая сопряженная начально-краевая задача является нелинейной и обратной относительно градиентов давлений вдоль оси цилиндрической трубки. Для ее решения применяется тау-метод. Показано, что с ростом времени численное решение нестационарной задачи выходит на стационарный режим. Установлено, что влияние приращений внутренней энергии межфазной поверхности существенно сказывается на динамике течения смесей в слоях.

Ключевые слова: нестационарное решение, бинарная смесь, поверхность раздела, энергетическое условие, внутренняя энергия, обратная задача, градиент давления, тау-метод, термодиффузия.

DOI: 10.17516/1997-1397-2022-15-5-635-640
УДК 512.554

Maximal Abelian Ideals and Automorphisms of Nonfinitary Nil-triangular Algebras

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Received 24.12.2021, received in revised form 15.04.2022, accepted 25.06.2022

Abstract. We study mutually connected automorphisms and abelian ideals of nonfinitary nil-triangular algebras.

Keywords: Chevalley algebra, nil-triangular subalgebra, unitriangular group, finitary and nonfinitary generalizations, automorphisms, abelian ideal.

Citation: E.A. Sotnikova, Maximal Abelian Ideals and Automorphisms of Nonfinitary Nil-triangular Algebras, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 635–640.
DOI: 10.17516/1997-1397-2022-15-5-635-640.

Introduction

We study mutually connected automorphisms and abelian ideals of nonfinitary nil-triangular algebras.

Choose an arbitrary chain (or a linearly ordered set) Γ with the order relation $\leqslant$. The matrix product in the module $M(\Gamma, K)$ of all Γ-matrices $\alpha = ||a_{ij}||_{i,j\in\Gamma}$ over K is not defined for an infinite chain Γ. However, the submodule $NT(\Gamma, K)$ of nil-triangular Γ-matrices is an algebra exactly when $\Gamma = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}\setminus\mathbb{N}$. [1–3]. See Sec. 1.

The Chevalley algebra over a field K is characterized by a root system Φ and a Chevalley basis consisting of elements e_r ($r \in \Phi$) and a base of suitable Cartan subalgebra, [4, Sec. 4.4]. The subalgebra $N\Phi(K)$ with the basis $\{e_r \mid r \in \Phi^+\}$ is said to be a nil-triangular subalgebra. The root automorphisms of the subalgebra $N\Phi(K)$ generate a unipotent subgroup $U\Phi(K)$ of the Chevalley group of type Φ over K, [5]. For nil-triangular subalgebras of Chevalley algebras classical types the nonfinitary generalizations $NG(K)$ of types $G = B_{\mathbb{N}}$, $C_{\mathbb{N}}$ and $D_{\mathbb{N}}$ were constructed in [1].

R. Slowik [6] investigated the automorphisms of the limit unitriangular group $UT(\infty, K)$ over a field K of order >2. By [2], this group was represented as the group $UT(\mathbb{N}, K)$ and also as the adjoint group of radical ring $NT(\mathbb{N}, K)$. For any ring K which has no zero divisors, the automorphism group of the associated Lie ring of $NT(\mathbb{N}, K)$ (i.e. type $A_{\mathbb{N}}$) were described in [2]; also, it coincides with the automorphism group of the adjoint group ($\simeq UT(\mathbb{N}, K)$).

Let K be an arbitrary domain. Our main Theorem 2.1 in Sec. 2 describes maximal abelian ideals of algebras $NG(K)$ of types $G = B_{\mathbb{N}}$, $C_{\mathbb{N}}$ and $D_{\mathbb{N}}$. See also a reduction of automorphisms in Sec. 3.

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1. Preliminaries

Unless otherwise specified, K denotes an associative commutative ring with a (nonzero) unity.

The Chevalley algebra over a field K is characterized by a root system Φ and a Chevalley basis consisting of elements e_r $(r \in \Phi)$ and a base of suitable Cartan subalgebra, [4, Sec. 4.4].

The subalgebra $N\Phi(K)$ with the base $\{e_r \mid r \in \Phi^+\}$ is said to be a nil-triangular subalgebra. For the type A_{n-1} it is represented by the Lie algebra $NT(n, K)$ of all nil-triangular $n \times n$-matrices over K with the matrix units e_{iv} $(1 \leqslant v < i \leqslant n)$.

In [5], the subalgebra $N\Phi(K)$ of other classical types B_n, C_n and D_n was represented by special matrices with a base of 'matrix units' e_{iv} with restrictions, respectively,

$$-i < v < i \leqslant n, \quad -i \leqslant v < i \leqslant n, \ v \neq 0, \quad 1 \leqslant |v| < i \leqslant n.$$

After appropriate numbering of roots $r = r_{iv}$ we obtain $e_r = e_{iv}$ and

$$e_{ij} * e_{uv} = 0 \quad (i \neq v, \ j \neq u, j \neq -v). \tag{1}$$

We represent any element $\alpha \in N\Phi(K)$ by the sum $\alpha = \sum a_{iv}e_{iv}$ and by Φ^+- matrix $||a_{iv}||$ which corresponds to the types. For example, the B_n^+- matrix has the form

$$\begin{array}{ccccccc} & & & a_{10} & & & \\ & & a_{2,-1} & a_{20} & a_{21} & & \\ & & \cdots & \cdots & \cdots & & \\ a_{n,-n+1} & \cdots & a_{n,-1} & a_{n0} & a_{n1} & \cdots & a_{n,n-1}, \end{array}$$

Removing the zero column, we obtain the D_n^+- matrix.

The following lemma is proved in [5, Lemma 1.1].

Lemma 1.1. *The signs of the structural constants of the Chevalley basis can be chosen so that we have $e_{ij} * e_{jv} = e_{iv}$ and (1), and also*

$$\Phi = B_n, D_n : \quad e_{jv} * e_{i,-v} = e_{i,-j} \quad (i > j >| v |> 0);$$

$$\Phi = C_n : \ e_{jm} * e_{i,-m} = e_{im} * e_{j,-m} = e_{i,-j} \quad (i > j > m \geqslant 1);$$

$$\Phi = B_n : \ e_{i0} * e_{j0} = 2e_{i,-j}, \quad \Phi = C_n : \ e_{ij} * e_{i,-j} = 2e_{i,-i} \quad (i > j \geqslant 1).$$

Now we choose an arbitrary chain (a linearly ordered set) Γ with the order relation $\leqslant$. All Γ-matrices $\alpha = ||a_{ij}||_{i,j\in\Gamma}$ over K with the usual multiplication by scalars from K and matrix addition form a K-module $M(\Gamma, K)$. For an infinite chain Γ, matrix multiplication in the module $M(\Gamma, K)$ is not defined. Denote by $\mathbb{N}$ a chain of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \cdots, n\}$ (or $\mathbb{N} = \{0, 1, 2, 3, \cdots, n\}$).

It is known ([1–3]) that the submodule $NT(\Gamma, K)$ of all nonfinitary (low) nil-triangular Γ-matrices with the usual matrix multiplication is an algebra if and only if $\Gamma = \mathbb{Z}$, $\mathbb{N}$ or $\mathbb{Z} \setminus \mathbb{N}$. Algebras $NG(K)$ for Lie type A_Γ (i.e. $NT(\Gamma, K)$) were studied in [1].

Also, they had been constructed for classical types $G = B_\mathbb{N}$, $C_\mathbb{N}$, and $D_\mathbb{N}$ in [1,7]. The matrix units $e_{im} \in NG(K)\,(i, m \in \Gamma)$ is determined with restrictions, respectively,

$$-i < m < i, \quad -i \leqslant m < i, \ m \neq 0, \quad 1 \leqslant |m| < i.$$

Note that each unit $e_{im} \in NG(K)$ is associated to the ideal

$$T_{im} = \langle \alpha = ||a_{uv}|| \in NG(K) \mid a_{uv} = 0 \ \text{ if } \ u < i \ \text{ or } \ v > m \rangle.$$

For any ideal J of K the congruence subring $NG(K, J)$ of all matrices over J from $NG(K)$ is determined, as usual.

2. Main Theorem

According to [1] and [2], the maximal abelian ideals and automorphisms of the Lie ring $NA_\Gamma(K)$ were described at $\Gamma = \mathbb{N}$. The aim of this section is to describe the maximal abelian ideals of nonfinitary algebras $NG(K)$ of other classical types $G = B_\mathbb{N}$, $C_\mathbb{N}$ and $D_\mathbb{N}$.

By Lemma 1.1, T_{im} for $m < 0$ is always an abelian ideal. Another way of constructing abelian ideals is known for type $D_\mathbb{N}$. Note that the centralizer of the ideal T_{21} in Lie ring $ND_\mathbb{N}(K)$ is $C(T_{21}) = T_{3,-2}$. Also, for any pair $(a, b) \neq (0, 0)$ over K and $t \in K$ the elements

$$ae_{i,-1} + be_{i1}, \ t(ae_{j,-1} + be_{j1}) \in ND_\mathbb{N}(K)$$

commute when $2ab = 0$. Thus, we obtain the abelian ideal

$$\mathcal{M}(K, a, b) = T_{3,-2} + \sum_{i\in\mathbb{N}} K(ae_{i,-1} + be_{i1}).$$

For any domain K we denote its field of fractions by Q_K.

Theorem 2.1. *Let M be a maximal abelian ideal of the Lie ring $NG(K)$ over a domain K. Then the following statements are fulfilled.*

i) $G = B_\mathbb{N}$: $M = T_{10}$ *for* $2K = 0$ *and* $M = T_{2,-1}$ *for* $2K \neq 0$.

ii) $G = C_\mathbb{N}$: $M = T_{1,-1}$.

iii) $G = D_\mathbb{N}$: $M = ND_\mathbb{N}(K) \cap \mathcal{M}(Q_K, a, b)$ *for* $(a, b) \neq (0, 0)$.

Proof. Any matrix from M can be correctly represented as a sum (possibly infinite) of elementary matrices. In this sense, we can assume that the ideal M is generated by elementary matrices.

Let $\alpha = \|a_{uv}\| \in NG(K)$. Denote by T_α, the principal ideal (α) of $NG(K)$. We need

Lemma 2.1. *Let a_{uv} be a nonzero element and either $v \geqslant 1$ or $G = D_\mathbb{N}$, $v > 1$. Let J be the principal ideal Ka_{uv} of K. Then $T_{m,v} \cap NG(K, J) \subset T_\alpha$ for all $m \geqslant u + 2$.*

Proof. It is sufficiently to prove the case of the matrix α with zeros all coordinates which are $(u + 2, v)$–coordinates above or to the right. The multiplications of α by $Ke_{u+1,u}$ and then by $Ke_{u+2,u+1}$ give matrices α_1 having zeros all rows with numbers $\neq (u + 2)$ and their $(u + 2, v)$–coordinates run ideal $J = Ka_{uv}$.

Further, multiplying the matrix α_1 by the elements $e_{vm}\,(m < v)$ in succession, we obtain inclusion $Je_{u+2,p} \subset T_\alpha$ and hence $Je_{m,p} \subset T_\alpha$ for all $p < v$ and $m \geqslant u + 2$. Thus, we arrive at the inclusion in T_α of the congruence subring $T_{mv} \cap NG(K, J)$ required in the lemma of each ideal T_{mv} in $NG(K)$ for $m \geqslant u + 2$. Lemma 2.1 is proved. □

By Lemma 1.1, all enumerated in i) – iii) ideals are abelian.

Let M be an arbitrary maximal abelian ideal in $NG(K)$. Assume that there is a matrix $\alpha = \|a_{uv}\| \in M$ with at least one non-zero coordinate a_{uv} for $v \geqslant 1$. By Lemma 2.1, the principal ideal $T_\alpha = (\alpha)$ contains intersections $T_{mv}(J) := T_{mv} \cap NG(K, J)$ for $m \geqslant u + 2$.

For algebras $NC_{\mathbb{N}}(K)$ the condition that the ideal $T_{mv}(J)$ be abelian is obviously equivalent to the condition $a_{uv}^2 = 0$. When K is a domain, we obtain the equality $a_{mv} = 0$. The obtained contradiction proves the inclusion $M \subset T_{1,-1}$.

Consider the case $G = B_{\mathbb{N}}$. We prove the inclusion $M \subset T_{10}$. Assume the opposite: M contains a matrix α with nonzero coordinate a_{uv} for $v > 0$. By Lemma 2.1, by analogy with the type of $C_{\mathbb{N}}$, we obtain the equality $a_{uv}^2 = 0$, contradicting the choice of principal ring K.

Now suppose that $M \subset T_{10}$ and α exists in M with nonzero (u,0)-coordinate a. By shifting the u-th row of the matrix (as in the proof of Lemma 2.1), we find the matrices β, $\gamma \in M$ with conditions:

$$\beta = ae_{i0} \ mod \ T_{2,-1}, \quad \gamma = ae_{s0} \ mod \ T_{2,-1} \quad (s > i > u+1).$$

From Lemma 1.1 it follows that equality $\gamma * \beta = 2a^2 e_{s,-i}$ and, since M is abelian, the equality $2a^2 = 0$. This is possible only for $2K = 0$ in which case $M = T_{10}$. For $2K \neq 0$, the centralizer of the ideal T_{10} is equal to $C(T_{10}) = T_{2,-1}$, whence $M = T_{2,-1}$.

For the type $G = D_{\mathbb{N}}$ (by analogy with the type $C_{\mathbb{N}}$) by Lemma 2.1 we obtain inclusion $M \subset T_{21}$. By the equality $C(T_{21}) = T_{3,-2}$ from § 1, we obtain inclusion $T_{3,-2} \subset M$ and we find the matrix $\alpha = \|a_{uv}\|$ in M with the pair $(a_{i,-1}, a_{i1}) \neq (0,0)$. Then the ideal M contains

$$K(ae_{m,-1} + be_{m1}) \quad (a := a_{i,-1},\, b := a_{i1})$$

for all $m > i$. Since M is abelian, we immediately obtain the conditions $2ab = 0$. When the domain K is a field, i.e. coincides with the field of fractions Q_K, the proof is complete.

When K is a proper subring in its field of fractions Q_K, we construct abelian ideals $\mathcal{M}(Q_K, a, b)$ for various $a, b \in Q_K$. Then the intersections

$$M = ND_{\mathbb{N}}(K) \cap \mathcal{M}(Q_K, a, b)$$

give all maximal abelian ideals of the Lie ring $ND_{\mathbb{N}}(K)$.

Therefore, Theorem 2.1 is completely proved. □

Example 2.1 Suppose a ring K has a nonzero element x with $x^2 = 0$. Choose a principal ideal $J = Kx$. (For example, $K = Z_n$ is the residue ring of integers modulo n, where n is a multiple of 4.) Then the congruence subring $NG(K, J)$ in $NG(K)$ is an abelian ideal that do not belong to any of the ideals T_{im} $(i > m)$.

3. Remark on the reduction of automorphisms

It is clear that an automorphism of a ring always induces an automorphism of a quotient ring with respect to the characteristic ideal.

As a corollary of Theorem 2.1 we easily obtain

Proposition 3.1. *When K is a domain, the ideal T_{10} is characteristic in the Lie ring $NB_{\mathbb{N}}(K)$. The ideals $T_{1,-1}$ and T_{21} are characteristic in the Lie rings $NC_{\mathbb{N}}(K)$ and $ND_{\mathbb{N}}(K)$ respectively.*

Proof. In the Lie ring $NB_{\mathbb{N}}(K)$ the ideal T_{10} is characteristic for $2K = 0$ as the only maximal abelian ideal, and for $2K \neq 0$ as the centralizer $C(T_{2,-1}) = T_{10}$ of the characteristic ideal $T_{2,-1}$. The ideal $T_{1,-1}$ is the only (and therefore characteristic) maximal abelian ideal in the Lie ring $NC_{\mathbb{N}}(K)$.

By Theorem 2.1, the ideal T_{21} in the Lie ring $ND_{\mathbb{N}}(K)$ generates all maximal abelian ideals; the ideal $T_{2,-1}$ and its image with respect to the graph automorphism are sufficient. Hence, the ideal T_{21} and its centralizer $C(T_{21}) = T_{3,-2}$ are characteristic. This finishes the proof of the proposition. $\square$

Note the following isomorphisms:

$$NB_{\mathbb{N}}(K)/T_{10} \simeq NB_{\mathbb{N}}(K)/T_{2,-1} \simeq NT(\mathbb{N}, K),$$

$$NC_{\mathbb{N}}(K)/T_{1,-1} \simeq NT(\mathbb{N}, K),$$

$$ND_{\mathbb{N}}(K)/T_{21} \simeq NT(\mathbb{N}, K).$$

The automorphisms of nonfinitary Lie rings $NG(K)$ of type $G = A_{\mathbb{N}}$, i.e., Lie rings $NT(\mathbb{N}, K)$, were interconnectedly studied with maximal abelian ideals earlier in [1, 7]. Along with the standard automorphisms, for Lie rings $NT(\Gamma, K)$ single out hypercentral and other non-standard automorphisms, [8–10].

In [2], standard automorphisms of Lie rings $NT(\mathbb{N}, K)$ are proved. On the other hand, for the Lie algebra $ND_{\mathbb{N}}(K)$ there are nonstandard automorphisms generalizing graph automorphisms.

In the group $SL(2, K)$ we choose a subgroup

$$S = \{A = ||a_{uv}|| \in SL(2, K) \mid 2a_{11}a_{12} = 2a_{21}a_{22} = 0\}.$$

Obviously, $S = SL(2, K)$ for $2K = 0$. By analogy with [10, 11] for any matrix $A = ||a_{uv}|| \in S$, we associate the automorphism $\widetilde{A}$ of the Lie algebra $ND_{\mathbb{N}}(K)$ according to the rule

$$\widetilde{A} : e_{2,-1} \to a_{11}e_{2,-1} + a_{12}e_{2,1}, \quad e_{2,1} \to a_{21}e_{2,-1} + a_{22}e_{2,1}, \quad e_{i+1,i} \to e_{i+1,i} \quad (i = 3, 4, \cdots). \quad (2)$$

The reduction of automorphisms leads to the following theorem.

Theorem 3.1. *Every automorphism of a nonfinitary algebra $NG(K)$ over a domain K is a standard automorphism of type $G = A_{\mathbb{N}}, B_{\mathbb{N}}$ or $C_{\mathbb{N}}$, and for type $G = D_{\mathbb{N}}$ there is a product of the suitable automorphism $\widetilde{A}$ and the standard automorphism.*

Remark 3.1 Theorem 2.1 and Theorem 3.1 transfer to Lie rings $NG(K)$ of classical types $G = A_{\Gamma}, B_{\Gamma}, C_{\Gamma}, D_{\Gamma}$ for $\Gamma = Z \setminus \mathbb{N}$; they are antiisomorphic to the Lie rings studied for $\Gamma = \mathbb{N}$.

The author expresses his gratitude to V. M. Levchuk for guidance and important corrections. This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement no. 075-02-2022-876).

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Максимальные абелевы идеалы и автоморфизмы нефинитарных нильтреугольных алгебр

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Аннотация. Для нефинитарных обобщений нильтреугольных подалгебр алгебр Шевалле в статье взаимосвязано исследуются автоморфизмы и максимальные абелевы идеалы.

Ключевые слова: алгебра Шевалле, нильтреугольная подалгебра, унитреугольная группа, финитарные и нефинитарные обобщения, автоморфизмы, абелев идеал.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 641–644

DOI: 10.17516/1997-1397-2022-15-5-641-644
УДК 512.5

Dehn Functions and the Space of Marked Groups

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Received 10.09.2022, received in revised form 10.11.2022, accepted 20.12.2022

Abstract. In the space of marked group, we suppose that a sequence (G_i, X_i) converges to (G, X), where G is finitely presented. We obtain an inequality which connects Dehn functions of G_is and G.

Keywords: space of marked groups, Gromov-Grigorchuk metric, finitely presented groups, Dehn functions.

Citation: M. Shahryari, Dehn Functions and the Space of Marked Groups, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 641–644. DOI: 10.17516/1997-1397-2022-15-5-641-644.

In the space of marked groups, consider the situation a sequence (G_i, X_i) in which converges to a finitely presented marked group (G, X). What can we say about the relation between corresponding Dehn functions of the groups G_i and G? Suppose $\Gamma_i = \langle X_i | R_i \rangle$ is an arbitrary presentation for G_i and $\Gamma = \langle X | R \rangle$ is an arbitrary finite presentation for G. Let $L = \max_{r \in R} \|r\|$. We will prove that for any $n \geqslant 0$,

$$\limsup_i \frac{\delta_i(n)}{\delta_i(L)} \leqslant \delta(n).$$

Here of course, δ_i is the Dehn function of G_i corresponding to Γ_i. Also δ is the Dehn function of G corresponding to a finite presentation Γ. As a result, it shows that if the set $\{\delta_i(L) : i \geqslant 1\}$ is finite, then so is the set $\{\delta_i(n) : i \geqslant 1\}$, for all $n \geqslant 0$.

1. Basic notions

The idea of Gromov-Grigorchuk metric on the space of finitely generated groups is proposed by M. Gromov in his celebrated solution to the Milnor's conjecture on the groups with polynomial growth (see [5]). For a detailed discussion of this metric, the reader can consult [2]. Here, we give some necessary basic definitions. A marked group (G, X) consists of a group G and an m-tuple of its elements $X = (x_1, \ldots, x_m)$ such that G is generated by X. Two marked groups (G, X) and (G', X') are *the same*, if there exists an isomorphism $G \to G'$ sending any x_i to x'_i. The set of all such marked groups is denoted by $\mathcal{G}_m$. This set can be identified by the set of all normal subgroup of the free group $\mathbb{F} = \mathbb{F}_m$. Since the later is a closed subset of the compact topological space $2^{\mathbb{F}}$ (with the product topology), so it is also a compact space. This space is in fact metrizable: let B_λ be the closed ball of radius λ in $\mathbb{F}$ (having the identity as the center) with respect to its word metric. For any two normal subgroups N and N', we say that they are in distance at most $e^{-\lambda}$, if $B_\lambda \cap N = B_\lambda \cap N'$. So, if Λ is the largest of such numbers, then we can define

$$d(N, N') = e^{-\Lambda}.$$

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This induces a corresponding metric on $\mathcal{G}_m$. To see what is this metric exactly, let (G, X) be a marked group. For any non-negative integer λ, consider the set of *relations* of G with length at most λ, i.e.

$$\mathrm{Rel}_\lambda(G, X) = \{w \in \mathbb{F} : \|w\| \leqslant \lambda, w_G = 1\}.$$

Then $d((G, X), (G', X')) = e^{-\Lambda}$, where Λ is the largest number such that $\mathrm{Rel}_\Lambda(G, X) = \mathrm{Rel}_\Lambda(G', X')$. Note that we identify here X and X' using the correspondence $x_i \to x'_i$. This metric on $\mathcal{G}_m$ is the so called Gromov-Grigorchuk metric.

Many topological properties of the space $\mathcal{G}_m$ is discussed in [2]. In this note, we need just one elementary fact: any finitely presented marked group (G, X) in $\mathcal{G}_m$ has a neighborhood, every element in which is a quotient of G.

We also need to review some basic notions concerning *Dehn* and *isoperimtry* functions. Let $\langle X|R\rangle$ be a presentation for a finitely generated group G (X is finite). Let $w \in \mathbb{F} = F(X)$ be a word such that $w_G = 1$. Clearly in this case w belongs to $\langle R^{\mathbb{F}}\rangle$, the normal closure of R in $\mathbb{F}$. Hence, we have

$$w = \prod_{i=1}^{k} u_i r_i^{\pm 1} u_i^{-1},$$

for some elements $u_1, \ldots, u_k \in \mathbb{F}$ and $r_1, \ldots, r_k \in R$. The smallest possible k is called the *area* of w and it is denoted by $\mathrm{Area}_R(w)$. A function $f : \mathbb{N} \to \mathbb{R}$ is an *isoperimetric* function for the given presentation, if for all $w \in \mathbb{F}$, with $w_G = 1$, we have

$$\mathrm{Area}_R(w) \leqslant f(\|w\|).$$

The corresponding Dehn function is the smallest isoperimetric function, i.e.

$$\delta(n) = \max\{\mathrm{Area}_R(w) : w \in \mathbb{F}, w_G = 1, \|w\| \leqslant n\}.$$

This function measures the complexity of the word problem in the case of finitely presented group G: the word problem for the presentation $\langle X|R\rangle$ is solvable, if and only if, the corresponding Dehn function is recursive. In fact the recursive Dehn functions measures the time complexity of fastest non-deterministic Turing machine solving word problem of G (see [3] and [7]). Also in the case of finitely presented groups, the *type* of Dehn function is a quasi-isometry invariant of groups. Although, in this note, we use the exact values of Dehn function, we give the definition of *type*. Let $f, g : \mathbb{N} \to \mathbb{N}$ be two arbitrary functions. We say that f is *dominated* by g, if there exists a positive number C, such that for all n,

$$f(n) \leqslant Cg(Cn + C) + Cn + C.$$

We denote the domination by $f \preceq g$. These two functions are said to be equivalent, if $f \preceq g$ and $g \preceq f$. The type of a Dehn function is its equivalence class with respect to this relation. Two Dehn functions of a fixed group with respect to different finite presentations have the same type. There are many classes of finitely presented groups having Dehn functions of type n^α for a dense set of exponents $\alpha \geqslant 2$ (see [1]). Hyperbolic groups are the only groups having linear type Dehn functions. Olshanskii proved that there is no group with Dehn function of type n^α with $1 < \alpha < 2$ (see [6]). For a study of Dehn functions of non-finitely presented groups, the reader can consult [4].

2. Main results

We work within the space of marked groups $\mathcal{G}_m$.

Theorem 1. *Let (G_i, X_i) be a sequence converging to (G, X), where G is finitely presented. Then for any n, we have*

$$\limsup_i \frac{\delta_i(n)}{\delta_i(L)} \leqslant \delta(n),$$

where δ_i is any Dehn function of G_i and δ is the Dehn function of G corresponding to any finite presentation $\Gamma = \langle X|R\rangle$ and $L = \max_{r\in R}\|r\|$.

Proof. As G is finitely presented, we may assume that all G_i is a quotient of G. We also identify X_i by X using the obvious correspondence. Let $\mathbb{F} = F(X)$ be the free group on X and assume that $w \in \mathbb{F}$. Suppose that $w_G = 1$ and $l = \mathrm{Area}_R(w)$. Then we have

$$w = \prod_{j=1}^{l} a_j r_j^{\pm 1} a_j^{-1},$$

where $a_1, \ldots, a_l \in \mathbb{F}$ and $r_1, \ldots, r_l \in R$. We know that $(r_j)_{G_i} = 1$, for all i and j, hence

$$r_j = \prod_{t_j=1}^{l_{ij}} u_{it_j} r_{it_j}^{\pm 1} u_{it_j}^{-1},$$

where $l_{ij} = \mathrm{Area}_{R_i}(r_j)$, $r_{i1}, \ldots, r_{il_{ij}} \in R_i$ and $u_{i1}, \ldots, u_{il_{ij}} \in \mathbb{F}$. Therefore, we have

$$w = \prod_{j=1}^{l} a_j (\prod_{t_j=1}^{l_{ij}} u_{it_j} r_{it_j}^{\pm 1} u_{it_j}^{-1})^{\pm 1} a_j^{-1} = \prod_{j=1}^{l}\prod_{t_j=1}^{l_{ij}} a_j u_{it_j} r_{it_j}^{\mp 1} u_{it_j}^{-1} a_j^{-1}.$$

This shows that

$$\mathrm{Area}_{R_i}(w) \leqslant \sum_{j=1}^{l} l_{ij} = \sum_{j=1}^{l} \mathrm{Area}_{R_i}(r_j).$$

Suppose $K_i = \max_{r\in R}(\mathrm{Area}_{R_i}(r))$. Then, we have

$$(*) \quad \mathrm{Area}_{R_i}(w) \leqslant K_i \cdot \mathrm{Area}_R(w).$$

Now, let $n \geqslant 1$. There exists an integer i_0 such that for any $i \geqslant i_0$, we have

$$d((G_i, X_i), (G, X)) \leqslant e^{-n}.$$

This shows that $\mathrm{Rel}_n(G_i, X_i) = \mathrm{Rel}_n(G, X)$, for $i \geqslant i_0$. In other words

$$\{w \in \mathbb{F} :\ \|w\| \leqslant n, w_{G_i} = 1\} = \{w \in \mathbb{F} :\ \|w\| \leqslant n, w_G = 1\}.$$

By $(*)$ and by the definition of Dehn function, we conclude $\delta_i(n) \leqslant K_i \cdot \delta(n)$. Hence, for $i \geqslant i_0$, we have

$$\frac{\delta_i(n)}{K_i} \leqslant \delta(n),$$

and therefore

$$\sup_{i\geqslant i_0} \frac{\delta_i(n)}{K_i} \leqslant \delta(n).$$

For any j, define

$$a_j(n) = \sup_{i\geqslant j} \frac{\delta_i(n)}{K_i} \leqslant \delta(n),$$

which a decreasing sequence in j. Since $a_{i_0}(n) \leqslant \delta(n)$, so $\lim_j a_j(n) \leqslant \delta(n)$. This shows that

$$\limsup_i \frac{\delta_i(n)}{K_i} \leqslant \delta(n).$$

Now, note that

$$K_i = \max_{r\in R} \mathrm{Area}_{R_i}(r) \;\leqslant\; \max_{r\in R, \|r\|=\|w\|} \mathrm{Area}_{R_i}(w) \;\leqslant\; \delta_i(L).$$

This completes the proof. □

As a result, we see that if the set $\{\delta_i(L) :\ i \geqslant 1\}$ is finite, then so is the set $\{\delta_i(n) :\ i \geqslant 1\}$, for all $n \geqslant 0$. This is because, if we put $M = \max_i \delta_i(L)$, then

$$\limsup_i \delta_i(n) \leqslant M \cdot \delta(n).$$

Now, if the second set is infinite, then the sequence $(\delta_i(n))_i$ has a divergent subsequence, which is a contradiction.

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Функции Дена и пространство отмеченных групп

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Аннотация. Предположим, что в пространстве отмеченной группы последовательность (G_i, X_i) сходится к (G, X), где G конечно представлена. Получаем неравенство, связывающее функции Дена G_is и G.

Ключевые слова: пространство отмеченных групп, метрика Громова–Григорчука, конечно определенные группы, функции Дена.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 645–650

DOI: 10.17516/1997-1397-2022-15-5-645-650
УДК 517.9

Local Asymptotic Normality of Statistical Experiments in an Inhomogeneous Competing Risks Model

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Received 30.03.2022, received in revised form 21.05.2022, accepted 08.07.2022

Abstract. In this paper we consider an inhomogeneous competing risks model. For the likelihood ratio statistics (LRS), proved the theorem on the locally asymptotic normality of statistical experiment.

Keywords: competing risks model, random censoring, local asymptotic normality, likelihood ratio statistics.

Citation: A. Abdushukurov, N. Nurmukhamedova, Local Asymptotic Normality of Statistical Experiments in an Inhomogeneous Competing Risks Model, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 645–650.
DOI: 10.17516/1997-1397-2022-15-5-645-650.

1. Introduction and preliminaries

In a inhomogeneous competing risks model (CRM) it's interesting to investigate the independent random variables (r.v.) $\{X_m,\, m \geqslant 1\}$ with a distribution function (d.f.) $\{H_m(x;\theta),\, m \geqslant 1\}$ with values in a measurable space $(\mathcal{X}_m, \mathcal{B}_m)$, $m \geqslant 1$, where $\mathcal{X}_m$ is set of possible values of r.v. X_m and $\mathcal{B}_m = \sigma(\mathcal{X}_m)$. D.f. of r.v. X_m also depends on the scalar parameter $\theta \in \Theta$, where Θ-parametrical space; is open set in R^1. In CRM with r.v. X_m pairwise disjoint and forming a complete group of events $\left\{A_m^{(1)}, \dots, A_m^{(k)}\right\}$ observed. We observe the sample of size *n:* $\left\{\left(X_m;\ \delta_m^{(0)}, \delta_m^{(1)}, \dots, \delta_m^{(k)}\right),\, m = 1, \dots, n\right\}$, where $\delta_m^{(i)} = I\left(A_m^{(i)}\right)$ is indicator of events $A_m^{(i)}$. Let $H_m^{(i)}(x;\theta) = P_\theta\left(X_m < x,\, \delta_m^{(i)} = 1\right)$ are subdistributions such that $H_m^{(1)}(x;\theta) + \dots + H_m^{(n)}(x;\theta) = H(x;\theta)$ for all $(x;\theta) \in R^1 \times \Theta$. By $h_m^{(i)}$ we define density of subdistributions $H_m^{(i)}$. Then note that there is a density $h_m^{(1)}(x;\theta) + \dots + h_m^{(k)}(x;\theta) = \dfrac{\partial H_m(x;\theta)}{\partial x} = h_m(x;\theta)$ for all $(x;\theta) \in R^1 \times \Theta$. We define $\Lambda_m^{(i)}(x;\theta) = \int\limits_{-\infty}^{x} (1 - H_m(u;\theta))^1\, dH_m(u;\theta)$ integral and $\lambda_m^{(i)}(x;\theta)$ density of failure rate of the i-th risk. Then

$$\lambda_m^{(1)}(x;\theta) + \dots + \lambda_m^{(k)}(x;\theta) = \frac{h_m(x;\theta)}{1 - H_m(x;\theta)},$$

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for all $(x;\theta) \in R^1 \times \Theta$.

We introduce the functional of the exponential intensity function [1,3]:

$$1 - F_m^{(i)}(x;\theta) = \exp(-\Lambda_m^{(i)}(x;\theta)),\ i = 1, \dots, k; \quad m = 1, \dots, n.$$

Note that this functional has all the properties of a d.f. [1].

By $f_m^{(i)}(x;\theta)$ we define density of $F_m^{(i)}(x;\theta)$. It is easy to see that

$$f_m^{(i)}(x;\theta) = \exp\left\{\Lambda^{(i)}(x;\theta)\right\} \frac{h_m^{(i)}(x;\theta)}{1 - H_m(x;\theta)},\ i = 1, \dots, k.$$

Denote by $\mathcal{F}_t^{(i)} = \sigma\left\{X_m I\left(X_m \leqslant t\right),\ \delta_m^{(i)} I\left(X_m \leqslant t\right),\ 1 \leqslant m \leqslant n\right\}$, flow of σ-algebra, generated by pairs of observations $\left\{\left(X_m, \delta_m^{(i)}\right),\ 1 \leqslant m \leqslant n\right\}$, $i = \overline{1,k}$. Let define a sequence of martingale processes for $m = \overline{1,\, n}$ and $i = \overline{1,k}$:

$$\mu_m^{(i)}(t) = I\left(X_m \leqslant t, \delta_m^{(i)} = 1\right) - \int_{-\infty}^{t} I\left(X_m > s\right) d\Lambda^{(i)}(s;\theta_0), \quad t \geqslant 0,$$

where θ_0 is true value of parameter θ, $\theta_0 \in \Theta$. These martingales have zero mea

$$E_{\theta_0}\mu_m^{(i)} = H_m^{(i)}(t;\theta_0) - \int_0^t \left(1 - H(s;\theta_0)\right) d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= H_m^{(i)}(t;\theta_0) - \int_0^t \frac{1 - H_m(s;\theta_0)}{1 - H_m(s;\theta_0)} dH_m^{(i)}(s;\theta_0) = H_m^{(i)}(t;\theta_0) - H_m^{(i)}(t;\theta_0) = 0,$$

$i = \overline{1,k};\ m = \overline{1,n}$. They are members of class $\mathcal{M}^2(\mathcal{F}_t^{(i)})$, i.e. square-integrable martingales with a predictable quadratic characteristics [4]:

$$< \mu_m^{(i)}, \mu_m^{(j)} >= \begin{cases} \Lambda^{(i)}(t), & i = j, \\ 0 \quad , & i \neq j. \end{cases}$$

Therefore, these martingales are orthogonal. According to this

$$\sum_{m=1}^{n} \mu_m^{(i)}(t) \in \mathcal{M}^2(\mathcal{F}_t^{(i)}), \quad \sum_{i=1}^{k}\sum_{m=1}^{n} \mu_m^{(i)}(t) \in \mathcal{M}^2(\mathcal{F}_t),$$

where $\mathcal{F}_t = \bigcap\limits_{i=1}^{k} \mathcal{F}_t^{(i)}$. To prove local asymptotic normality (LAN) we need some regularity conditions for $f_m^{(i)}(x;\theta)$:

(C1) The supports $N_{f_m^{(i)}} = \{x : f_m^{(i)}(x;\theta) > 0\}$ are independent from θ, $i = \overline{1,k};\ m = \overline{1,n}$;

(C2) There exist and are continuous for all x derivatives $\dfrac{\partial^l}{\partial\theta^l} f_m^{(i)}(x;\theta)$, $l = 1,2;\ i = \overline{1,k}$; $m = \overline{1,n}$;

(C3) Fisher information $\sum\limits_{m=1}^{n} I_m(\theta) = \sum\limits_{m=1}^{n}\sum\limits_{i=1}^{k} I_m^{(i)}(\theta)$ is finite and continuous in θ and positive at the point θ_0, where

$$I_m^{(i)}(\theta) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta}\log\lambda^{(i)}(s;\theta)\right)^2 dH_m^{(i)}(s;\theta).$$

Such a representation of Fisher information in terms of intensity density $\lambda_m^{(i)}(x;\theta)$ was established by the authors of [2]. Let $\varphi(n) = \left(\sum\limits_{m=1}^{n} I_m(\theta_0)\right)^{-1/2}$. Let for every $u \in R^1$, $\theta_n = \theta_0 + u\varphi(n) \in \Theta$. We have a likelihood ratio statistics (LRS) of the sample $Z^{(n)} = (X_1, \ldots, X_n)$: $dQ_{\theta_n}^{(n)}\left(Z^{(n)}\right)/dQ_{\theta_0}^{(n)}\left(Z^{(n)}\right)$, where

$$Q_\theta^{(n)}(Z^{(n)}) = p_n(Z^{(n)};\theta) = \prod_{m=1}^{n}\prod_{i=1}^{k}\left\{f_m^{(i)}(X_m;\theta)\cdot\prod_{\substack{j=1\\ j\neq i}}^{k}[1-F_m^{(i)}(X_m;\theta)]^{\delta_m^{(i)}}\right\}.$$

2. LAN for LRS.

The following theorem asserts LAN for LRS.

Theorem. *Under regularity conditions (C1)–(C3) for LRS we have representation*

$$\frac{dQ_{\theta_n}^{(n)}(Z^{(n)})}{dQ_{\theta_0}^{(n)}(Z^{(n)})} = \exp\left\{u\Delta_n(\theta_0) + \frac{u^2}{2} + R_n(u)\right\}, \tag{1}$$

where $\Delta_n = \varphi(n)\sum\limits_{m=1}^{n}\sum\limits_{i=1}^{k}\int\limits_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\lambda_m(s;\theta_0+u\varphi(n)\gamma)\right)d\mu_m^{(i)}(s)$, $0<\gamma<1$, $\mathrm{L}\left(\Delta_n/Q_{\theta_0}^{(n)}\right) \to N(0;1)$ *and* $R_n(u) \overset{Q_{\theta_0}^{(n)}}{\to} 0$ *at* $n\to\infty$.

Proof of Theorem. Let represent logarithm of LRS in terms of martingale-processes as follows:

$$\begin{aligned}
L_n(u) &= \log\left\{\frac{dQ_{\theta_n}^{(n)}(Z^{(n)})}{dQ_{\theta_0}^{(n)}(Z^{(n)})}\right\} = \log\left[\frac{p_n(Z^{(n)};\theta_n)}{p_n(Z^{(n)};\theta_0)}\right] = \\
&= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left\{\frac{f^{(i)}(X_m;\theta_n)\prod_{j=1}^{k}\left(1-F^{(j)}(X_m;\theta_n)\right)}{f^{(i)}(X_m;\theta_0)\prod_{j=1}^{k}\left(1-F^{(j)}(X_m;\theta_0)\right)}\right\} = \\
&= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left[\frac{\lambda_m^{(i)}(X_m;\theta_n)}{\lambda_m^{(i)}(X_m;\theta_0)}\right] + \sum_{m=1}^{n}\log\left[\frac{1-H_m(X_m;\theta_n)}{1-H_m(X_m;\theta_0)}\right] = \\
&= \sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left[\frac{\lambda_m^{(i)}(X_m;\theta_n)}{\lambda_m^{(i)}(X_m;\theta_0)}\right] + \sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n)-\Lambda_m(X_m;\theta_0)\right) = \qquad (2)\\
&= \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right]d\mu_m^{(i)}(s) + \\
&\quad + \left\{\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right]I(X_m>s)d\Lambda_m^{(i)}(s;\theta_0) - \right.\\
&\quad \left. - \sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n)-\Lambda_m(X_m;\theta_0)\right)\right\} = A_n(u) + R_n(u).
\end{aligned}$$

By the mean value theorem, from (2) we have

$$A_n(u) = \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left[\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)) - \log\lambda_m^{(i)}(s;\theta_0)\right]d\mu_m^{(i)}(s) =$$

$$= u\varphi(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\tfrac{\partial}{\partial\theta}\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)d\mu_m^{(i)}(s) = u\Delta_n(\theta_0),$$

where

$$\Delta_n(\theta_0) = \varphi(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\tfrac{\partial}{\partial\theta}\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)d\mu_m^{(i)}(s). \tag{3}$$

Integrating function in (3) is a continuous real-valued function of s, hence it is a predictable and is a square-integrable martingale: $\Delta_n \in \mathcal{M}^2(\mathcal{F}_t)$. We need to establish that

$$\mathcal{L}\left(\Delta_n(\theta_0)/Q_{\theta_0}^{(n)}\right) \to N(0;1). \tag{4}$$

It is easy to see that $E_{\theta_0}\Delta_n(\theta_0)=0$ (since $E_{\theta_0}\mu_m^{(i)}(t)=0$). Moreover, the quadratic variation of the martingale $\Delta_n(\theta_0)$ is

$$< \Delta_n(\theta_0), \Delta_n(\theta_0) >=$$

$$>= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2(1-H_m(s;\theta_0))\,d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= \left(\sum_{m=1}^{n} I_m(\theta_0)\right)^{-1}\sum_{m=1}^{n} I_m(\theta_0+u\varphi(n)\gamma) \to 1, \quad n\to\infty,$$

in view of condition (C3). Consider the Lindberg conditions

$$\varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right) d < \mu_m^{(i)}(t), \mu_m^{(j)}(t) >=$$

$$= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right)(1-H_m(s;\theta_0))d\Lambda_m^{(i)}(s;\theta_0) =$$

$$= \varphi^2(n)\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left(\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right)^2\times$$

$$\times I\left(\left|\frac{\partial\log\lambda_m^{(i)}(s;\theta_0+u\varphi(n)\gamma)}{\partial\theta}\right| > \varepsilon\varphi(n)\right) dH_m^{(i)}(s;\theta_0) \to 0, \quad n\to\infty,$$

where the convergence to zero of the integral follows from the requirement that the Fisher information $\sum\limits_{m=1}^{n} I_m(\theta)$ is finite in view of condition (C3). Consequently, weak convergence (4) follows from the central limit theorem for martingales [5]. Consider second addition in (2). Second addition in (2) converges in probability to $-\dfrac{u^2}{2}$: $R_n(u) \overset{Q_{\theta_0}^{(n)}}{\to} -\dfrac{u^2}{2}$, $n \to \infty$. Therefore, it remains to show that

$$R_n(u) + \frac{u^2}{2} \overset{Q_{\theta_0}^{(n)}}{\to} 0, \quad n \to \infty. \tag{5}$$

We investigate second addition inside the large curly bracket in (2). After elementary transformations, we have

$$\begin{aligned}
&\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[\frac{\lambda_m^{(i)}(s;\theta_n)}{\lambda_m^{(i)}(s;\theta_0)}\right] I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) - \sum_{m=1}^{n}\left(\Lambda_m(X_m;\theta_n) - \Lambda_m(X_m;\theta_0)\right) = \\
&\quad = \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left[1 + \frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)}\right] I(X_m > s)d\Lambda_m(s;\theta_0) + \\
&\quad\quad + \sum_{m=1}^{n}\int_{-\infty}^{+\infty}\left(\Lambda_m(X_m;\theta_n) - \Lambda_m(X_m;\theta_0)\right)dI(X_m > s) = \\
&\quad = \sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\log\left(1 + \frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)}\right) - \\
&\quad\quad - \int_{-\infty}^{\infty} I(X_m > s)\frac{\lambda_m^{(i)}(s;\theta_n) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0)}d\Lambda_m(s;\theta_0) = \\
&\quad = -\frac{1}{2}\sum_{m=1}^{n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left[\frac{\lambda_m^{(i)}(s;\theta_0 + u\varphi(n)\gamma^*) - \lambda_m^{(i)}(s;\theta_0)}{\lambda_m^{(i)}(s;\theta_0 + u\varphi(n)\gamma^*)}\right] I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) = \\
&\quad = -\frac{u^2\varphi^2(n)}{2}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\lambda(s;\theta_0 + u\varphi(n)\gamma^*)\right)^2 I(X_m > s)d\Lambda_m^{(i)}(s;\theta_0) = \\
&\quad = -\frac{u^2}{2}\left(\sum_{m=1}^{n} I_m(\theta_0)\right)^{-1}\sum_{m=1}^{n} I_m(\theta_0 + u\varphi(n)\gamma^*)\left(1 + o_p(1)\right) \to -\frac{u^2}{2}, \ \ n \to \infty,
\end{aligned} \tag{6}$$

where $0 < \gamma^* < 1$. Now (6) implies (5). The theorem is proved. □

Mutual contiguity of probability measures $Q_{\theta_n}^{(n)}$ and $Q_{\theta_0}^{(n)}$ follows from the theorem.

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Локальная асимптотическая нормальность статистических экспериментов в неоднородной модели конкурирующих рисков

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Аннотация. В статье рассматривается неоднородная модель конкурирующих рисков. Для статистики отношения правдоподобия доказана теорема о локально асимптотической нормальности статистического эксперимента.

Ключевые слова: модель конкурирующих рисков, случайное цензурирование, локальная асимптотическая нормальность, статистика отношения правдоподобия.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 651–662

DOI: 10.17516/1997-1397-2022-15-5-651-662
УДК 517.958

Two-dimensional Inverse Problem for an Integro-differential Equation of Hyperbolic Type

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Received 10.09.2021, received in revised form 10.05.2022, accepted 20.07.2022

Abstract. A multidimensional inverse problem of determining the kernel of the integral term of an integro-differential wave equation is considered. In the direct problem it is required to find the displacement function from the initial-boundary value problem. In the inverse problem it is required to determine the kernel of the integral term that depends on both the temporal and one spatial variable. Local unique solvability of the problem posed in the class of functions continuous in one of the variables and analytic in the other variable is proved with the use of the method of scales of Banach spaces of real analytic functions.

Keywords: integro-differential equation, inverse problem, delta function, integral equation, Banach theorem.

Citation: J.Sh. Safarov, Two-dimensional Inverse Problem for an Integro-differential Equation of Hyperbolic Type, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 651–662.
DOI: 10.17516/1997-1397-2022-15-5-651-662.

1. Introduction. Formulation of the problem

Let us consider the integro-differential equation

$$u_{tt} = \Delta u + \int_0^t k(x,\alpha)u(x,z,t-\alpha)\,d\alpha, \qquad x \in R,\ z \in (0,l),\ t \in R, \tag{1}$$

with initial conditions

$$u|_{t<0} = 0, \tag{2}$$

and boundary conditions

$$u_z|_{z=0} = \delta'(t), \qquad u_z|_{z=l} = 0. \tag{3}$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplace operator, $\delta'(t)$ is the derivative of Dirac delta function, $l > 0$ is a finite real number.

Finding function $u(x,z,t) \in D$ (from the class of generalized functions) for known $k(x,t)$ is called the direct problem. The inverse problem consists in determination of function $k(x,t) \in C(\Pi)$ with respect to the solution of the direct problem and

$$u(x,0,t) = g(x,t), \tag{4}$$

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where $g(x,t)$ is a given smooth function, $\Pi = \{(x,t) : x \in R,\ t > 0.\}$

One-dimensional inverse problems for the differential equations were studied in [1–5]. Inverse problem (1)–(4) is a multidimensional inverse problem for differential equations. The idea to extend the method of scales of Banach spaces of analytic functions developed by L.V. Ovsyannikov [6] and L. Nirenberg [7] to multidimensional inverse problems belongs to Romanov. This method was applied, with some modifications, to study local solvability of multidimensional inverse problems [8–10]. A similar problem was studied when $z > 0$ [11]. A special feature of this work is that equation (1) is studied in a bounded domain with respect to the variable z, i.e, $z \in (0,l)$. It is proved in this paper that formulated problem is locally uniquely solvable in the class of functions analytic with respect to the variable x.

2. Study of the direct problem

First, let us consider direct problem (1)–(3), that is, we assume that function $k(x,t)$ is known. In what follows, this problem is considered in the domain $B = R \times G$, where $G = \{(z,t) :\ 0 < z < l, 0 < t < 2l - z\}$ is a combination of areas B_1 and B_2. Areas B_1 and B_2 are described as follows

$$B_1 = R \times G_1, \quad G_1 = \{(z,t) :\ 0 < z < l,\ 0 < t < z\},$$

$$B_2 = R \times G_2, \quad G_2 = \{(z,t) :\ 0 < z < l,\ z < t < 2l - z\}.$$

Lemma 2.1. *Solution of equation (1) in domain B_1 with conditions (2), (3)*

$$u(x,z,t) \equiv 0.$$

Proof. Using the d'Alembert formula, we obtain in the region $B_0 \subset B_1$ the following integral equation

$$u(x,z,t) = \frac{1}{2} \iint\limits_{\Omega(z,t)} \left[u_{xx}(x,\xi,\tau) + \int_0^\tau k(x,\alpha) u(x,\xi,\tau-\alpha) d\alpha \right] d\xi d\tau,$$

where $\Omega(z,t) = \{(z,t) : z - t + \tau \leqslant \xi \leqslant z + t - \tau,\ 0 \leqslant \tau \leqslant t\}$, $B_0 = R \times G_0$, and $G_0 = \Big\{(z,t) :\ 0 < z < l,\ 0 < t \leqslant \dfrac{l}{2} - |z - \dfrac{l}{2}| \Big\}$.

Since the obtained equation is a homogeneous equation of the Volterra type of the second kind it has only zero solution.

Therefore, $u(x,z,t) \equiv 0$ in the domain G_0. Let us take an arbitrary point $(x,z,t) \in B_1 \backslash B_0$. Let us put the term u_{zz} in equation (1) to the left side and represent the wave operator $\left(\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial z^2}\right)$ as $\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial z}\right)$.

Integrating the obtained relation along the characteristic $dz/dt = 1$, from the point $(x, z-t, 0)$ to the point (x,z,t), we obtain

$$(u_t - u_z)(x,z,t) - (u_t - u_z)(x, z-t, 0) = \int_0^t \left[u_{xx}(x,z,\tau) + \int_0^\tau k(x,\tau-\alpha) u(x,\tau - t + x, \alpha) d\alpha \right] d\tau.$$

Using condition (2), we rewrite the last relation in the form

$$(u_t - u_x)|_{x=l} = \int_{\frac{t}{2}}^t \left[u_{xx}(x,z,\tau) + \int_0^\tau k(x,\tau-\alpha) u(x,\tau - t + x, \alpha) d\alpha \right] d\tau, \qquad t \in (0,l).$$

Further, using boundary condition (3) for $z = l$, we find

$$u(x,l,t) = \int_0^t \int_{\frac{\tau}{2}}^{\tau} \left[u_{xx}(x,\tau,\theta) + \int_0^{\theta} k(x,\theta-\alpha)u(x,\theta-\tau+l,\alpha)d\alpha \right] d\theta d\tau.$$

Changing variables in the inner integral by the formula $\theta - \tau + l = \xi$, we rewrite the last equation as

$$u(x,l,t) = \int_0^t \int_{l-\frac{\tau}{2}}^{l} \left[u_{xx}(x,\tau-l+\xi,\tau) + \int_0^{\tau-l+\xi} k(x,\tau-l+\xi-\alpha)u(x,\xi,\alpha)d\alpha \right] d\xi d\tau. \quad (5)$$

Integrating equation (1) along the characteristic $dz/dt = 1$ from the point $(x, z-t, 0)$ to the point (x,z,t),, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) u(x,z,t) = \int_{\frac{l+z-t}{2}}^{z} \left[u_{xx}(x,\xi,t) + \int_0^{\xi+t-z} k(x,\xi+t-z-\alpha)u(x,\xi,\alpha)d\alpha \right] d\xi.$$

Further, integrating this equality along the characteristic $dz/dt = -1$ from the point (x,z,t) to the point $(x,l,t+zl)$ and using (5), we find the equation for $u(x,z,t)$ in domain $B_1 \backslash B_0$

$$u(x,z,t) = \int_0^{t+z-l} \int_{l-\frac{\tau}{2}}^{l} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi+\tau-l} k(x,\xi+\tau-l-\alpha)u(\xi,\alpha,\tau)d\alpha \right] d\xi d\tau +$$

$$+ \int_{t+z-l}^{t} \int_{\frac{l+z-t-2\tau}{2}}^{-\tau+z+t} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi+2\tau-z-t} k(x,\xi+2\tau-z-t-\alpha)u(\xi,\alpha,\tau)d\alpha \right] d\xi d\tau,$$

where $(x,z,t) \in B_1 \backslash B_0$.

This equation is also a homogeneous equation of the Volterra type. Hence,

$$u(x,z,t) \equiv 0, \qquad (x,t) \in B_1 \backslash B_0$$

.

□

Using the d'Alembert formula in B_2 area, we obtain

$$u(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right) + \frac{1}{2}\left(\delta(t+z) - \delta(t-z)\right) +$$

$$+ \frac{1}{2} \int_0^z \int_{\tau-z+t}^{-\tau+z+t} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi-\tau} k(x,\alpha)u(x,\xi,\tau-\alpha)d\alpha \right] d\xi d\tau, \quad (x,z,t) \in B_2.$$

Let us introduce the function

$$\widetilde{u}(x,z,t) = u(x,z,t) - \frac{1}{2}\left(\delta(t-z) - \delta(t+z)\right). \quad (6)$$

For function $\widetilde{u}(x,z,t)$ we have the following equation

$$\widetilde{u}(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right) +$$

$$+\frac{1}{2} \int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\widetilde{u}_{xx}(x,\xi,\tau) + \frac{1}{2}k(x,\tau-\xi) + \int_0^{\tau-\xi} k(x,\alpha)\widetilde{u}(x,\xi,\tau-\alpha)d\alpha \right] d\tau d\xi, \quad (7)$$

where $(x,z,t) \in B_2$.

Substituting function (6) into equations (1)–(3) and equating the coefficients of the singularities as in [12–14], we obtain

$$\widetilde{u}(x,z,t)\mid_{t=z+0}=0. \tag{8}$$

Taking into account (8), we take the limit $t \to z+0$, in equation (7) and obtain

$$\frac{1}{2}\left(g(x,2z)+g(x,0)\right)=$$

$$=-\frac{1}{2}\int_0^z\int_\xi^{2z-\xi}\left[\widetilde{u}_{xx}(x,\xi,\tau)+\frac{1}{2}k(x,\tau-\xi)+\int_0^{\tau-\xi}k(x,\alpha)\widetilde{u}(x,\xi,\tau-\alpha)d\alpha\right]d\tau d\xi, \quad (x,z,t)\in B_2.$$

Let us differentiate the obtained relation with respect to z

$$g_t(x,2z)=-\int_0^z\left[\widetilde{u}_{xx}(x,\xi,2z-\xi)+\frac{1}{2}k(x,2z-2\xi)+\int_0^{2z-2\xi}k(x,\alpha)\widetilde{u}(x,z,2z-\xi-\alpha)d\alpha\right]d\xi$$

, where $(x,z,t)\in B_2$.

Let us differentiate this equation with respect to z once more. Making a preliminary change of variables in the second integral $2z-2\xi=\eta$ and solving the resulting equation for function $k(x,z)$, we find

$$k(x,z)=-4g_{tt}(x,z)-4\int_0^{\frac{z}{2}}\left[\widetilde{u}_{xxt}(x,\xi,z-\xi)+2\int_0^{z-2\xi}k(x,\alpha)\widetilde{u}(x,z,z-\xi-\alpha)d\alpha\right]d\xi, \tag{9}$$

where $(x,z,t)\in B_2$. To obtain the equation for $\widetilde{u}_t(x,z,t)$ we differentiate equation (7) with respect to t

$$\begin{aligned}\widetilde{u}_t(x,z,t)&=\frac{1}{2}\left(g_t(x,t+z)+g_t(x,t-z)\right)-\frac{1}{2}k(x,t-z)z+\\&+\frac{1}{2}\int_0^z\left[\widetilde{u}_{xx}(x,\xi,t+z-\xi)-\widetilde{u}_{xx}(x,\xi,t-z+\xi)+\frac{1}{2}k(x,t+z-2\xi)+\right.\\&\left.+\int_0^{t+z-2\xi}k(x,\alpha)\widetilde{u}(x,\xi,t+z-\xi-\alpha)d\alpha-\int_0^{t-z}k(x,\alpha)\widetilde{u}(x,\xi,t-z+\xi-\alpha)d\alpha\right]d\xi.\end{aligned} \tag{10}$$

□

3. Theorem on solvability of the inverse problem

Let us introduce the Banach space $A_s(r), s>0$ of analytic functions $h(x), x\in R$ for which the norm is finite

$$\|h\|_s(r)=\sup_{|x|<r}\sum_{|\alpha|=0}^{\infty}\frac{s^{|\alpha|}}{\alpha!}\left\|\frac{\partial^\alpha}{\partial x^\alpha}h(x)\right\|<\infty$$

.

Here $r>0, s>0$, α is a non-negative integer.

In what follows, parameter r is fixed while parameter s is treated as a variable parameter. Further, parameter r is omitted for simplicity in the notation for the norms of the space A_s. When parameter s is changed the scale of Banach spaces A_s appears. The following property is obvious: if $h(x)\in A_s$ then $h(x)\in A_{s'}$ for all $s'\in(0,s)$. Hence $A_s\subset A_{s'}$ if $s'<s$ and the following inequality holds

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha}h(x)\right\|_{s'}\leqslant \alpha^\alpha\frac{\|h\|_s}{(s-s')^\alpha}\quad \forall\alpha\quad 0<s'<s\leqslant s_0. \tag{11}$$

In what follows, parameter r is fixed. The norm of function $f(x,z,t)$ in A_{s_0} for fixed z and t is denoted by $\|f\|_{s_0}(z,t)$. This norm in $C_{(z,t)}(G_2,A_{s_0})$ is defined by the equality

$$\|f\|_{C_{(z,t)}(G_2,A_{s_0})} = \sup_{(z,t)\in G_2} \|f\|_{s_0}(z,t),$$

where $C_{(z,t)}(G_2,A_{s_0})$ denotes the class of functions continuous with respect to variables z and t in domain G_2 with values in A_{s_0}.

Theorem 3.1. *Let*

$$(g(x,+0), g_t(x,+0)) \in A_{s_0}, \quad (g(x,z), g_t(x,z), g_{tt}(x,z)) \in (C_l[0,2l], A_{s_0})$$

$$\max\{\|g\|_{s_0}(t), \|g_t\|_{s_0}(t), \|g_{tt}\|_{s_0}(t)\} \leqslant R, \quad t\in[0,2l].$$

where $R>0$ is given number.

Then there is $a\in(0,l)$ such that for any $s\in(0,s_0)$ in domain $\Gamma_{sl} = B_2 \cap \{(x,z,t) : 0\leqslant z\leqslant a(s_0-s)\}$ there is a unique solution of the system of equations (7), (9), (10) for which

$$(\widetilde{u}(x,z,t), \widetilde{u}_t(x,z,t)) \in C(A_{s_0}, F), \quad k(x,t) \in C(A_{s_0}, [0, a(s_0-s)])$$

$$F = \{(z,t,s) : (z,t)\in D_{0l}, \quad 0<z<a(s_0-s)\}$$

. Moreover,

$$\|\widetilde{u}-\widetilde{u}_0\|_s(z,t) \leqslant R; \quad \|\widetilde{u}_t - \widetilde{u}_{0t}\|_s(z,t) \leqslant \frac{R}{s_0-s}; \; \|k-k_0\|_s(z) \leqslant \frac{R}{(s_0-s)^2}.$$

Proof. It is convenient to introduce the notation

$$\varphi_1(x,z,t) = \widetilde{u}(x,z,t), \quad \varphi_2(x,z,t) = \widetilde{u}_t(x,z,t) + \frac{1}{2}k(x,t-z)z, \quad \varphi_3(x,z) = k(x,z)$$

$$\varphi_1^0(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right), \quad \varphi_2^0(x,z,t) = \frac{1}{2}\left(g_t(x,t+z) + g_t(x,t-z)\right),$$

$$\varphi_3^0(x,z) = -4g_{zz}(x,z).$$

Then we obtain from equations (6), (9), (10) that

$$\varphi_1(x,z,t) = \varphi_1^0(x,z,t) +$$

$$+\frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}(x,\xi,\tau) + \frac{1}{2}\varphi_3(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3(x,\alpha)\varphi_1(x,\xi,\tau-\alpha)d\alpha\right] d\tau d\xi, \quad (12)$$

$$\varphi_2(x,z,t) = \varphi_2^0(x,z,t) + \frac{1}{2}\int_0^z \left[\varphi_{1xx}(x,\xi,t+z-\xi) - \varphi_{1xx}(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3(x,t+z-2\xi) + \right.$$

$$\left. + \int_0^{t+z-2\xi} \varphi_3(x,\alpha)\varphi_1(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3(x,\alpha)\varphi_1(x,\xi,t-z+\xi-\alpha)d\alpha\right] d\xi, \quad (13)$$

$$\varphi_3(x,z) = \varphi_3^0(x,z) -$$

$$-4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}(x,\xi,z-\xi) - \frac{1}{2}\varphi_3(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3(x,\alpha)\varphi_1(x,z,z-\xi-\alpha)d\alpha\right] d\xi. \quad (14)$$

Let us assume that numbers $a_1, a_2, \ldots, a_n$ are determined by the recurrence relation

$$a_{n+1} = a_n \frac{(n+1)^2}{(n+1)^2+1}$$

. They form a decreasing numerical sequence and a is the limit of this sequence:

$$a = \lim_{n\to\infty} a_n = a_0 \prod_{n=1}^{\infty} \frac{(n+1)^2}{(n+1)^2+1}.$$

The positive number $a_0 < \dfrac{l}{s_0}$ will be selected later. Let us construct successive approximations as follows

$$\varphi_1^{n+1}(x,z,t) = \varphi_1^0(x,z,t) +$$
$$+ \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}^n(x,\xi,\tau) + \frac{1}{2}\varphi_3^n(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,\tau-\alpha)d\alpha \right] d\tau d\xi, \tag{15}$$

$$\varphi_2^{n+1}(x,z,t) = \varphi_2^0(x,z,t) +$$
$$+\frac{1}{2}\int_0^z \Big[\varphi_{1xx}^n(x,\xi,t+z-\xi) - \varphi_{1xx}^n(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3^n(x,t+z-2\xi) + \tag{16}$$
$$+ \int_0^{t+z-2\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,t-z+\xi-\alpha)d\alpha \Big] d\xi,$$

$$\varphi_3^{n+1}(x,z) = \varphi_3^0(x,z) -$$
$$-4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}^n(x,\xi,z-\xi) - \frac{1}{2}\varphi_3^n(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,z,z-\xi-\alpha)d\alpha \right] d\xi. \tag{17}$$

We define function $s'(z)$ by the formula

$$s'(z) = \frac{s+\nu^n(z)}{2}, \qquad \nu^n(z) = s_0 - \frac{z}{a_n}. \tag{18}$$

Let us introduce the notation $\psi_i^n = \varphi_i^{n+1} - \varphi_i^n,\ i = 1,2,3$. For $n = 0$ the following relations hold

$$\psi_1^0(x,z,t) =$$
$$= \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}^0(x,\xi,\tau) + \frac{1}{2}\varphi_3^0(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,\tau-\alpha)d\alpha \right] d\tau d\xi,$$

$$\psi_2^0(x,z,t) =$$
$$= \frac{1}{2}\int_0^z \Big[\varphi_{1xx}^0(x,\xi,t+z-\xi) - \varphi_{1xx}^0(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3^0(x,t+z-2\xi) +$$
$$+ \int_0^{t+z-2\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,t-z+\xi-\alpha)d\alpha \Big] d\xi,$$

$$\psi_3^0(x,z) =$$
$$= 4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}^0(x,\xi,z-\xi) - \frac{1}{2}\varphi_3^0(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,z,z-\xi-\alpha)d\alpha \right] d\xi$$

.

For $n = 1$ we have

$$\psi_1^1(x,z,t) = \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\psi_{1xx}^0(x,\xi,\tau) + \frac{1}{2}\psi_3^0(x,\tau-\xi) +$$

$$+ \int_0^{\tau-\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,\tau-\alpha + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,\tau-\alpha)\big)\, d\alpha\Big] d\tau d\xi,$$

$$\psi_2^1(x,z,t) = \frac{1}{2}\int_0^z \Big[\psi_{1xx}^0(x,\xi,t+z-\xi) - \psi_{1xx}^0(x,\xi,t-z+\xi) + \frac{1}{2}\psi_3^0(x,t+z-2\xi) +$$

$$+ \int_0^{t+z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,t+z-\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,t+z-\xi-\alpha)\big)\, d\alpha -$$

$$- \int_0^{t-z} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,t-z+\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,t-z+\xi-\alpha)\big)\, d\alpha\Big] d\xi,$$

$$\psi_3^1(x,z) = -4\int_0^{\frac{z}{2}} \Big[\psi_{2xx}^0(x,\xi,z-\xi) - \frac{1}{2}\psi_3^0(x,t-2\xi) +$$

$$+2\int_0^{z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,z+\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,z+\xi-\alpha)\big)\, d\alpha\Big] d\xi.$$

Thus, for any n we obtain

$$\psi_1^n(x,z,t) = \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\psi_{1xx}^{n-1}(x,\xi,\tau) + \frac{1}{2}\psi_3^{n-1}(x,\tau-\xi) +$$

$$+ \int_0^{\tau-\xi} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,\tau-\alpha + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,\tau-\alpha)\big)\, d\alpha\Big] d\tau d\xi,$$

$$\psi_2^n(x,z,t) = \frac{1}{2}\int_0^z \Big[\psi_{1xx}^{n-1}(x,\xi,t+z-\xi) - \psi_{1xx}^{n-1}(x,\xi,t-z+\xi) + \frac{1}{2}\psi_3^{n-1}(x,t+z-2\xi) +$$

$$+ \int_0^{t+z-2\xi} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,t+z-\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,t+z-\xi-\alpha)\big)\, d\alpha -$$

$$- \int_0^{t-z} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,t-z+\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,t-z+\xi-\alpha)\big)\, d\alpha\Big] d\xi,$$

$$\psi_3^n(x,z) = -4\int_0^{\frac{z}{2}} \Big[\psi_{2xx}^{n-1}(x,\xi,z-\xi) - \frac{1}{2}\psi_3^{n-1}(x,t-2\xi) +$$

$$+2\int_0^{z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^n(x,\xi,z+\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,z+\xi-\alpha)\big)\, d\alpha\Big] d\xi.$$

Next we show that if $a \in (0,l)$ is chosen in a suitable way then for any $n = 1, 2, \ldots$ the following inequalities hold

$$\lambda_n = \max\Big\{\sup_{(z,t,s)\in F_n}\Big[\|\psi_1^n\|_s(z,t)\frac{\nu^n(z)-s}{z}\Big], \sup_{(z,t,s)\in F_n}\Big[\|\psi_2^n\|_s(z,t)\frac{(\nu^n(z)-s)^2}{z}\Big],$$
$$\sup_{(z,t,s)\in F_n}\Big[\|\psi_3^n\|_s(z)\frac{(\nu^n(z)-s)^3}{z}\Big]\Big\} < \infty \tag{19}$$

$$\left\|\varphi_i^{n+1} - \varphi_0^{n+1}\right\|_s(z,t) \leqslant \frac{R_0}{(s_0-s)^{i-1}},\ i = 1, 2, \quad \left\|\varphi_3^{n+1} - \varphi_0^{n+1}\right\|_s(z) \leqslant \frac{R_0}{(s_0-s)^2}. \tag{20}$$

where
$$F_n = \{(z,t,s) : (z,t) \in G_l,\ 0 < z < a_n(s_0 - s),\ 0 < s < s_0\}.$$
Let $n = 0$. Then, taking into account (11), we obtain
$$\|\psi_1^0\|_s(z,t) \leqslant$$
$$\frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\|\varphi_{1xx}^0\|_s(\xi,\tau) + \frac{1}{2}\|\varphi_3^0\|_s(\tau-\xi) + \int_0^{\tau-\xi} \|\varphi_3^0\|_s(\alpha)\|\varphi_1^0\|_s(\xi,\tau-\alpha)d\alpha\Big]d\tau d\xi \leqslant$$
$$\leqslant \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t}\Big[\frac{4R}{(s(\xi)-s')^2} + \frac{R}{2} + R^2 t\Big]d\tau d\xi.$$
Let us use formula (18) for $n = 0$:
$$\|\psi_1^0\|_s(z,t) \leqslant \int_0^z (z-\xi)\Big[\frac{16R}{(\nu^0(z)-s)^2} + \frac{R}{2} + R^2 t\Big]d\xi \leqslant$$
$$\leqslant \Big[16 + s_0^2(0,5+2Rl)\Big]\int_0^z \frac{(z-\xi)d\xi}{(\nu^0(z)-s)^2} \leqslant a_0 R\Big[16 + s_0^2(0,5+2Rl)\Big]\frac{z}{\nu^0(z)-s}, \quad (z,t,s) \in F_0.$$

Let us estimate other components in a similar way:
$$\|\psi_2^0\|_s(z,t) \leqslant \frac{1}{2}\int_0^z \Big[\|\varphi_{1xx}^0\|(x,\xi,t+z-\xi) + \|\varphi_{1xx}^0\|(x,\xi,t-z+\xi) + \frac{1}{2}\|\varphi_3^0\|(x,t+z-2\xi) +$$
$$+ \int_0^{t+z-2\xi} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,\xi,t-z+\xi-\alpha)d\alpha\Big]d\xi \leqslant$$
$$\leqslant \frac{a_0}{2}R\Big[32 + s_0^2(0,5+4Rl)\Big]\frac{z}{(\nu^0(z)-s)^2}, \quad (z,t,s) \in F_0,$$

$$\|\psi_3^0\|_s(z) \leqslant 4\int_0^{\frac{z}{2}} \Big[\|\varphi_{2xx}^0\|(x,\xi,z-\xi) + \frac{1}{2}\|\varphi_3^0\|(x,t-2\xi) +$$
$$+ 2\int_0^{z-2\xi} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,z,z-\xi-\alpha)d\alpha\Big]d\xi \leqslant a_0 R\Big[402 + s_0^2(2+16Rl)\Big]\frac{z}{(\nu^0(z)-s)^3},$$
$$(z,t,s) \in F_0.$$

To obtain these estimates the following inequalities are used
$$\frac{1}{\nu^0(\xi)-s} \leqslant \frac{1}{\nu^0(z)-s}, \qquad \nu^0(z) - s < s_0$$
. They are true for $\xi \in (0,z)$, $s \in (0,s_0)$, $(z,t,s) \in F_0$. The obtained estimates show the validity of inequality (19) for $n = 0$. Further, for $(z,t,s) \in F_1$ we find that
$$\left\|\varphi_i^1 - \varphi_0^0\right\|_s(z,t) = \|\psi_i^0\|_s(z,t) \leqslant \frac{a_0\lambda_0 z}{(\nu^0(z)-s)^i} \leqslant \frac{2^{i-1}a_0\lambda_0}{(s_0-s)^{i-1}}, \ \ i = 1,2.$$
$$\left\|\varphi_3^1 - \varphi_0^0\right\|_s(z) = \|\psi_3^0\|_s(z) \leqslant \frac{a_0\lambda_0 z}{(\nu^0(z)-s)^3} \leqslant \frac{4a_0\lambda_0}{(s_0-s)^2}$$
. Thus, if a_0 is chosen so that $4a_0\lambda_0 \leqslant R$ then inequalities (20) are true $n = 0$.

Let us show by the method of mathematical induction that inequalities (19), (20) also hold for other n if a_0 is chosen appropriately. Let us assume that inequalities (19), (20) are true for $n = 1, 2, \dots j$. Then for $(z,t,s) \in F_{j+1}$ we have

$$\|\psi_1^{j+1}\|(z,t) = \frac{1}{2}\int_0^z \int\limits_{\xi-z+t}^{-\xi+z+t} \Big[\|\psi_{1xx}^j\|(x,\xi,\tau) + \frac{1}{2}\|\psi_3^j\|(x,\tau-\xi)+$$

$$+\int\limits_0^{\tau-\xi} \Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,\tau-\alpha + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,\tau-\alpha)\Big)\,d\alpha\Big]d\tau d\xi \leqslant$$

$$\leqslant \frac{1}{2}\int\limits_0^z \int\limits_{\xi-z+t}^{-\xi+z+t} \Big[\frac{4a_0\lambda_j\xi}{(s(\xi)-s')^2(\nu^j(\xi)-s)} + \frac{a_0\lambda_j\xi}{(\nu^j(\xi)-s)^3} + \frac{a_0\lambda_j\xi Rt}{(\nu^j(\xi)-s)^3} + \frac{a_0\lambda_j\xi Rt(1+s_0^2)}{(\nu^j(\xi)-s)(s_0-s)^2}\Big]d\tau d\xi \leqslant$$

$$\leqslant \lambda_j a_0^2\Big[17 + 6Rl + 2Rls_0^2\Big]\frac{z}{\nu^{j+1}(z)-s} =: \lambda_j a_0\eta_1(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)}, \quad (z,t,s)\in F_0.$$

Here function $s'(\xi)$ is taken in form (18) with $n = j$ and inequalities

$$\|\varphi_1^{j+1}\|_s(z,t) \leqslant 2R, \quad \|\varphi_2^j\|_s(z) \leqslant R\frac{1+s_0^2}{(s_0-s)}, \quad \|\varphi_3^j\|_s(z) \leqslant R\frac{1+s_0^2}{(s_0-s)^2},$$

are valid according to the inductive hypothesis as well as the obvious inequalities $a_j \leqslant a_0$, $\nu^{j+1}(z) < \nu^j(z)$.

Similar reasoning for ψ_2^{j+1}, ψ_3^{j+1} leads to the inequalities

$$\|\psi_2^{j+1}\|_s(z,t) \leqslant \frac{1}{2}\int_0^z \big[\|\psi_{1xx}^j\|(x,\xi,t+z-\xi) + \|\psi_{1xx}^j\|(x,\xi,t-z+\xi) + \frac{1}{2}\|\psi_3^j\|(x,t+z-2\xi)+$$

$$+\int_0^{t+z-2\xi}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,t+z-\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,t+z-\xi-\alpha)\Big)\,d\alpha+$$

$$+\int_0^{t-z}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,t-z+\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,t-z+\xi-\alpha)\Big)\,d\alpha\big]d\xi \leqslant$$

$$\leqslant \lambda_j\frac{a_0}{2}\big[33 + 6Rl + 2Rls_0^2\big]\frac{z}{(\nu^{j+1}(z)-s)^2} =: \lambda_j a_0\eta_2(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)^2}, \quad (z,t,s)\in F_0,$$

$$\|\psi_3^{j+1}\|_s(z) \leqslant 4\int_0^{\frac{z}{2}}\Big[\|\psi_{2xx}^j\|(x,\xi,z-\xi) + \frac{1}{2}\|\psi_3^j\|(x,t-2\xi)+$$

$$+2\int_0^{z-2\xi}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,z+\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,z+\xi-\alpha)\Big)\,d\alpha\Big]d\xi \leqslant$$

$$\leqslant \lambda_j a_0\big[402 + 48Rl + 12Rls_0^2\big]\frac{z}{(\nu^{j+1}(z)-s)^3} =: \lambda_j a_0\eta_3(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)^3}, \quad (z,t,s)\in F_0,$$

It follows from the obtained estimates that

$$\lambda_{j+1} \leqslant \lambda_j\rho, \quad \lambda_{j+1} < \infty, \quad \rho := \max a_0\{\eta_1,\eta_2,\eta_3\}. \tag{21}$$

At the same time for $(x,t,s) \in F_{j+2}$ we have

$$\|\varphi_i^{j+2} - \varphi_i^0\|_s(z,t) \leqslant \sum_{n=0}^{j+1}\|\varphi_i^{n+1} - \varphi_i^n\|_s(z,t) = \sum_{n=0}^{j+1}\|\psi_i^n\|_s(z,t) \leqslant \sum_{n=0}^{j+1}\frac{\lambda_n z}{(\nu^n(z)-s)^i} \leqslant$$

$$\leqslant \frac{1}{(s_0-s)^{i-1}}\sum_{n=0}^{j+1}\frac{\lambda_n a_n^i a_{j+2}}{(a_n-a_{j+2})^i} \leqslant \frac{\lambda_0 a_0}{(s_0-s)^{i-1}}\sum_{n=0}^{j+1}\rho^n(n+1)^{2i}, \quad i=1,2,3.$$

Let us choose $a_0 \in (0, l)$ in such a way that

$$\rho < 1, \quad \lambda_0 a_0 \sum_{n=0}^{\infty} \rho^n (n+1)^6 \leqslant R.$$

Then

$$\|\varphi_i^{j+2} - \varphi_i^0\|_s(z,t) \leqslant \frac{R}{(s_0 - s)^{i-1}}, \ \ (x,t,s) \in F_{j+2}, \ \ i = 1,2,3.$$

Since the choice does not depend on the number of approximations the successive approximations $\varphi_i^n, \ i = 1,2,3$, belong to

$$C(F, A_s), \quad F = \bigcap_{n=0}^{\infty} F_n$$

and the following inequalities

$$\|\varphi_i^n - \varphi_i^0\|_s(z,t) \leqslant \frac{R}{(s_0 - s)^{i-1}}, \ \ (x,t,s) \in F, \ \ i = 1,2,3.$$

are true. For $s \in (0, s_0)$ the series

$$\sum_{n=0}^{\infty} (\varphi_i^n - \varphi_i^{n-1})$$

converge uniformly in the norm of space $C(F, A_s)$ therefore $\psi_i^n \to \psi_i$. The limit functions are elements of $C(F, A_s)$ and satisfy equations (12), (13), (14).

Let us now prove the uniqueness of the found solution. Let us assume that $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$ are any two solutions that satisfy the inequalities

$$\|\varphi_i^{(k)} - \varphi_i^0\|_s(z,t) \leqslant R, \quad i = 1,2,3, \quad ,k = 1,2, \quad (x,t,s) \in F.$$

Let us introduce $\widetilde{\varphi}_i = \varphi_i^{(1)} - \varphi_i^{(2)} \ \ i = 1,2,3$ and

$$\lambda = \max_{1 \leqslant i \leqslant 3} \left\{ \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_1\|_s(z,t) \frac{\nu(z) - s}{z} \right], \quad \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_2\|_s(z,t) \frac{(\nu(z) - s)^2}{z} \right], \right.$$

$$\left. \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_3\|_s(z) \frac{(\nu(z) - s)^3}{z} \right] \right\} < \infty,$$

where

$$\nu(z) = s_0 - \frac{z}{a}, \quad a = a_0 \prod_{n=0}^{\infty} \frac{(n+1)^2}{(n+1)^2 + 1}.$$

Then, from equations (15), (16), (17) one can obtain the following relations for functions $\widetilde{\varphi}_i$

$$\widetilde{\varphi}_1(x,z,t) =$$

$$= \frac{1}{2} \int_0^z \int_{\xi - z + t}^{-\xi + z + t} \left[\widetilde{\varphi}_{1xx}(x,\xi,\tau) + \frac{1}{2} \widetilde{\varphi}_3(x, \tau - \xi) + \int_0^{\tau - \xi} \widetilde{\varphi}_3(x,\alpha) \widetilde{\varphi}_1(x,\xi,\tau - \alpha) d\alpha \right] d\tau d\xi,$$

$$\widetilde{\varphi}_2(x,z,t) = \frac{1}{2} \int_0^z \left[\widetilde{\varphi}_{1xx}(x,\xi,t+z-\xi) - \widetilde{\varphi}_{1xx}(x,\xi,t-z+\xi) + \frac{1}{2} \widetilde{\varphi}_3(x,t+z-2\xi) + \right.$$

$$\left. + \int_0^{t+z-2\xi} \widetilde{\varphi}_3(x,\alpha) \widetilde{\varphi}_1(x,\xi,t+z-\xi-\alpha) d\alpha - \int_0^{t-z} \widetilde{\varphi}_3(x,\alpha) \widetilde{\varphi}_1(x,\xi,t-z+\xi-\alpha) d\alpha \right] d\xi,$$

$$\widetilde{\varphi}_3(x,z) = -4 \int_0^{\frac{z}{2}} \left[\widetilde{\varphi}_{2xx}(x,\xi,z-\xi) - \frac{1}{2} \widetilde{\varphi}_3(x,t-2\xi) + 2 \int_0^{z-2\xi} \widetilde{\varphi}_3(x,\alpha) \widetilde{\varphi}_1(x,z,z-\xi-\alpha) d\alpha \right] d\xi$$

which are similar to equalities for ψ_i^n, $i = 1, 2, 3$.

Applying the estimates given above, we find an analogue of inequality (21) in the form

$$\lambda < \lambda\rho',$$

where $\rho' := \max a\{\eta_1, \eta_2, \eta_3\}$. Since $a < a_0$ then $\rho' < \rho < 1$. Therefore $\lambda = 0$ and $\varphi_i^{(1)} = \varphi_i^{(2)}$, $i = 1, 2, 3$. The theorem is proved.

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Двумерная обратная задача для интегро-дифференциального уравнения гиперболического типа

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Аннотация. Рассматривается двумерная обратная задача определения ядра интегрального члена в интегро дифференциальном уравнении гиперболического типа. В прямой задаче требуется найти функцию смещения из начально-краевой задачи.В обратной задаче требуется определение ядра интегрального члена зависящего как от временной, так и от одной пространственной переменной. Доказывается, локальная, однозначная разрешимость поставленной задачи в классе функций непрерывных по одной из переменных и аналитический по другой переменной, на основе метода шкал банаховых пространств вещественных аналитических функций.

Ключевые слова: интегро-дифференциальное уравнение, обратная задача, дельта функция, интегральное уравнение, теорема Банаха.

DOI: 10.17516/1997-1397-2022-15-5-663-671
УДК 517.9

Second Hankel Determinant for Bi-univalent Functions Associated with q-differential Operator

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Received 21.09.2021, received in revised form 10.03.2022, accepted 20.07.2022

Abstract. The objective of this paper is to obtain an upper bound to the second Hankel determinant denoted by $H_2(2)$ for the class $S_q^*(\alpha)$ of bi-univalent functions using q-differential operator.

Keywords: Hankel determinant, bi-univalent functions, q-differential operator, Fekete-Szegö functional.

Citation: M.G. Shrigan, Second Hankel Determinant for Bi-univalent Functions Associated with q-differential Operator, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 663–671.
DOI: 10.17516/1997-1397-2022-15-5-663-671.

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=n}^{\infty} a_k\, z^k \quad (a_k \geqslant 0;\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and univalent in the open unit disk given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The Koebe one-quarter theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geqslant \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all functions f which are bi-univalent in $\mathbb{U}$ and are given by the Taylor–Maclaurin series expansion (1). The behavior of the coefficients is unpredictable when the biunivalency condition is imposed on the function $f \in \mathcal{A}$. A systematic study of the class Σ of bi-univalent function in $\mathbb{U}$, which is introduced in 1967 by Lewin [12]. For a brief history and interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties of the bi-univalent function class Σ, one can refer to the work of

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Srivastava *et al.* [21] and references therein. Ever since then, several authors investigated various subclasses of the class Σ of bi-univalent functions. For some more recent works see [22–27]. The class of bi-starlike functions is introduce by Brannan and Taha [2] (see also [14]). For $0 \leqslant \alpha < 1$, a function $f \in \mathcal{A}$ is in the class $S_q^*(\alpha)$ of bi-starlike function of order α if both f and f^{-1} are starlike in $\mathbb{U}$ and obtained estimates on the initial coefficients conjectured that $|a_2| \leqslant \sqrt{2}$. It may be noted that for $\alpha = 0$, $q \longrightarrow 1^-$, $S_q^*(\alpha) = S^*$, the familiar subclass of starlike functions in $\mathbb{U}$.

For the univalent function in the class $\mathcal{A}$, it is well known that the n^{th} coefficient a_n is bounded by n. The bounds for the coefficients gives information about the geometric properties of these functions. For example growth and distortion properties of normalized univalent function are obtained by using the bounds of its second coefficient a_2. In 1966, Pommerenke [15] define the Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

A good amount of literature is available about the importance of Hankel determinant. It plays an important role in the study of singularities as well as in the study of power series with integral coefficients ([3,4]). In 1916, Bieberbach proved that if $f \in S$, then $|a_2^2 - a_3| \leqslant 1$. In 1933, Fekete and Szegö [5] proved that

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} 4\mu - 3 & \text{if } \ \mu \geqslant 1, \\ 1 + 2exp[-2\mu/(1-\mu)] & \text{if } \ 0 \leqslant \mu < 1, \\ 3 - 4\mu & \text{if } \ \mu \leqslant 0. \end{cases} \tag{3}$$

The Hankel functional $H_2(1) = \left|a_3 - a_2^2\right|$ and $H_2(2) = \left|a_2 a_4 - a_3^2\right|$ is also known as Fekete–Szegö functional and second Hankel determinant respectively. The Hankel functional has many applications in functional theory. For example $|a_3 - a_2^2|$ is equal to $S_f(z)/6$, where $S_f(z)$ is the schwarzian derivative of the locally univalent function defined $S_f(z) = (f''(z)/f'(z))' - 1/2(f''(z)/f(z))^2$ (See [19]). In 1969, Keough and Merkers [11] solved Fekete–Szegö problem for the classes of starlike and convex functions. Lee *et al.* [13] established the sharp bounds to $|H_2(2)|$ by generalizing several classes defined by subordination. Janteng *et al.* [9] (see also [1,18]) provided a brief survey on Hankel determinants and obtained bounds for $|H_2(2)|$ for the classes of starlike and convex functions.

The theory of q-calculus in recent years has attracted the attention of researchers. The q-analogy of the ordinary derivative was initiated at the beginning of century by Jackson [8]. Ismail *et al.* [7] first introduce and explore class of generalized complex functions via q-calculus on the open unit disk $\mathbb{U}$. Recently many newsworthy results related to subclass of analytic functions and q-operators are meticulously studied by various authors (see [10,17,20]). For $0 < q < 1$, the q-derivative of a function f given by (1) is defined as

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & \text{for } \ z \neq 0, \\ f'(0) & \text{for } \ z = 0. \end{cases} \tag{4}$$

We note that $\lim\limits_{q\to 1^-} D_q f(z) = f'(z)$. From (4), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \, z^{k-1}, \tag{5}$$

where as $q \to 1^-$

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^k \longrightarrow k. \tag{6}$$

In this connection, our aim is to study upper bounds for functional $|a_2a_4 - a_3^2|$ for functions belonging to the class $f \in S_q^*(\alpha)$, which is defined as follows.

Definition 0.1. *A function $f(z)$ given by (1) is said to be in the class $f \in S_q^*(\alpha)$, $0 < q < 1$, $0 \leqslant \alpha < 1$ if the following conditions are satisfied:*

$$\begin{aligned} f \in \Sigma, \qquad & \frac{z(D_q f(z))}{f(z)} > \beta \qquad (0 \leqslant \beta < 1; z \in \mathbb{U}) \\ \text{and} \qquad & \frac{z(D_q g(w))}{g(w)} > \beta \qquad (0 \leqslant \beta < 1; z \in \mathbb{U}), \end{aligned} \tag{7}$$

where the function g is the extension of f^{-1} to $\mathbb{U}$.

In order to derive our main results, we have to recall here the following lemma.

Lemma 0.1 ([16]). *If $h \in \mathcal{H}$, then $|B_k| \leqslant 2$, for each $k \geqslant 1$ and the inequality is sharp for the function $\dfrac{1+z}{1-z}$.*

Lemma 0.2 ([6]). *If $p \in \mathcal{P}$, $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$ then $2c_2 = c_1^2 + x(4 - c_1^2)$, $4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$, for some x, z with $|x| \leqslant 1$ and $|z| \leqslant 1$.*

Another result that will required is the maximum value of a quadratic expression. Stranded computation shows

$$\max_{(0\leqslant t\leqslant 4)} \left(Pt^2 + Qt + R\right) = \begin{cases} (4PR - Q^2)/4P & \text{if } Q > 0,\ P \leqslant -Q/8, \\ R & \text{if } Q \leqslant 0,\ P \leqslant -Q/4, \\ 16P + 4Q + R & \text{if } Q \geqslant 0,\ P \geqslant -Q/8 \ \text{ or } \ Q \leqslant 0,\ P \geqslant -Q/4. \end{cases} \tag{8}$$

1. Main results

In this section, we investigate second Hankel determinant $|H_2(2)|$ for functions belonging to the class $S_q^*(\alpha)$ using q-differential operator. For convenience, in the sequel we use the abbreviation $q_2 = [2]_q - 1$, $q_3 = [3]_q - 1$, $q_4 = [4]_q - 1$.

Theorem 1.1. *Let $0 \leqslant \alpha < 1$, $0 < q < 1$. If function $f \in \mathcal{A}$ given by (1) belongs to the class $\mathcal{S}_q^*(\alpha)$ then*
i. For $Q > 0$, $P \leqslant -Q/8$

$$\left|a_2a_4 - a_3^2\right| \leqslant T\left(R - \frac{Q^2}{4P}\right). \tag{9}$$

ii. For $Q \leqslant 0$, $P \leqslant -Q/4$

$$\left|a_2a_4 - a_3^2\right| \leqslant TR. \tag{10}$$

iii. For $Q \geqslant 0$, $P \geqslant -Q/4$

$$\left|a_2a_4 - a_3^2\right| \leqslant T\left(16P + 4Q + R\right), \tag{11}$$

where

$$\begin{aligned} &P = 4\beta^2 L + \beta M + N, \quad Q = U - 4\beta V, \\ &R = 64q_2^4 q_4, \quad L = (q_4 - q_3)q_3^2, \quad M = q_2^2 q_4 + 8q_3^2 - 8q_3q_4, \\ &N = 4(q_4 - q_3) - q_2^2 q_3 q_4 + 4q_2^4 q_4, \quad U = 4q_2^2 q_3 q_4 + 12q_2^3 q_3^2 - 32q_2^4 q_4, \quad V = q_2^2 q_3 q_4 \quad \text{and} \\ &T = \frac{(1-\beta)^2}{4q_2^4 q_3^2 q_4}. \end{aligned} \tag{12}$$

Proof. If $f \in \mathcal{S}_q^*(\alpha)$ and $g \in f^{-1}$. Then

$$\frac{z(D_q f(z))}{f(z)} = \beta + (1-\beta)p(z)$$

and

$$\frac{w(D_q g(w))}{g(w)} = \beta + (1-\beta)q(w). \tag{13}$$

We obtain

$$\frac{z(D_q f(z))}{f(z)} = 1 + q_2a_2z + \left[q_3a_3 - q_2a_2^2\right]z^2 + \left[q_4a_4 - (q_2+q_3)a_3a_2 + q_2a_2^3\right]z^3 + \cdots. \tag{14}$$

Also

$$\begin{aligned} \frac{w(D_q g(w))}{g(w)} = 1 - q_2a_2z + \left[q_3\left(2a_2^2 - a_3\right) - q_2a_2^2\right]w^2 + \\ + \left[(q_2+q_3)a_2\left(2a_2^2 - a_3\right) - q_4\left(5a_2^3 - 5a_2a_3 + a_4\right) - q_2a_2^3\right]w^3 + \cdots. \end{aligned} \tag{15}$$

From (13), (14) and (15), it is easily seen that

$$a_2 = \frac{(1-\beta)c_1}{q_2}, \tag{16}$$

$$a_3 = \frac{(1-\beta)^2 c_1^2}{q_2^2} + \frac{(1-\beta)(c_2 - d_2)}{2q_3} \tag{17}$$

and

$$a_4 = \frac{q_3(1-\beta)^3 c_1^3}{q_2^3 q_4} + \frac{5(1-\beta)^2 c_1 (c_2 - d_2)}{4q_2q_3} + \frac{(1-\beta)(c_3 - d_3)}{2q_4}. \tag{18}$$

Upon simplification, we easily establish

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| \frac{(q_3 - q_4)(1-\beta)^4}{q_2^4 q_4} c_1^4 + \frac{(1-\beta)^3}{4q_2^2 q_3} c_1^2 (c_2 - d_2) + \\ + \frac{(1-\beta)^2}{2q_2q_4} c_1(c_3 - d_3) - \frac{(1-\beta)^2}{4q_3^2}(c_2 - d_2)^2 \Bigg|. \end{aligned} \tag{19}$$

According to Lemmas 1 and 2, we write

$$c_2 - d_2 = \frac{4-c_1^2}{2}(x-y) \tag{20}$$

and

$$4c_3 - 4d_3 = \frac{c_1^3}{2} + \frac{c_2\left(4-c_1^2\right)}{2}(x+y) - \frac{c_1\left(4-c_1^2\right)}{2}(x^2+y^2) + \\ + \frac{\left(4-c_1^2\right)}{2}\left(\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right), \tag{21}$$

for some x, y, z and w with $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$ and $|w| \leqslant 1$. Substituting values of c_2, c_3, d_2 and d_3 from (20), (21) on the right side of (19), we have

$$\left|a_2a_4 - a_3^2\right| \leqslant \mathrm{M}_1 + \mathrm{M}_2(\varrho_1+\varrho_2) + \mathrm{M}_3(\varrho_1^2+\varrho_2^2) + \mathrm{M}_4(\varrho_1+\varrho_2) := F(\varrho_1,\varrho_2), \tag{22}$$

where

$$\mathrm{M}_1 = \frac{(q_4-q_3)(1-\beta)^4}{q_2^4 q_4}c_1^4 + \frac{(1-\beta)^2}{4q_2q_4}c_1^4 + \frac{(1-\beta)^2}{2q_2q_4}c_1\left(4-c_1^2\right), \tag{23}$$

$$\mathrm{M}_2 = \left[\frac{(1-\beta)^3}{8q_2^2q_3}c_1^2\left(4-c_1^2\right) + \frac{(1-\beta)^2}{4q_2q_4}c_1^2\left(4-c_1^2\right)\right](|x|+|y|), \tag{24}$$

$$\mathrm{M}_3 = \left[\frac{(1-\beta)^2}{8q_2q_4}c_1^2\left(4-c_1^2\right) - \frac{(1-\beta)^2}{4q_2q_4}c_1\left(4-c_1^2\right)\right]\left(|x|^2+|y|^2\right), \tag{25}$$

$$\mathrm{M}_4 = \frac{(1-\beta)^2}{8q_3^2}\left(4-c_1^2\right)^2](|x|+|y|)^2. \tag{26}$$

Applying Lemma 1, without loss of generality assume $c_1 \equiv c \in [0,2]$ for $\varrho_1 = |x| \leqslant 1$ and $\varrho_2 = |y| \leqslant 1$ and using triangle inequality, we have

$$\mathrm{M}_1 = \frac{(1-\beta)^2}{4q_2^4q_4}\left[4(q_4-q_3)(1-\beta)^2 - 2c^3 + 8c + q_2^3\right]c^4 \geqslant 0, \tag{27}$$

$$\mathrm{M}_2 = \frac{(1-\beta)^2}{8q_2^2q_3q_4}\left[(1-\beta)q_4 + 2q_2q_3\right]c^2\left(4-c^2\right) \geqslant 0, \tag{28}$$

$$\mathrm{M}_3 = \frac{(1-\beta)^2}{8q_2q_4}\left(4-c^2\right)c(c-2) \leqslant 0, \tag{29}$$

$$\mathrm{M}_4 = \frac{(1-\beta)^2}{4q_3^2}\left(4-c^2\right)^2 \geqslant 0. \tag{30}$$

To maximize the function $F(\varrho_1,\varrho_2)$ on the closed region $\mathfrak{S} = \{(\varrho_1,\varrho_2) : 0 \leqslant \varrho_1 \leqslant 1, 0 \leqslant \varrho_2 \leqslant 1\}$. Differentiating $F(\varrho_1,\varrho_2)$ partially with respect to ϱ_1 and ϱ_2, we get

$$F_{\varrho_1\varrho_1} \cdot F_{\varrho_2\varrho_2} - \left(F_{\varrho_1\varrho_2}\right)^2 < 0. \tag{31}$$

This shows that the function $F(\varrho_1,\varrho_2)$ cannot have local maximum in the interior of the region $\mathfrak{S}$. Now we investigate the maximum of $F(\varrho_1,\varrho_2)$ on the boundary of the region $\mathfrak{S}$. For $\varrho_1 = 0$ and $0 \leqslant \varrho_2 \leqslant 1$ (similarly $\varrho_2 = 0$ and $0 \leqslant \varrho_1 \leqslant 1$), we obtain

$$F(0,\varrho_2) = \Omega(\varrho_2) = (\mathrm{M}_3+\mathrm{M}_4)\varrho_2^2 + \mathrm{M}_2\varrho_2 + \mathrm{M}_1. \tag{32}$$

i. $\mathrm{M}_3+\mathrm{M}_4 \geqslant 0$: In this case for $0 \leqslant \varrho_2 \leqslant 1$ and any fixed c with $0 \leqslant c \leqslant 2$, it is clear that $\Omega'(\varrho_2) = 2(\mathrm{M}_3+\mathrm{M}_4)\varrho_2+\mathrm{M}_2 > 0$, that is $\Omega(\varrho_2)$ is an increasing function hence for fixed $c \in [0,2)$, the maximum of $\Omega(\varrho_2)$ occurs at $\varrho_2 = 1$ and maximum of $\varrho_2 = \mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3+\mathrm{M}_4$.

ii. $\mathrm{M}_3+\mathrm{M}_4 < 0$: Since $\mathrm{M}_2+2(\mathrm{M}_3+\mathrm{M}_4) \geqslant 0$ for $0 < \varrho_2 < 1$ and for any fixed c with $0 \leqslant c < 2$, it is clear that $\mathrm{M}_2+2(\mathrm{M}_3+\mathrm{M}_4) < 2(\mathrm{M}_3+\mathrm{M}_4)\varrho_2+\mathrm{M}_2 < \mathrm{M}_2$ and so $\Omega'(\varrho_2) > 0$. Hence for fixed c with $0 \leqslant c < 2$, the maximum $\Omega'(\varrho_2)$ occurs at $\varrho_2 = 1$. Also for $c = 2$ we obtain

$$F(\varrho_1,\varrho_2) = \frac{4(1-\beta)^2(q_4-q_3)}{q_2^4 q_4}\left[(1-\beta)^2+\frac{q_2^3}{(q_4-q_3)}\right]. \tag{33}$$

For $\varrho_1 = 1$ and $0 \leqslant \varrho_2 < 1$ (similarly $\varrho_2 = 1$ and $0 \leqslant \varrho_1 \leqslant 1$), we obtain

$$F(1,\varrho_2) = \mho(\varrho_2) = (\mathrm{M}_3+\mathrm{M}_4)\varrho_2^2+(\mathrm{M}_2+2\mathrm{M}_4)\varrho_2+\mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3+\mathrm{M}_4. \tag{34}$$

Thus from above cases of $\mathrm{M}_3+\mathrm{M}_4$ we get that

$$\max\mho(\varrho_2) = \mho(1) = \mathrm{M}_1+2\mathrm{M}_2+2\mathrm{M}_3+4\mathrm{M}_4. \tag{35}$$

Since $\Omega(1) \leqslant \mho(1)$ for $c \in [0,2]$, we obtain $\max F(\varrho_1,\varrho_2) = F(1,1)$ on the boundary of the square $\mathfrak{S}$. Thus , the maximum of F occurs at $\varrho_1 = 1$ and $\varrho_2 = 1$ in the closed square $\mathfrak{S}$.

Let $\Bbbk : [0,2] \longrightarrow \mathbb{R}$ defined by

$$\Bbbk(c) = \max(\varrho_1,\varrho_2) = F(1,1) = \mathrm{M}_1+2\mathrm{M}_2+2\mathrm{M}_3+4\mathrm{M}_4. \tag{36}$$

Substituting the values of $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3,\mathrm{M}_4$ in the function $\Bbbk$ defined by (36), we get

$$\begin{aligned}\Bbbk(c) =&\frac{(1-\beta)^2}{4q_2^4q_3^2q_4}\Big(\left|4(q_4-q_3)(1-\beta)^2q_3^2-(1-\beta)q_2^2q_3q_4+4q_2^4q_4\right|c^4+\\&+\left|4(1-\beta)q_2^2q_3q_4+12q_2^3q_3^2-32q_2^4q_4\right|c^2+\left|64q_2^4q_4\right|\Big)\end{aligned} \tag{37}$$

which is quadratic in c^2. Using the standard computation, we get

$$\left|a_2a_4-a_3^2\right| \leqslant T\begin{cases}(4PR-Q^2)/4P & \text{if } Q>0,\ P\leqslant -Q/8,\\ R & \text{if } Q\leqslant 0,\ P\leqslant -Q/4,\\ 16P+4Q+R & \text{if } Q\geqslant 0,\ P\geqslant -Q/8 \text{ or } Q\leqslant 0,\ P\geqslant -Q/4\end{cases} \tag{38}$$

where P, Q, R and T are given by (12).
This completes the proof. □

Theorem 1.2. *Let $0<q<1,\ 0\leqslant\alpha<1$ and $f\in S_q^*(\alpha)$. Then for complex μ*

$$\left|a_3-\mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2}{q_2^2}. \tag{39}$$

Proof. Letting $c := c_1 > 0$. Then for complex μ, using (16) and (17), we have

$$\begin{aligned}a_3-\mu a_2^2 &= \frac{(1-\beta)^2c^2}{q_2^2}+\frac{(1-\beta)(c_2-d_2)}{2q_3}-\mu\frac{(1-\beta)^2c^2}{q_2^2} =\\ &= \frac{(2-\mu)(1-\beta)^2c^2q_3+(1-\beta)(c_2-d2)}{2q_2^2q_3}.\end{aligned} \tag{40}$$

By (16), we obtain

$$a_3 - \mu a_2^2 = \frac{2(2-\mu)(1-\beta)^2 c^2 q_3 + (1-\beta)(4-c^2)(x-y)}{4q_2^2 q_3}, \tag{41}$$

where x and y satisfying $|x| \leqslant 1$, $|y| \leqslant 1$

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2 c^2}{4q_2^2}, \tag{42}$$

using $c \leqslant 2$, we get

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2}{q_2^2}. \tag{43}$$

This completes the proof. □

I am grateful to the reviewer(s) of this article who gave valuable suggestions in order to improve and revise the paper in present form.

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Второй определитель Ганкеля для биунивалентных функций, ассоциированных с q-дифференциальным оператором

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Аннотация. Целью данной статьи является получение верхней оценки второго определителя Ганкеля, обозначаемого $H_2(2)$, для класса $S_q^*(\alpha)$ биунивалентных функций используя q-дифференциальный оператор.

Ключевые слова: определитель Ганкеля, биоднолистные функции, q-дифференциальный оператор, функционал Фекете-Сегаö.

DOI: 10.17516/1997-1397-2022-15-5-672-678
УДК 532.5+517.9

Some Solutions of the Euler System of an Inviscid Incompressible Fluid

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Received 10.04.2022, received in revised form 02.06.2022, accepted 08.08.2022

Abstract. We consider a system of two-dimensional Euler equations describing the motions of an inviscid incompressible fluid. It reduces to one non-linear equation with partial derivatives of the third order. A group of point transformations allowed by this equation is found. Some invariant solutions and solutions not related to invariance are constructed. The solutions found describe vortices, jet streams, and vortex-like formations.

Keywords: Euler equations, group of point transformations, invariant solutions, vortices, jets.

Citation: O.V. Kaptsov, Some Solutions of the Euler System of an Invisci Incompressible Fluid, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 672–678. DOI: 10.17516/1997-1397-2022-15-5-672-678.

Introduction

It is well known that the system of two-dimensional Euler equations

$$u_t + uu_x + vu_y + p_x = 0, \qquad v_t + uv_x + vv_y + p_y = 0, \qquad u_x + v_y = 0 \tag{1}$$

describes plane motions of an inviscid incompressible fluid [1]. Here u, v are the components of the velocity vector, p is the pressure. The symmetry group of the system (1) was found by A. Rodionov and V. Andreev. They found a new non-local operator and constructed some [2] invariant solutions. Moreover. they studied the invariant properties of the system in Lagrangian coordinates. Very non-trivial and interesting solutions in Lagrangian variables are constructed in the monograph [3]. At present, the question of the integrability of the system (1) by the method of the inverse scattering problem remains open.

It is very interesting to study axisymmetric flows with swirl [1]. The transformation group is admitted by these equations in Euler and Lagrangian coordinates is also presented in [2]. Few solutions are known for this model.

In this paper, the Euler system (1) is converted to one equation for the stream function. An infinite group of symmetries for this equation is found, and invariant solutions describing single vortices and kinks are constructed. Two kinks solutions and ones corresponding to the infinite group are given. A new solution of the stationary equation of axisymmetric flows with swirl, known in the physical literature as the Grad-Shafranov equation [4], is found.

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1. Symmetry groups and invariant solutions

It is well known [1] that the system of equations (1) can be reduced to one equation with partial derivatives of the third order

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = 0, \tag{2}$$

where ψ is a stream function, Δ is the two-dimensional Laplacian operator and the lower indices denote differentiation by the corresponding variables. If the solution of the equation (2) is known, then the components of the velocity vector are reconstructed by the formulas $u = \psi_y,\ v = -\psi_x$.

Standard methods [5] can be used to find an Lie symmetry algebra of the equation (2). It is generated by the following operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\psi\frac{\partial}{\partial \psi},$$

$$X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_5 = 2ty\frac{\partial}{\partial x} - 2tx\frac{\partial}{\partial y} + (x^2+y^2)\frac{\partial}{\partial \psi},$$

$$X_6 = f(t)\frac{\partial}{\partial \psi}, \quad X_7 = g(t)\frac{\partial}{\partial x} + yg(t)'\frac{\partial}{\partial \psi}, \quad X_8 = h(t)\frac{\partial}{\partial y} - xh(t)'\frac{\partial}{\partial \psi},$$

where f, g and h are arbitrary functions of t. The first four operators generate well-known transformations: the translation in the t-direction, two scaling symmetries and rotation in $\mathbb{R}^2(x,y)$. The operator X_5 is responsible for the transition to a coordinate system rotating with a constant angular velocity. It generates the transformation

$$\tilde t = t, \quad \tilde x = x\cos at - y\sin at, \quad \tilde y = x\sin at + y\cos at, \quad \tilde\psi = \psi + a(x^2+y^2)/2, \ \forall a \in \mathbb{R}.$$

The operator X_6 defines the time shift of the function ψ: $\psi \longrightarrow \psi + f(t)$, and the operators X_7, X_8 give the generalized Galilean boosts

$$\tilde t = t,\ \tilde y = y,\ \tilde x = x + g,\ \tilde\psi = \psi + yg',$$

$$\tilde t = t,\ \tilde x = x, \ \ \tilde y = y + h,\ \tilde\psi = \psi - xh'.$$

The infinite subalgebra generated by the three operators X_6, X_7, X_8 induces action on the solutions of the equation (2)

$$\psi(t,x,y) \longrightarrow \psi(t, x-g, y-h) + yg' - xh' + m, \tag{3}$$

where m is an arbitrary function of t.

Let us consider the stationary equation (2)

$$\psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = 0. \tag{4}$$

The left-hand side of this equation is the Jacobian determinant of ψ and $\Delta\psi$. Therefore, any solution to the equation

$$\Delta\psi = \omega(\psi)$$

satisfies (4). It can be shown that equation (4) admits a symmetry algebra generated by six operators

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \frac{\partial}{\partial y}, \qquad Y_3 = \frac{\partial}{\partial \psi}, \qquad Y_4 = \psi\frac{\partial}{\partial \psi},$$

$$Y_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Let us proceed to the construction of solutions related to the symmetries of the equations.
Vortices. A solution of the equation (4), which is invariant under the rotation transformation, has the form $\psi = F(x^2+y^2)$, where F is an arbitrary smooth function. Hence, according to (3), the function

$$\psi = F((x-g)^2 + (y-h)^2) + yg' - xh'$$

also the solution of this equation for any smooth functions $g(t), f(t)$.

Define the functions F, g, h as follows

$$F = \frac{1}{(x^2+y^2)^6 + 0.2}, \qquad g = \sin(t) + 0.5t, \qquad h = 3\cos(4t) + 0.1t.$$

Then the pattern of streamlines at time $t = 0$ looks like in Fig. 1, where we see a single vortex. At $t = 1.2$ the vortex disappears (Fig. 2), but at time $t = 2.4$ it appears again. We get a

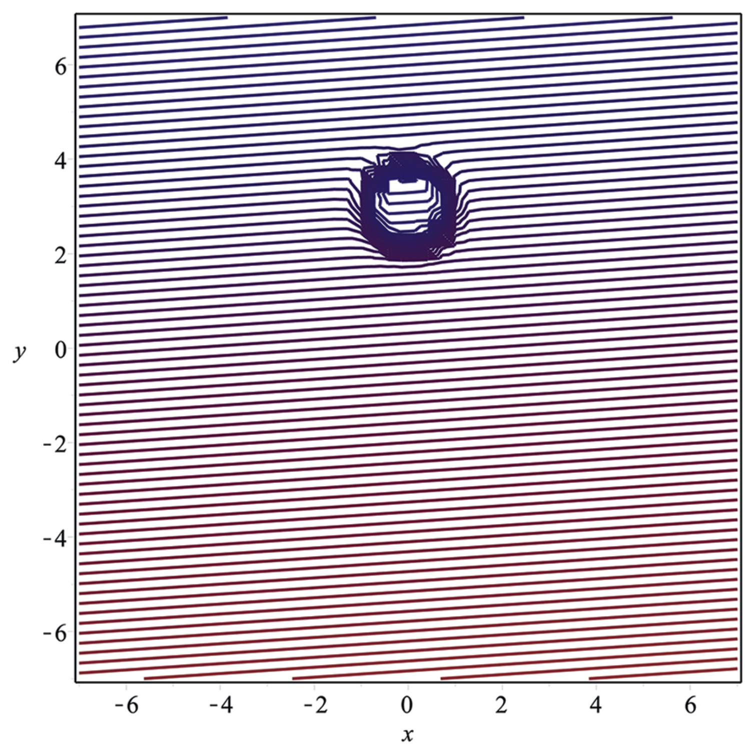

Fig. 1. Vortex at time t=0.

"flickering" vortex: it either appears or disappears. It is easy to choose the functions F, f, g so that the vortex will exist all the time. To do this, it is enough to leave the function F the same, and change the functions g and h a little: $g = \sin(t) + 1.5t,\ h = \cos(4t) + t$.

If one chooses the function

$$\psi = \frac{1}{x^2+y^2},$$

then we obtain a steady flow with a singularity at the origin of coordinates. This singular solution is analogous to the classical point vortex in a irrotational flow [1], but this flow is rotational.

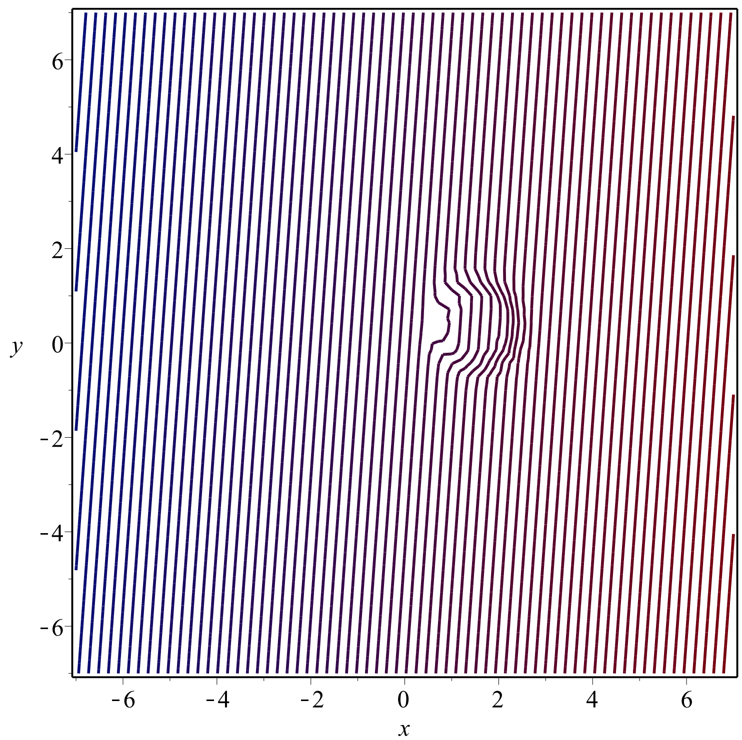

Fig. 2. The disappearance of the vortex at t=1.2

Using the action of the transformation group (3), it is not difficult to obtain a nonstationary vortex with a singular point.

Kink and soliton. The solution of the equation (4) invariant under the combination of translations is of the form

$$\psi = F(ax + by), \quad a, b \in \mathbb{R},$$

where F is an arbitrary smooth function. If we take the function F equal to $\arctan(ax + by)$, then its graph is a two-dimensional kink (step). Using the Galilean transformations, we obtain a nonstationary solution of the equation (2)

$$\psi = yg' - xh' + \arctan(\exp(x - g + y - h)),$$

where $g = 0.5t, h = 0.1t$. In this case, the graphs of the velocity components at different times are similar to a soliton and an antisoliton with variable amplitudes, respectively.

"Two soliton" solution. Using computer algebra systems, it is not difficult to find the following stationary solution

$$\psi = \arctan\left(\frac{f_1 + f_2}{1 + s_{12} f_1 f_2}\right) \tag{5}$$

of the equation (4). Here f_1, f_2 are functions equal to $\exp(k_i x + n_i y + m_i)$ $(i = 1, 2)$, and k_i, n_i, m_i, s_{12} are parameters that satisfy two relations

$$n_2^2 + k_2^2 = n_1^2 + k_1^2, \qquad s_{12} = \frac{(n_1 - n_2)^2}{(k_1 + k_2)^2}.$$

A typical graph of the function ψ given by the formula (5) looks like two stationary kinks for real parameter values (Fig.3). The streamline pattern represents the interaction of two jets. Using the

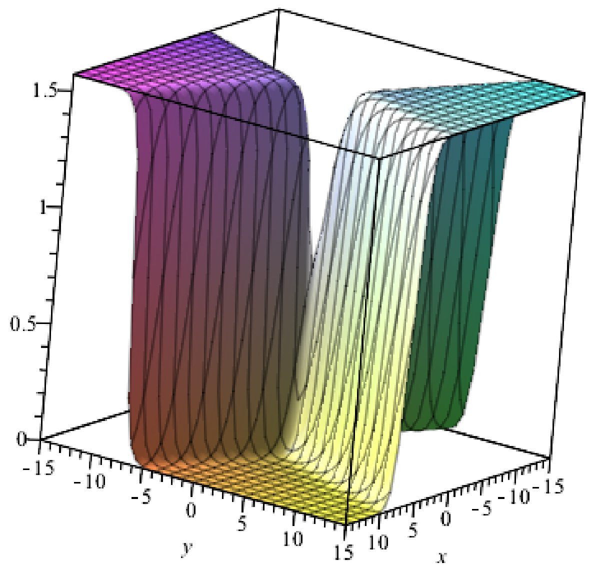

Fig. 3. Two stationary kinks

generalized Galilean subgroup, one can construct non-stationary solutions of the equation (2).

Another way to construct stationary solutions of the equation (2) follows from the next statement. Let ϕ be the solution Laplace equation $\Delta\phi = 0$, then the function

$$\psi = \log(s\Delta(\log\phi)), \quad \forall s \in \mathbb{R},$$

satisfies the equation (2). One can obtain nonstationary solutions using the symmetry group of the equation (2). This representation for the stream function is due to the fact that the Liouville equation

$$\Delta\psi = \exp(\psi),$$

admits an infinite group of transformations.

2. Additional solutions

We proceed to the construction of other non-stationary solutions of the equation (2). We look for the function ψ in the form

$$\psi = F(kx + ny + m(t)) + r_0(t)x + r_1(t)y + \sum_{i=2}^{N} r_i(t)(kx + ny)^i, \tag{6}$$

where F is an arbitrary function, k, n are arbitrary constants, and m, r_j $(j = 0, \ldots, N)$ are unknown functions on t. Substituting the representation (6) into equation (2), we have a system of ordinary differential equations on functions $m(t), r_j(t)$. Solving this system, we obtain recurrence formulas for the functions m, r_i

$$m = \int S dt, \qquad r_N = C, \qquad r_{i-1} = i \int r_i S dt, \quad i = 2, \ldots, N,$$

where C is an arbitrary constant, $S = -kr_0 + nr_1$, and r_0, r_1 are arbitrary smooth functions on t.

One can look for a solution to the equation (2) in the form

$$\psi = F(k(t)x + n(t)y + m(t)) + \sum_{i+j>0} r_{ij}(t)x^i y^j.$$

Substituting the latter expression into equation (2) results in a system of nonlinear ordinary differential equations on functions $k(t), n(t), m(t), r_{ij}(t)$. Finding its solutions remains an open problem.

Let us now consider the stationary equation for the stream function, which describes an axisymmetric swirling flow [1],

$$\psi_{xx} + \psi_{rr} - \psi_r/r = r^2 G + H, \tag{7}$$

where G, H are arbitrary functions of ψ. In plasma physics, this equation is called the Grad-Shafranov equation [4]. Some of its solutions are presented in the monograph [2], where there is also a group classification of this equation. Shan'ko [6] found some functionally-invadant solutions of the equation (7).

We will look for a solution to the equation (7) in the form

$$\psi = S(x^2 + ar^2), \qquad a \in \mathbb{R},$$

where S is the function to be found. Substituting this representation into the equation (7), we obtain the relation

$$2S' + 4x^2 S'' + r^2(4a^2 S'' - G) - H = 0.$$

We introduce a new variable $q = x^2 + ar^2$ and rewrite the last relation as

$$2S' + 4qS'' - H + r^2(4a(a-1)S'' - G) = 0.$$

Two equations follow from this

$$2S' + 4qS'' = H(S), \qquad 4a(a-1)S'' = G(S). \tag{8}$$

If the function S is given, then from the last two equations (8) one can find the functions G and H. Suppose, for example, $S = 1/q$. Then from the first equation of the system (8) we have $H(q^{-1}) = 6q^{-2}$. So the function $H(\psi)$ is $6\psi^2$. Similarly, from the second equation of the system (8) we find the function $G(\psi) = 8a(a-1)\psi^3$. Therefore, the equation (7), with the found functions G, H, has a solution

$$\psi = \frac{1}{x^2 + ar^2}.$$

The components of the velocity vector, according to [1], are

$$u = \frac{-2a}{(x^2 + ar^2)^2}, \qquad v = \frac{2x}{r(x^2 + ar^2)^2}, \qquad w = \frac{\sqrt{A - 4\psi^3}}{r}, \qquad A \in \mathbb{R}.$$

This solution has a singularity at $r = 0$ and tends to zero at infinity.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement no. 075-02-2022-873).

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Некоторые решения системы Эйлера невязкой несжимаемой жидкости

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Аннотация. В работе изучается система двумерных уравнений Эйлера, описывающая движения невязкой несжимаемой жидкости. Она сводится к одному нелинейному уравнению с частными производными третьего порядка. Найдена группа точечных преобразований, допускаемых этим уравнением. Построены некоторые инвариантные решения и решения не связанные с инвариантностью. Найденные решения описывают вихри, струйные течения и вихреподобные образования.

Ключевые слова: уравнения Эйлера, группы преобразований, инвариантные решения, вихри, струи.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 679–682

DOI: 10.17516/1997-1397-2022-15-5-679-682
УДК 512.554.3

On Centralizers of the Graph Automorphisms of Niltriangular Subalgebras of Chevalley Algebras

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Received 10.09.2022, received in revised form 10.11.2022, accepted 20.12.2022

Abstract. Graph automorphisms of a Chevalley group correspond to each type of reduced indecomposable root system Φ, which Coxeter graph has a non-trivial symmetry. It is well-known, that a Chevalley algebra and its niltriangular subalgebra N has a graph automorphism θ exaclty when Φ is of type A_n, D_n or E_6. We note connections with homomorphisms of root systems introduced in 1982.

The main theorem on the centralizers in N of the automorphism θ gives new representations of niltriangular subalgebras, using also the unique series of unreduced indecomposable root system of type BC_n.

Keywords: Chevalley algebra, niltriangular subalgebra, homomorphisms of root systems.

Citation: V.M. Levchuk, G.S. Suleimanova, On Centralizers of the Graph Automorphisms of Niltriangular Subalgebras of Chevalley Algebras, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 679–682. DOI: 10.17516/1997-1397-2022-15-5-679-682.

1. Preliminary notes

The semisimple complex Lie algebras are classified in Cartan–Killing theory parallel with the root systems of a Euclidean space V.

For any indecomposable root system Φ in V and for any field K we have corresponding Chevalley algebra $L_\Phi(K)$. The elements e_r ($r \in \Phi$) and the basis of the suitable Cartan subalgebra H with the condition $He_r \subseteq Ke_r$ [1, Sec. 4.2] form a basis of $L_\Phi(K)$. The elements e_r ($r \in \Phi^+$) for the positive root system Φ^+ in Φ form a basis of niltriangle subalgebra $N\Phi(K)$.

The system of fundamental roots Π, which is a basis in Φ^+, is unique. The Cartan numbers $A_{rs} = 2(r,s)/(r,s)$ ($r, s \in \Pi$) are integer, they form the Cartan matrix. We call a graph with nodes, one associated with each fundamental root, such that the ith node is joined to the jth node by a bond of strength $A_{rs}A_{sr}$, a Coxeter graph of the root system Φ, by the terminology of J.-P. Serre [2] (see also remark in [3, Sec. 1]). If we mark each node by a number (r, r) we obtain the Dynkin diagram.

The classification of simple complex Lie algebras, to within isomorphism, is connected with the classification of indecomposable root systems, to within equivalence. There exist 9 series of reduced root systems [4, Tables I– IX] – the classical types A_n, B_n, C_n, D_n, and the exceptional types G_2, F_4, E_6, E_7 and E_8.

Co-roots $h_r = 2r/(r,r)$ ($r \in \Phi$) give also *a dual root system* Φ^* with base Π^* and, also, the systems Φ, $(\Phi^*)^*$ are equivalence. Note that the systems of type B_n, C_n are dual.

On the other hand, all indecomposable unreduced root systems, to within equivalence, are exhausted by the systems of type BC_n. For $n = 1$ we have

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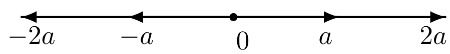


Type BC_1.

In 1982 in [5] homomorphisms of root systems were introduced. Denote by $L_0(\Phi)$ the additive subgroup in V generated by the roots of Φ (the lattice of roots).

Definition 1. *We call a mapping $\Phi \to \Phi'$ a homomorphism of Φ into Φ' if this mapping may be extended to a homomorphism of the group $L_0(\Phi)$ into the group $L_0(\Phi')$.*

We consider homomorphisms of the root system Φ, which are not isomorphisms. Such homomorphisms exist only if the Coxeter graph of the system Φ has a non-trivial symmetry and all roots in Φ have an equal length [5].

The symmetry may be linearly extended to a permutation $^-$ on the root system Φ. Under the same restrictions on Φ the Chevalley algebra $L_\Phi(K)$ has a graph automorphism

$$\theta : \; e_r \to e_{\bar{r}} \quad (r \in \Pi),$$

according to the proof of Proposition 12.2.3 in [1]. By [6], the same conclusion is true for the induced automorphism of subalgebra $N\Phi(K)$.

By [7] and [8], an arbitrary (not necessary associative) algebra A is said to be an exact enveloping algebra for a Lie algebra L if L is isomorphic to the associated algebra $A^{(-)}$. The both algebras L and A may be defined by structure constants in the same basis, in contrast to the universal associative enveloping algebra.

The existance and the structure of exact enveloping algebras for the Lie algebras $N\Phi(K)$ were considered in [7]. Centralizers of graph automorphisms of Lie algebras $N\Phi(K)$ play an important role in the investigation of uniqueness (problem of I. P. Shestakov, 2017).

For the classical types on this way special exact enveloping algebras

$$RA_{n-1}(K) = NT(n, K), \quad RB_n(K), \quad RC_n(K), \quad RD_n(K).$$

were discovered (the associative algebra $NT(n, K)$ of all (lower) niltriangular matrices of degree n over K exists only for the type A_{n-1}):

The problem on non-trivial homomorphismd of root systems and embeddings of the centralizers is studied for Φ of type D_n and $\theta^2 = 1$, [9]. This problem was represented at the student conference in SFU (2022) by students of IMFI (M. V. Dektyarev, D. R. Khismatulin, A. D. Pakhomova and I. V. Salmina).

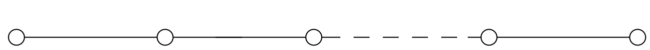
2. Centralizers of graph automorphisms

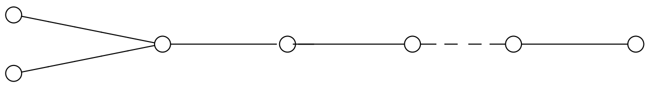
It is well-known that the number

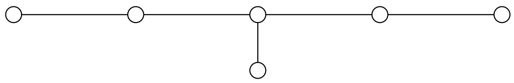
$$p(\Phi) := \max\{(r, r)/(s, s) \mid r, s \in \Phi\}$$

is equal to 1 or 2 or Φ of type G_2 and $p(\Phi) = 3$.

The root systems of the same length (i.e. with $p(\Phi)$=1), having a non-trivial symmetry of order 2, exhausted by the types A_n, D_n and E_6 with Coxeter graphs, correspondingly,

A_n :  (n nodes, $n \geqslant 1$)

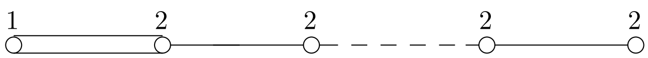
D_n :  (n nodes, $n \geqslant 4$)

E_6 : 

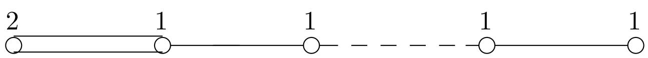
In this cases the graph automorphism θ is defined for the Chevalley algebra and for its subalgebra $N\Phi(K)$. In addition, either θ is of order 2, or Φ of type D_4 and $\theta^3 = 1$.

By [2, Ch. V, Sec. 15], a Coxeter graph gives *Dynkin diagram*, if we mark each node r by a number (r, r) (we assume that short roots have a lengh 1).

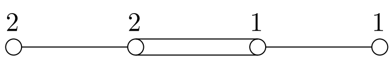
The root systems of type B_n are C_n dual and they have the same Coxeter graph. However, Dynkin diagrams for these types coinsides when $n = 2$. In this case the root systems are equivalent and the Coxeter graph have a non-trivial symmetry of order 2, as for types F_4 and G_2:

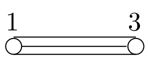
B_n : 

($n \geqslant 2$)

C_n : 

($n \geqslant 2$)

F_4 : 

G_2 : 

Note that Chevalley algebras (in contrast to Chevalley groups) of types F_4, G_2 and $B_2 = C_2$ doesn't have a graph automorphism [10].

We study the centralizer $C(\theta)$ of the graph automorphism θ of the Lie algebra $N\Phi(K)$, i.e. the subalgebra of all θ-stationary elements. Note, that the root system of type A_{2n}, by [5, Lemma 7], has a homomorphism to the unreduced root system of type BC_n.

The main result of the article is

Theorem 1. *Let θ be a graph automorphism of a Lie algebra $N\Phi(K)$. Then one of the following statements is valid.*

(a) $\theta^3 = 1$, Φ of type D_4 and $C(\theta) \simeq NG_2(K)$;

(b) $\theta^2 = 1$, Φ type D_n $(n \geqslant 4)$ and $C(\theta) \simeq NB_{n-1}(K)$;

(c) $\theta^2 = 1$, Φ of type A_{2n-1} $(n \geqslant 3)$ and $C(\theta) \simeq NC_n(K)$;

(d) $\theta^2 = 1$, Φ of type E_6 and $C(\theta) \simeq NF_4(K)$;

(e) $\theta^2 = 1$, Φ of type A_{2n} $(n \geqslant 2)$ and the centralizer $C(\theta)$ in $NA_{2n}(K)$ is a subalgebra, associated with the unreduced root system of type BC_n.

A special case was considered in [9, Lemma 3.6].

Lemma 1. *The algebra $RB_n(K)$ is represented in the algebra $RD_{n+1}(K)$ by a centralizre of the graph automorphism θ of order 2 of the Lie algebra $RD_{n+1}(K)^{(-)}$.*

The authors showed that, using this lemma, an increasing sequence with uniquely defined isomorphic embeddings of algebras

$$RB_{n-1}(K) \subset RD_n(K) \subset RB_n(K) \subset RD_{n+1}(K) \subset \cdots, \quad n = 3, 4, 5, \dots .$$

may be obtained.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement 075-02-2022-867).

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О централизаторах графовых автоморфизмов нильтреугольных подалгебр алгебр Шевалле

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Аннотация. Для группы Шевалле графовые автоморфизмы связывают с каждым типом ассоциированной приведенной неразложимой системы корней Ф, граф Кокстера которой допускает нетривиальную симметрию. Известно, что алгебра Шевалле и, аналогично, ее нильтреугольная подалгебра N обладает графовым автоморфизмом θ точно когда Ф — типа A_n, D_n или E_6. Мы отмечаем связь с введенными в 1982 году гомоморфизмами систем корней.

Основная теорема о централизаторах в N автоморфизма θ приводит к новым представлениям нильтреугольных подалгебр, использующим и единственную серию неприведенных неразложимых систем корней типа BC_n.

Ключевые слова: алгебра Шевалле, нильтреугольная субалгебра, гомоморфизмы систем корней.