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CONTENTS

СОДЕРЖАНИЕ

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Summation of Functions and Polynomial Solutions to a Multidimensional Difference Equation

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Abstract. We define a set of polynomial difference operators which allows us to solve the summation problem and describe the space of polynomial solutions for these operators in equations with the polynomial right-hand side. The criterion describing these polynomial difference operators was obtained. The theorem describing the space of polynomial solutions for the operators was proved.

Keywords: Bernoulli numbers, Bernoulli polynomials, summation problem, multidimensional difference equation, Euler–Maclaurin formula, Todd operator.

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1. Introduction and preliminaries

The summation of functions is one of the main problems of the theory of finite differences, and the answer was given in the famous Euler–Maclaurin formula obtained by Euler in 1733 and independently by Maclaurin in 1738 (see [6, 7, 21]).

In [1, 2, 13] the problem of rational summation was studied, that is, finding sums of the form

$$S(x) = \sum_{t=0}^{x} \varphi(t), \tag{1}$$

where the function $\varphi(t)$ is a rational function. The solution to the problem consists in finding a solution in symbolic form, that is, explicitly in the form of a mathematical function (formula) and is called the *indefined summation problem* (see also [8, 9]).

In the *definite summation problem*, the function φ can depend not only on the summation index, but also on the summation boundary x, that is, $S(x) = \sum\limits_{t=0}^{x} \varphi(t, x)$ (see, for example, [11, 20]).

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The problem of indefinite summation is reduced to solving the so-called (see [8,9]) telescopic equation — the inhomogeneous difference equation

$$(\delta - 1)f(x) = \varphi(x), \tag{2}$$

where δ is a shift operator: $\delta f(x) := f(x+1)$.

By analogy with the problem of integrating functions, the solution $f(x)$ to equation (2) is called *the discrete antiderivative of the function* $\varphi(x)$. If $f(x)$ is the discrete antiderivative function $\varphi(x)$, then the required sum is

$$S(x) = f(x+1) - f(0). \tag{3}$$

Formula (3) is called the *discrete analogue of the Newton–Leibniz formula.*

Euler's approach to the problem of finding a discrete antiderivative is based on the operator equality $\delta = e^D$, which allows us to write (2) in the form

$$Df(x) = \left[\frac{D}{e^D - 1}\right]\varphi(x),$$

where D is a differentiation operator.

The expression in square brackets on the right-hand side of the last equality is called the *Todd operator* and is understood as follows: $\left[\frac{D}{e^D - 1}\right] = \sum\limits_{m=0}^{\infty} \frac{B_m}{m!} D^m$, where b_m are Bernoulli numbers (see, for example, [3,6,10,17,19]). Thus, we obtain the Euler–Maclaurin formula

$$\sum_{t=0}^{x} \varphi(t) = \int_0^{x+1} \varphi(t)dt + \sum_{m=1}^{\infty} \frac{B_m}{m!}\left[\varphi^{(m-1)}(x+1) - \varphi^{(m-1)}(0)\right],$$

in which the required sum is expressed in terms of the derivatives and the integral of the function $\varphi(t)$.

Remark 1. *In the summation problem we can use other operators instead of* $\delta - 1$. *For example, we can consider the operator* $(\delta - 1)(\delta - 2)$ *and solve the difference equation*

$$f(x+2) - 3f(x+1) + 2f(x) = \varphi(x), \quad x = 0, 1, 2, \ldots.$$

If a solution to this equation is found then the sum $S(x)$ *can be written as* $S(x) = f(x+2) - 2f(x+1) - [f(1) - 2f(0)]$. *For* $n = 1$ *polynomial difference operators* $P(\delta) = c_0 + c_1\delta + \cdots + c_m\delta^m$, *where* $c_0 + \cdots + c_m = 0$, *has a similar property (effect), see Theorem 2.3.*

Euler's approach to the problem of indefinite summation of a function $\varphi(t) = \varphi(t_1, \ldots, t_n)$ of several variables suggests that you need to find a multidimensional analogue of (2), and compute a discrete antiderivative to obtain an analogue of the Newton-Leibniz formula (3). In Section 2 we implemented it to sum a function over the integer points in an n-dimensional parallelepiped (Lemma 2.2 and Theorem 2.3).

Bernoulli numbers and polynomials play an important role in classical one-dimensional summation theory and various branches of combinatorial analysis. Bernoulli polynomials are solutions of difference equation (2) with polynomial right-hand side $\varphi(t) = t^\mu$:

$$B_\mu(t+1) - B_\mu(t) = \mu t^{\mu-1}.$$

In the third section of this paper, we use spaces of polynomial solutions (generalized Bernoulli polynomials) to sum functions of several discrete arguments.

2. Operators with a summing effect and a discrete analogue of the Newton–Leibniz formula

To formulate the main result of the paper (Theorem 2.3), we need the following definitions and notations. For a given function of several discrete arguments $\varphi(t) = \varphi(t_1, \dots, t_n)$, we consider the problem of finding the sum of its values over all integer points of an n-dimensional parallelepiped with a "variable" vertex $x \in \mathbb{Z}^n_{\geqslant}$:

$$\Pi(x) = \{t \in \mathbb{R}^n_{\geqslant} : 0 \leqslant t_j \leqslant x_j, j = 1, \dots, n\}. \tag{4}$$

This sum can be written as follows:

$$S(x) = \sum_{t_1=0}^{x_1} \cdots \sum_{t_n=0}^{x_n} \varphi(t_1, \dots, t_n) = \sum_{t \in \Pi(x)} \varphi(t). \tag{5}$$

To solve the summation problem means to find a formula expressing (5) in terms of a (finite) number of terms independent of x.

Operating on the complex-valued functions $f(x)$ of integer arguments $x = (x_1, \dots, x_n)$, we define the shift operator δ_j with respect to the j-th variable

$$\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n), \quad \delta_j^{\alpha_j} = \underbrace{\delta_j \circ \dots \circ \delta_j}_{\alpha_j \ times},$$

where δ_j^0 is the identity operator. Some properties of the shift operator were studied in [12]. Denote $P(\delta) = \sum\limits_{0 \leqslant \alpha \leqslant l} c_\alpha \delta^\alpha$ — polynomial difference operator with constant coefficients c_α, $\alpha = (\alpha_1, \dots, \alpha_n)$, $l = (l_1, \dots, l_n) \in \mathbb{Z}^n_{\geqslant}$, and the inequality $l \geqslant \alpha$ means $l_j \geqslant \alpha_j, j = 1, \dots, n$. We will also use the notation $l \ngeqslant \alpha$, if there is at least one j_0 for which $l_{j_0} < \alpha_{j_0}$.

The difference equation for the unknown function $f(x)$ is written as follows:

$$P(\delta) f(x) = \varphi(x), \ x \in \mathbb{Z}^n_{\geqslant}. \tag{6}$$

Definition 2.1. *A polynomial difference operator $P(\delta)$ of the difference equation* (6) *is called an operator with a summing effect if the sum* (5) *can be represented through solutions $f(x)$ to this equation at finite set of points regardlessly of the numbers of summands in $S(x)$.*

In this case, naturally, $f(x)$ can be called *the discrete antiderivative of the function $\varphi(x)$*, and the corresponding expression solving the summation problem (5) is *a discrete analogue of the Newton–Leibniz formula.*

For any point x, we define the projection operator π_j along the x_j axis:

$$\pi_j x := (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$$

and define its action: $\pi_j f(x) := f(\pi_j x)$.

Let $\mathcal{P}(A)$ be the power set of A and $V := \mathcal{P}(\{1, \dots, n\}), J = \{j_1, \dots, j_k\} \in V$. If we denote $\pi_J = \pi_{j_1} \circ \ldots \circ \pi_{j_k}$, then the set of vertices of the parallelepiped $\Pi(x)$ can be written as $\{\pi_J x, J \in V\}$. Note that $\pi_\varnothing x = x$.

Lemma 2.2. *In (6) let $P(\delta) = R(\delta)(\delta - I)$, where $R(\delta)$ is a polynomial operator. Then for any solution f of (6), the discrete analogue of the Newton–Leibniz formula is*

$$\sum_{t \in \Pi(x)} \varphi(t) = R(\delta) \sum_{J \in V} (-1)^{\#J} f(\pi_J(x + I)),$$

where $\#J$ is a number of elements of the set J.

Proof. Since

$$\sum_{t_j=0}^{x_j}(\delta_j-1)f(t)=\sum_{t_j=0}^{x_j}(\delta_j-1)\delta_j^{t_j}\pi_j f(t)=(\delta_j-1)\Big(\sum_{t_j=0}^{x_j}\delta_j^{t_j}\Big)\pi_j f(t)=$$
$$=(\delta_j-1)\frac{\delta_j^{x_j+1}-1}{\delta_j-1}\pi_j f(t)=(\delta_j^{x_j+1}-1)\pi_j f(t),$$

we get

$$\sum_{0\leqslant t\leqslant x}\varphi(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}-1)\pi_j f(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)f(t),$$

hense, since π_j and δ_k permute for $j\neq k$, we have

$$\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)=\sum_{J\in V}(-1)^{\#J}\delta_{\overline{J}}^{x_{\overline{J}}+I}\pi_{\overline{J}}\pi_J,$$

where $\overline{J}=\{1,\dots,n\}\backslash J,\ \delta=(\delta_1,\cdots,\delta_n)$.

Thus we conclude that

$$\sum_{t\in\Pi(x)}\varphi(t)=R(\delta)\prod_{j=1}^{n}(\delta_j^{x_j+1}\pi_j-\pi_j)f(t)=R(\delta)\sum_{J\in V}(-1)^{\#J}f(\pi_J(x+I)).$$

□

Note that the case $R(\delta)\equiv 1$ was proved in [18].

We see that in Lemma 2.2 finding the value of (5) is reduced to calculating the values of the function $f(x)$ at the vertices of the parallelepiped $\Pi(x+I)$, the number of which is 2^n and does not depend on x. Thus, the operator $P(\delta)=R(\delta)(\delta-I)$ has a summing effect.

We denote $\partial=(\partial_1,\dots,\partial_n)$, where ∂_j is the differenctiation operators with respect to the j-th variable, $j=1,\dots,n$, and $\partial^\mu=\partial_1^{\mu_1}\dots\partial_n^{\mu_n}$.

Theorem 2.3. *In the summation problem (5), the polynomial difference operators*

$$P(\delta)=R(\delta)\prod_{j=1}^{n}(\delta_j-1)=R(\delta)(\delta-I)$$

and only they have a summing effect, where $R(\delta)$ is a polynomial.

Proof. We transform (5), assuming that $f(t)$ is a solution to the difference equation (6) and using the equality $f(t)=\delta^t f(0)$ yields

$$S(x)=\sum_{t\in\Pi(x)}\varphi(t)=\sum_{t\in\Pi(x)}P(\delta)f(t)=\sum_{t\in\Pi(x)}\delta^tP(\delta)f(0). \tag{7}$$

Next, we use the multiple geometric progression formula $\sum\limits_{t\in\Pi(x)}\delta^t=\dfrac{\delta^{x+I}-I}{\delta-I}$ and expand the characteristic polynomial in a Taylor series at the point $I=(1,1,\dots,1)$: $P(z)=\sum\limits_{0\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^\alpha$. Then we transform the resulting expression

$$P(z)=\sum_{\substack{\alpha\geqslant 0\\ \alpha\ngeqslant I}}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^\alpha+(z-I)\sum_{I\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(z-I)^{\alpha-I}$$

and to express (5) as

$$S(x)=\left(\sum_{\substack{\alpha\geqslant 0\\ \alpha\ngeqslant I}}\frac{\partial^\alpha P(I)}{\alpha!}(\delta-I)^\alpha\right)\sum_{t\in\Pi(x)}f(t)+$$
$$+\left(\sum_{I\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(\delta-I)^{\alpha-I}\right)\left(\delta^{x+I}-I\right)f(0),\quad (8)$$

where $\delta^{x+I}-I=(\delta_1^{x_1+1}-1)\cdots(\delta_n^{x_n+1}-1)$.

Note that the number of summands in the second sum of the right-hand side of (8) does not depend on numbers of summands in $S(x)$, but in the first sum it does. If $P(\delta)=R(\delta)(\delta-I)$, then the first term is absent and $P(\delta)$ has a summing effect.

On the other hand, if $P(\delta)$ has a summing effect, then $\sum\limits_{\substack{\alpha\geqslant 0\\ \alpha\ngeqslant I}}\frac{\partial^\alpha P(I)}{\alpha!}(\delta-I)^\alpha\equiv 0$, but then

$$P(\delta)=\sum_{I\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(\delta-I)^\alpha=(\delta-I)R(\delta),$$

where $R(\delta)=\sum\limits_{I\leqslant\alpha\leqslant l}\frac{\partial^\alpha P(I)}{\alpha!}(\delta-I)^{\alpha-I}$. □

Example. Find the sum

$$S(x_1,x_2)=\sum_{t_1=0}^{x_1}\sum_{t_2=0}^{x_2}\varphi(t_1,t_2)$$

for the function

$$\varphi(t_1,t_2)=\frac{1}{(t_1+t_2+1)(t_1+t_2+2)(t_1+t_2+3)}.$$

We note that the function

$$f(t_1,t_2)=\frac{1}{2}\frac{1}{t_1+t_2+1}$$

is a solution to the difference equation $(\delta_1-1)(\delta_2-1)f(t)=\varphi(t)$. Since $P(\delta)=(\delta_1-1)(\delta_2-1)$, $R\equiv 1$, the sum is

$$S(x)=f(x_1+1,x_2+1)-f(x_1+1,0)-f(0,x_2+1)+f(0,0)=$$
$$=\frac{1}{2}\left(\frac{1}{x_1+x_2+3}-\frac{1}{x_1+2}-\frac{1}{x_2+2}+1\right).$$

3. Polynomial solutions to a multidimensional difference equation

Bernoulli numbers and polynomials play an important role in the classical one-dimensional summation theory. Bernoulli polynomials are solutions of the difference equation (2) with the polynomial right-hand side $\varphi(t)=t^{\mu-1}$:

$$\frac{1}{\mu}\left(B_\mu(t+1)-B_\mu(t)\right)=t^{\mu-1}.\quad (9)$$

Bernoulli numbers and polynomials are well studied (see, for example, [6,19]) and have numerous applications in various branches of mathematics (see [5,15,16]).

One of the options for finding the Bernoulli polynomials is to use the operator equality $\delta = e^D$. From (9) we find the formula for the Bernoulli polynomials

$$B_\mu(t) = \frac{\mu}{\delta - 1} t^{\mu-1} = \frac{\mu}{e^D - 1} t^{\mu-1},$$

whence we get

$$B_\mu(t) = \frac{D}{e^D - 1} t^\mu, \tag{10}$$

where $\dfrac{D}{e^D-1} = \sum\limits_{\nu=0}^{\infty} B_\nu \dfrac{D^\nu}{\nu!}$ is a differential operator of infinite order, $B_\nu = B_\nu(0)$ are Bernoulli numbers.

The action of the operator $\dfrac{D}{e^D-1}$ on polynomials is well defined. We obtain a formula for finding the Bernoulli polynomials

$$B_\mu(t) = \sum_{\nu=0}^{\mu} \frac{B_\nu}{\nu!} D^\nu t^\mu.$$

Remark. The above scheme for finding the Bernoulli polynomials can be viewed as a method for finding a particular solution of the equation (2) in the case when the right-hand side of $\varphi(t)$ is a polynomial.

We are interested in computing polynomial solutions to difference equation (6) with polynomial right-hand sides. In this case, without loss of generality, we can consider the case $\varphi(t) = t^\mu = t_1^{\mu_1} \dots t_n^{\mu_n}$. In addition, we are interested in polynomial difference operators $P(\delta)$ with a summing effect, which, by virtue of Theorem 2.3, can be written in the form

$$P(\delta) = R(\delta) \prod_{j=1}^{n} (\delta_j - 1)^{k_j}, \tag{11}$$

where $R(\delta)$ is a polynomial difference operator with constant coefficients, $R(I) \neq 0$.

We consider the difference equation

$$R(\delta) \prod_{j=1}^{n} (\delta_j - 1)^{k_j} f(t) = t^\mu, \quad t \in \mathbb{Z}^n_{\geqslant}, \tag{12}$$

and find its particular polynomial solutions by analogy with the one-dimensional case, that is, we use the operator equalities $\delta_j = e^{D_j}$, $j = 1, 2, \dots, n$.

The function $\operatorname{Td}(\xi) = \dfrac{1}{R(e^\xi)} \prod\limits_{j=1}^{n} \dfrac{\xi_j^{k_j}}{(e^{\xi_j} - 1)^{k_j}}$ is holomorphic at the point $\xi = 0$ and therefore admits its expansion in some neighborhood of zero as a power series

$$\operatorname{Td}(\xi) = \sum_{m \geqslant 0} \frac{\tilde{b}_{k,m}}{m!} \xi^m. \tag{13}$$

Substituting the differentiation operator D_j into (13) in place of the variable ξ_j, we define the *differential operator of infinite order*:

$$\operatorname{Td}(D) = \sum_{m \geqslant 0} \frac{\tilde{b}_{k,m}}{m!} D^m. \tag{14}$$

For $k_1 = \ldots = k_n = 1$ and $R(\delta) \equiv 1$, the operator defined in (14) is called the *Todd operator* (see, for example, [4, 14]). In the general case, it is natural to call it *the generalized Todd operator*, and the numbers $\tilde{b}_{k,m}$ — *generalized Bernoulli numbers.* Any polynomial solution to equation (12) is called the Bernoulli polynomial associated with the polynomial difference operator (11).

The case $R(\delta) \equiv 1$ was considered in [18].

We set $\mu^{(m)} = \mu(\mu-1)(\mu-2)\cdots(\mu-(m-1))$.

Theorem 3.1. *Let $P(\delta)$ be an operator with summing effect of the form (11). Then the set of Bernoulli polynomials associated with this operator is described by the formula*

$$f(x) = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} \frac{\mu^{(m)} x^{\mu+k-m}}{(\mu+k-m)^{(k)}} + \sum_{i=1}^{n} \sum_{m_i=1}^{k_i} x_i^{k_i - m_i} q_{m_i}(x_1, \ldots [i] \ldots, x_n), \tag{15}$$

where q_{m_i} are arbitrary polynomials in $(n-1)$-th variables $x_1, \ldots, [i], \ldots, x_n$.

Proof. From the difference equation (12), using $\delta_j = e^{D_j},\ j = 1, 2, \ldots, n$, and the definition of the Todd operator, we obtain

$$D^k f(x) = \mathrm{Td}(D) x^\mu = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} D^m x^\mu = \sum_{0 \leqslant m \leqslant \mu} \frac{\tilde{b}_{k,m}}{m!} \mu^{(m)} x^{\mu-m}. \tag{16}$$

Integrating (16) k_j times over the variable x_j for all $j = 1, \ldots, n$, we get (15). □

Example. As an illustration of the application of (15), we present the solution of the difference equation

$$(\delta_1 - 1)(\delta_2 - 1) f(x, y) = xy. \tag{17}$$

We have $P(\delta) = (\delta_1 - 1)(\delta_2 - 1), R \equiv 1, (\mu_1, \mu_2) = (1, 1), (k_1, k_2) = (1, 1)$, and $f(x, y) = \tilde{B}_{11,11}(x) + Q(x) + S(y)$, where $\tilde{B}_{11,11}(x) = \dfrac{\tilde{b}_{11,00}}{2 \cdot 2} x^2 y^2 + \dfrac{\tilde{b}_{11.01}}{2 \cdot 1} x^2 y + \dfrac{\tilde{b}_{11.10}}{1 \cdot 2} x y^2 + \dfrac{\tilde{b}_{11,11}}{1 \cdot 1} xy$ is the generalized Bernoulli polynomial, $\tilde{b}_{11,m}$ are the expansion coefficients of the generating function

$$\frac{D_1 D_2}{(e^{D_1} - 1)(e^{D_2} - 1)}$$

into the Taylor series at the point $D = 0$; $Q(x), S(y)$ are arbitrary polynomials in one variable.

Calculations give: $\tilde{b}_{11.00} = 1$, $\tilde{b}_{11.01} = -\dfrac{1}{2}$, $\tilde{b}_{11.10} = -\dfrac{1}{2}$, $\tilde{b}_{11,11} = \dfrac{1}{4}$.

Thus, any polynomial solution to (17) has the form

$$f(x, y) = \frac{1}{4}(x^2 y^2 - x^2 y - x y^2 + xy) + Q(x) + S(y).$$

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Сумма функций и полиномиальные решения многомерного разностного уравнения

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Аннотация. Определен набор полиномиальных разностных операторов, позволяющий решить задачу суммирования, и описано пространство полиномиальных решений этих операторов в уравнениях с полиномиальной правой частью. Получен критерий, описывающий эти полиномиальные разностные операторы. Доказана теорема, описывающая пространство полиномиальных решений для операторов.

Ключевые слова: числа Бернулли, многочлены Бернулли, задача суммирования, многомерное разностное уравнение, формула Эйлера–Маклорена, оператор Тодда.

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The Cauchy Problem for Equation of Elasticity Theory

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Abstract. A problem on the analytic continuation of the solution of equation of elasticity theory in a spatial domain is considered. Continuation is based on the values of the solution and stresses on a part of the boundary of this domain. Hence the problem presents the Cauchy problem.

Keywords: Cauchy problem,Lame equation, elliptic system,ill-posed problem, Carleman matrix, regularization.

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1. Introduction and preliminaries

It is well known that Cauchy problem for an elliptic equation is ill-posed. The solution of the problem is unique but unstable (Hadamard‘s example). For ill-posed problems the existence of a solution and it belonging to the correctness class is usually assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, that is usually a compact subset [1]. The uniqueness of the solution follows from the general Holmgren theorem [2].

The Cauchy problem for elliptic equations was the subject of study for mathematicians throughout the twentieth century and it continues to attract the attention of researchers to this day.

The development of special methods that allows one to deal with ill-posed Cauchy problems was stimulated by practical demands. Such problems can be found in hydrodynamics, signal transmission theory, tomography, geological exploration, geophysics, elasticity theory, and so on.

A solution of the Cauchy problem for the one-dimensional system of Cauchy–Riemann equations was first obtained in 1926 by Carleman [3]. He proposed the idea of introducing an additional function into the Cauchy integral formula which allows one to take the limit in order to damp the influence of integrals over that part of the boundary where the values of the function to be continued are not given. The idea of Carleman was developed in 1933 by Goluzin and Krylov [4]. They found a general way to obtain Carleman formulas for the one-dimensional system of Cauchy–Riemann equations.

Resting on the results of Carleman and Goluzin–Krylov, Lavrent'ev introduced the concept of the Carleman function for the one-dimensional system of Cauchy–Riemann equations. The

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method proposed by Lavrent'ev [5] consists in approximation of the Cauchy kernel on the additional part of the domain boundary outside the support of the data of the Cauchy problem.

The Carleman function of the Cauchy problem for the Laplace equation is a fundamental solution that depends on a positive parameter. It tends to zero together with its normal derivative on the part of the domain boundary outside the Cauchy data support as the parameter tends to infinity. Using the Carleman function and Green's integral formula, a Carleman formula is produced. It gives an exact solution of the Cauchy problem when the data are specified exactly. Construction of the Carleman function allows one to construct a regularization if the Cauchy data are given approximately. The existence of the Carleman function follows from the Mergelyan approximation theorem [6].

Fock and Kuni [7] found in 1959 an application of the Carleman formula to the one-dimensional system of Cauchy–Riemann equations. When part of the domain boundary is a segment of the real axis they used the Carleman formula to establish a criterion for the solvability of the Cauchy problem for the system of Cauchy–Riemann equations on the plane. An analog of the Carleman formula and criteria for the solvability of the Cauchy problem were obtained for analytic functions of several variables [8,9], for harmonic functions [10–12] and also [13–16].

A fairly complete survey on Carleman formulas can be found in [5,11,17,18].

In the present paper, a regularized solution of the Cauchy problem for the system of elasticity equations is constructed on the basis of the Carleman function method.

Let us assume that $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ are points in R^m, D_ρ is a bounded simple connected domain in R^m. Its boundary is a cone surface:

$$\Sigma: \quad \alpha_1 = \tau y_m, \;\; \alpha_1^2 = y_1^2 + \ldots + y_{m-1}^2, \;\; \tau = tg\frac{\pi}{2\rho}, \;\; y_m > 0, \;\; \rho > 1.$$

Let us also consider a smooth surface S that lies inside the cone.

Let us consider in domain D_ρ the system of equations of elasticity theory

$$\mu\Delta U(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,U(x) = 0;$$

here $U = (U_1, \dots, U_m)$ is the displacement vector, Δ is the Laplace operator, λ and μ are the Lame constants. For brevity, it is convenient to use matrix notation. Let us introduce the matrix differential operator

$$A(\partial_x) = \|A_{ij}(\partial_x)\|_{m\times m},$$

where

$$A_{ij}(\partial_x) = \delta_{ij}\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the elliptic system of equations can be written in matrix form

$$A(\partial_x)U(x) = 0. \tag{1}$$

Statement of the problem. Let us assume that Cauchy data of a solution U are given on S,

$$U(y) = f(y), \quad y \in S,$$
$$T(\partial_y, n(y))U(y) = g(y), \quad y \in S, \tag{2}$$

where $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_m)$ are prescribed continuous vector functions on S, $T(\partial_y, n(y))$ is the strain operator, i.e.,

$$T(\partial_y, n(y)) = \|T_{ij}(\partial_y, n(y))\|_{m\times m} = \left\|\lambda n_i\frac{\partial}{\partial y_j} + \mu n_j\frac{\partial}{\partial y_i} + \mu\delta_{ij}\frac{\partial}{\partial n}\right\|_{m\times m},$$

δ_{ij} is the Kronecker delta, and $n(y) = (n_1(y), \ldots, n_m(y))$ is the unit normal vector to the surface S at the point y.

It is required to determine function $U(y)$ in D, i.e., to find an analytic continuation of the solution of the system of equations in the domain from the values of f and g on a smooth part of S of the boundary.

In this paper, the Cauchy problem for system of static equations of elasticity theory is solved for cone type domains by the method of regularization of the solution according to Lavrentiev.

In earlier works [14–16], this problem was considered either in two or three-dimensional spaces or for other special domains for which it is required to construct special matrices of fundamental solutions in explicit form that depends on the domain and dimension of the space.

Similar problems were considered for an arbitrary domain, by expanding the fundamental solution into a series in terms of spherical functions [12, 19].

Let us suppose that instead of $f(y)$ and $g(y)$ their approximations $f_\delta(y)$ and $g_\delta(y)$ with accuracy δ, $0 < \delta < 1$ (in the metric of C) are given. They do not necessarily belong to the class of solutions. In this paper, a family of functions $U(x, f_\delta, g_\delta) = U_{\sigma\delta}(x)$ that depends on parameter σ is constructed. It is also proved that under certain conditions and special choice of parameter $\sigma(\delta)$ the family $U_{\sigma\delta}(x)$ converges in the usual sense to the solution $U(x)$ of problem (1), (2) as $\delta \to 0$.

Following A. N. Tikhonov, $U_{\sigma\delta}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem [1].

2. Construction of the matrix of fundamental solution for the system of equations of elasticity

Definition 2.1. *Matrix $\Gamma(y, x) = ||\Gamma_{ij}(y, x)||_{m\times m}$, is called the matrix of fundamental solutions of system (1), where*

$$\Gamma_{ij}(y,x) = \frac{1}{2\mu(\lambda+2\mu)}((\lambda+3\mu)\delta_{ij}q(y,x) - (\lambda+\mu)(y_j - x_j)\frac{\partial}{\partial x_i}q(y,x)), \quad i,j = 2,\ldots,m,$$

$$q(y,x) = \begin{cases} \dfrac{1}{(2-m)\omega_m}\cdot\dfrac{1}{|y-x|^{m-2}}, & m > 2 \\ \dfrac{1}{2\pi}\ln|y-x|, & m = 2, \end{cases}$$

and ω_m is the area of unit sphere in R^m.

Matrix $\Gamma(y, x)$ is symmetric and its columns and rows satisfy equation (1) at an arbitrary point $x \in R^m$, except $y = x$. Thus, we have

$$A(\partial_x)\Gamma(y,x) = 0, \quad y \neq x.$$

Developing idea of Lavrent'ev on the notion of Carleman function of the Cauchy problem for the Laplace equation [5], the following notion is introduced.

Definition 2.2. *The Carleman matrix of problem (1), (2) is $(m\times m)$ matrix $\Pi(y, x, \sigma)$ that satisfies the following two conditions*

1) $\Pi(y, x, \sigma) = \Gamma(y, x) + G(y, x, \sigma)$, where σ is a positive parameter, and matrix $G(y, x, \sigma)$ satisfies system (1) everywhere in domain D with respect to the variable y.

2) The following relation holds

$$\int\limits_{\partial D\setminus S} (|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|)ds_y \leqslant \varepsilon(\sigma),$$

where $\varepsilon(\sigma) \to 0$ *as* $\sigma \to \infty$ *uniformly in* x *on compact subsets of* D*. Here and elsewhere* $|\Pi|$ *denotes the Euclidean norm of matrix* $\Pi = ||\Pi_{ij}||$*, i.e.,* $|\Pi| = \Big(\sum\limits_{i,j=1}^{m} \Pi_{ij}^2\Big)^{\frac{1}{2}}$*. In particular,* $|U| = \Big(\sum\limits_{i=1}^{m} U_i^2\Big)^{\frac{1}{2}}$ *for a vector* $U = (U_1, \dots, U_m)$.

Definition 2.3. *A vector function* $U(y) = (U_1(y), \dots, U_m(y))$ *is said to be regular in* D *if it is continuous together with its partial derivatives of second order in* D *and partial derivatives of first order in* $\overline{D} = D\bigcup\partial D$.

In the theory of partial differential equations solution functions of potential type play an important role. As an example of such representation, the formula of Somilian–Bettis is considered below [20].

Theorem 2.1. *Any regular solution* $U(x)$ *of equation (1) in the domain* D *is represented as*

$$U(x) = \int_{\partial D} (\Gamma(y,x)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Gamma(y,x)\}^* U(y))ds_y, \;\; x \in D, \tag{3}$$

here A^* *is conjugate to* A.

Suppose that Carleman matrix $\Pi(y,x,\sigma)$ of problem (1), (2) exists. Then for the regular functions $v(y)$ and $u(y)$ the following relation holds

$$\int_{\partial D_\rho} [v(y)\{A(\partial_y)U(y)\} - \{A(\partial_y)v(y)\}^* U(y)]dy =$$

$$= \int_{\partial D_\rho} [v(y)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)v(y)\}^* U(y)]ds_y.$$

Substituting $v(y) = G(y,x,\sigma)$ and $u(y) = U(y)$ into the above relation, we obtain

$$\int_{\partial D_\rho} [G(y,x,\sigma)\{A(\partial_y)U(y)\} - \{A(\partial_y)G(y,x,\sigma)\}^* U(y)]dy = 0. \tag{4}$$

The theorem follows from (3) and (4).

Theorem 2.2. *Any regular solution* $U(x)$ *of equation (1) in domain* D_ρ *is represented as*

$$U(x) = \int_{\partial D_\rho} (\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^* U(y))ds_y, \;\; x \in D_\rho, \tag{5}$$

where $\Pi(y,x,\sigma)$ *is the Carleman matrix.*

Suppose that $K(\omega)$, $\omega = u + iv$ (u and v are real) is an entire function that takes real values on the real axis. It satisfies the following conditions

$$K(u) \neq 0, \;\; \sup_{v\geqslant 1} |v^p K^{(p)}(\omega)| = M(p,u) < \infty, \;\; p = 0, \dots, m, \;\; u \in R^1.$$

Let
$$s = \alpha^2 = (y_1 - x_1)^2 + \cdots + (y_{m-1} - x_{m-1})^2.$$

For $\alpha > 0$ function $\Phi(y, x)$ is defined by the following relations. If $m = 2$ then

$$-2\pi K(x_2)\Phi(y,x) = \int_0^\infty Im\left[\frac{K(i\sqrt{u^2+\alpha^2}+y_2)}{i\sqrt{u^2+\alpha^2}+y_2-x_2}\right]\frac{udu}{\sqrt{u^2+\alpha^2}}. \tag{6}$$

If $m = 2n + 1$, $n \geqslant 1$ then

$$C_m K(x_m)\Phi(y,x) = \frac{\partial^{n-1}}{\partial s^{n-1}}\int_0^\infty Im\left[\frac{K(i\sqrt{u^2+\alpha^2}+y_m)}{i\sqrt{u^2+\alpha^2}+y_m-x_m}\right]\frac{du}{\sqrt{u^2+\alpha^2}}, \tag{7}$$

where $C_m = (-1)^{n-1} \cdot 2^{-n}(m-2)\pi\omega_m(2n-1)!$. If $m = 2n$, $n \geqslant 2$ then

$$C_m K(x_m)\Phi(y,x) = \frac{\partial^{n-2}}{\partial s^{n-2}} Im \frac{K(\alpha i + y_m)}{\alpha(\alpha + y_m - x_m)}, \tag{8}$$

where $C_m = (-1)^{n-1}(n-1)!(m-2)\omega_m$.

The following theorem is valid [10]

Theorem 2.3. *Function $\Phi(y, x)$ can be expressed as*

$$\Phi(y,x) = \frac{1}{2\pi} ln\frac{1}{r} + g_2(y,x), \ \ m = 2, \ \ r = |y-x|,$$

$$\Phi(y,x) = \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y,x), \ \ m \geqslant 3, \ \ r = |y-x|,$$

where $g_m(y, x)$, $m \geqslant 2$ is a functions defined for all values of y, x and it is harmonic with respect to variable y in R^m.

Using function $\Phi(y, x)$, the following matrix is constructed

$$\Pi(y,x) = \|\Pi_{ij}(y,x)\|_{m\times m} = \Big\| \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\delta_{ij}\Phi(y,x) - \\ - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i}\Phi(y,x)\Big\|_{m\times m}, i,j = 1,2,\ldots,m. \tag{9}$$

3. The solution of problems (1), (2) in domain D_ρ

I. Let $x_0 = (0, \ldots, 0, x_m) \in D_\rho$. Let us introduce the following designations

$$\beta = \tau y_m - \alpha_0, \ \ \gamma = \tau x_m - \alpha_0, \ \ \alpha_0^2 = x_1^2 + \ldots + x_{m-1}^2, \ \ r = |x-y|,$$

$$s = \alpha^2 = (y_1 - x_1)^2 + \ldots + (y_{m-1} - x_{m-1})^2, \ \ w = i\tau\sqrt{u^2+\alpha^2} + \beta, \ \ w_0 = i\tau\alpha + \beta.$$

Let us construct a Carleman matrix for problem (1), (2) for domain D_ρ. The Carleman matrix is explicitly expressed in terms of the Mittag–Löffler entire function. It is defined by series [21]

$$E_\rho(w) = \sum_{n=0}^{\infty}\frac{w^n}{\Gamma\left(1+\frac{n}{\rho}\right)}, \quad \rho > 0, \quad E_1(w) = \exp w,$$

where $\Gamma(\cdot)$ is the Euler function.

Let us denote the contour in complex plane w by $\gamma = \gamma(1,\theta)$, $0 < \theta < \dfrac{\pi}{\rho}$, $\rho > 1$. It is in the direction of nondecreasing $argw$ and it consists of the following parts.

1) ray $argw = -\theta$, $|w| \geqslant 1$,
2) arc $-\theta \leqslant argw \leqslant \theta$ circle $|w| = 1$,
3) ray $argw = \theta$, $|w| \geqslant 1$.

Contour γ divides complex plane on two parts: D^- and D^+. They are on the left and the right sides of γ, respectively. Suppose that $\dfrac{\pi}{2\rho} < \theta < \dfrac{\pi}{\rho}$, $\rho > 1$. Then the following relation holds

$$E_\rho(w) = \exp w^\rho + \Psi_\rho(w), \quad w \in D^+$$

$$E_\rho(w) = \Psi_\rho(w), \quad E'_\rho(w) = \Psi'_\rho(w), \quad w \in D^-, \tag{10}$$

where

$$\Psi_\rho(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{\zeta - w}d\zeta, \qquad \Psi'_\rho(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{(\zeta - w)^2}d\zeta. \tag{11}$$

$$\begin{aligned}
Re\Psi_\rho(w) &= \frac{\Psi_\rho(w) + \Psi_\rho(\overline{w})}{2} = \frac{\rho}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho(\zeta - Rew)}{(\zeta - w)(\zeta - \overline{w})}d\zeta,\\
Im\Psi_\rho(w) &= \frac{\Psi_\rho(w) - \Psi_\rho(\overline{w})}{2i} = \frac{\rho Imw}{2\pi i}\int_\gamma \frac{\exp\zeta^\rho}{(\zeta - w)(\zeta - \overline{w})}d\zeta,\\
\frac{Im\Psi'_\rho(w)}{Imw} &= \frac{\rho}{2\pi i}\int_\gamma \frac{2\exp\zeta^\rho(\zeta - Rew)}{(\zeta - w)^2(\zeta - \overline{w})^2}d\zeta.
\end{aligned} \tag{12}$$

In what follows, $\theta = \dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2}$, $\rho > 1, \varepsilon_2 > 0$. It is clear that if $\dfrac{\pi}{2\rho} + \varepsilon_2 \leqslant |argw| \leqslant \pi$ then $w \in D^-$ and $E_\rho(w) = \Psi_\rho(w)$.

Let us set

$$E_{k,q}(w) = \frac{\rho}{2\pi i}\int_\gamma \frac{\zeta^q \exp\zeta^\rho}{(\zeta - w)^k(\zeta - \overline{w})^k}d\zeta, \quad k = 1, 2, \ldots, \quad q = 0, 1, 2, \ldots.$$

If $\dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2} \leqslant |argw| \leqslant \pi$ then the following inequalities are valid

$$|E_\rho(w)| \leqslant \frac{M_1}{1 + |w|}, \quad |E'_\rho(w)| \leqslant \frac{M_2}{1 + |w|^2},$$

$$|E_{k,q}(w)| \leqslant \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots, \tag{13}$$

where M_1, M_2, M_3 are constants.

Suppose that $\theta = \dfrac{\pi}{2\rho} + \dfrac{\varepsilon_2}{2} < \dfrac{\pi}{\rho}$, $\rho > 1$ in (10). Then $E_\rho(w) = \Psi_\rho(w)$, $\cos\rho\theta < 0$ and

$$\int_\gamma |\zeta|^q \exp(\cos\rho\theta|\zeta|^q)|d\zeta| < \infty, \quad q = 0, 1, 2, \ldots. \tag{14}$$

In this case for sufficiently large $|w|$ ($w \in D^+$, $\overline{w} \in D^-$) we have

$$\min_{\zeta\in\gamma}|\zeta - w| = |w|\sin\frac{\varepsilon_2}{2}, \quad \min_{\zeta\in\gamma}|\zeta - \overline{w}| = |w|\sin\frac{\varepsilon_2}{2}. \tag{15}$$

Now from (10) and

$$\frac{1}{\zeta - w} = -\frac{1}{w} + \frac{\zeta}{w(\zeta - \overline{w})},$$
$$\frac{1}{\zeta - \overline{w}} = -\frac{1}{\overline{w}} + \frac{\zeta}{\overline{w}(\zeta - \overline{w})}, \tag{16}$$

for large $|w|$ we obtain

$$\left| E_\rho(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w}\right| \leqslant \frac{\rho}{2\pi \sin\frac{\varepsilon_2}{2}}\frac{1}{|w|^2}.$$

$$\int_\gamma |\zeta| \exp\left[\cos\rho\theta|\zeta|^\rho\right]|d\zeta| \leqslant \frac{const}{|w|^2},$$

$$\Gamma^{-1}\left(1 - \frac{1}{\rho}\right) = \frac{\rho}{2\pi i}\int_\gamma \exp\left(\zeta^\rho\right) d\zeta.$$

It follows from this that

$$|E_\rho(w)| \leqslant \frac{M_1}{1 + |w|}.$$

From (11), (15) and

$$\frac{1}{(\zeta - w)^2} = \frac{1}{w^2} - \frac{2\zeta}{w^2(\zeta - w)} + \frac{\zeta^2}{w^2(\zeta - w)^2}$$

for large $|w|$ we obtain

$$\left| E'_\rho(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w^2}\right| \leqslant \frac{const}{|w|^3}$$

or

$$|E'_\rho(w)| = \frac{M_2}{1 + |w|^2}.$$

Considering (16), for $k = 1, 2, \ldots$ we have

$$\frac{1}{(\zeta - w)^k(\zeta - \overline{w})^k} = \left[\frac{(-1)^k}{w^k} + \ldots + \frac{\zeta^k}{w^k(\zeta - w)^k}\right]\left[\frac{(-1)^k}{\overline{w}^k} + \ldots + \frac{\zeta^k}{\overline{w}^k(\zeta - \overline{w})^k}\right] =$$
$$= \frac{1}{|w|^{2k}} - \frac{k}{|w|^{2k+1}|\zeta - w|} + \ldots.$$

Taking into account previous relations and (14), (15), for large $|w|$ we obtain

$$\left| E_{k,q}(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{|w|^{2k}}\right| \leqslant \frac{const}{|w|^{2k+1}}$$

or

$$|E'_{k,q}(w)| = \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots.$$

Therefore, since

$$(\zeta - w)(\zeta - \overline{w}) = \zeta^2 - 2\zeta(y_m - x_m) + u^2 + \alpha^2 + (y_m - x_m)^2, \quad \alpha^2 = s,$$

then

$$\frac{\partial^{n-1}}{\partial s^{n-1}}\frac{1}{(\zeta - w)(\zeta - \overline{w})} = \frac{(-1)^{n-1}(n-1)!}{(\zeta - w)^n(\zeta - \overline{w})^n}.$$

Now we obtain from (11) that

$$\frac{d^{n-1}}{ds^{n-1}}ReE_\rho(w) = \frac{(-1)^{n-1}(n-1)!\rho}{2\pi i}\int_\gamma \frac{(\zeta-(y_m-x_m))\exp\zeta^\rho}{(\zeta-w)^n(\zeta-\overline{w})^n}d\zeta,$$

$$\frac{d^{n-1}}{ds^{n-1}}\frac{ImE_\rho(w)}{\sqrt{u^2+\alpha^2}} = \frac{(-1)^{n-1}(n-1)!\rho}{\pi i}\int_\gamma \frac{\exp\zeta^\rho}{(\zeta-w)^n(\zeta-\overline{w})^n}d\zeta,$$

Then from (3.) we have

$$\left|\frac{d^{n-1}}{ds^{n-1}}ReE_\rho(w)\right| \leqslant \frac{const\cdot r}{1+|w|^2}$$

$$\left|\frac{d^{n-1}}{ds^{n-1}}\frac{ImE_\rho(w)}{\sqrt{u^2+\alpha^2}}\right| \leqslant \frac{const\cdot r}{1+|w|^2}.$$

For $\sigma > 0$ we set in formulas (6)-(9)

$$K(w) = E_\rho(\sigma^{\frac{1}{\rho}}w), \quad K(x_m) = E_\rho(\sigma^{\frac{1}{\rho}}\gamma). \tag{17}$$

Then, for $\rho > 1$ we obtain

$$\Phi(y,x) = \Phi_\sigma(y,x) = \frac{\varphi_\sigma(y,x)}{c_mE_\rho(\sigma^{\frac{1}{\rho}}\gamma)}, \quad y \neq x,$$

where $\varphi_\sigma(y,x)$ is defined as follows:
if $\frac{1}{\rho}$ $m = 2$ then

$$\varphi_\sigma(y,x) = \int_0^\infty Im\frac{E_\rho(\sigma w)}{i\sqrt{u^2+\alpha^2}+y_2-x_2}\frac{udu}{\sqrt{u^2+\alpha^2}},$$

if $m = 2n+1$, $n \geqslant 1$ then

$$\varphi_\sigma(y,x) = \frac{d^{n-1}}{ds^{n-1}}\int_0^\infty Im\frac{E_\rho(\sigma^{\frac{1}{\rho}}w)}{i\sqrt{u^2+\alpha^2}+y_m-x_m}\frac{udu}{\sqrt{u^2+\alpha^2}}, y \neq x,$$

if $m = 2n$, $n \geqslant 2$ then

$$\varphi_\sigma(y,x) = \frac{d^{n-2}}{ds^{n-2}}Im\frac{E_\rho\left(\sigma^{\frac{1}{\rho}}w\right)}{\alpha(i\alpha+y_m-x_m}, y \neq x.$$

Let us define matrix $\Pi(y,x,\sigma)$ using (9) for $\Phi(y,x) = \Phi_\sigma(y,x)$.
It was proved [10]

Theorem 3.1. *Function $\Phi_\sigma(y,x)$ can be expressed as*

$$\Phi_\sigma(y,x) = \frac{1}{2\pi}ln\frac{1}{r} + g_2(y,x,\sigma), \ \ m = 2, \ \ r = |y-x|,$$

$$\Phi_\sigma(y,x) = \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y,x,\sigma), \ \ m \geqslant 3, \ \ r = |y-x|,$$

where $g_m(y,x,\sigma)$, $m \geqslant 2$ is a function defined for all y,x and it is harmonic with spect to variable y in R^m.

We obtain from this theorem

Theorem 3.2. *Matrix $\Pi(y,x,\sigma)$ defined in (7)–(9) and (17) is a Carleman matrix for problem (1), (2).*

Let us first consider some properties of function $\Phi_\sigma(y,x)$.

I. Let $m=2n+1,\ \ n\geqslant 1,\ \ x\in D_\rho,\ \ y\neq x, \sigma\geqslant\sigma_0>0$ then
1) for $\beta\leqslant\alpha$ the following inequalites hold:

$$|\Phi_\sigma(y,x)|\leqslant C_1(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_2(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_3(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho),\ \ i=1,\dots,m, \tag{18}$$

2) for $\beta>\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant C_4(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_5(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant C_6(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ i=1,\dots,m. \tag{19}$$

II. Let $m=2n,\ \ n\geqslant 2,\ \ x\in D_\rho,\ \ x\neq y,\ \ \sigma\geqslant\sigma_0>0$ then
1) for $\beta\leqslant\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant \widetilde{C}_1(\rho)\frac{\sigma^{m-3}}{r^{m-2}}\exp(-\sigma\gamma^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_2(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_3(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho),\ \ y\in\partial D_\rho,\ \ i=1,\dots,m, \tag{20}$$

2) for $\beta>\alpha$ the following inequalities hold:

$$|\Phi_\sigma(y,x)|\leqslant \widetilde{C}_4(\rho)\frac{\sigma^{m-2}}{r^{m-2}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_5(\rho)\frac{\sigma^{m}}{r^{m-1}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial\Phi_\sigma}{\partial n}(y,x)\right|\leqslant \widetilde{C}_6(\rho)\frac{\sigma^{m+2}}{r^{m}}\exp(-\sigma\gamma^\rho+\sigma Re\omega_0^\rho),\ \ y\in\partial D_\rho,\ \ i=1,\dots,m. \tag{21}$$

III. Let $m=2,\ \ x\in D_\rho,\ \ x\neq y,\ \ \sigma\geqslant\sigma_0>0$ then
1) if $\beta\leqslant\alpha$ then

$$|\Phi_\sigma(y,x)|\leqslant C_7(\rho)E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)\ln\frac{1+r^2}{r^2},$$

$$\left|\frac{\partial\Phi_\sigma}{\partial y_i}(y,x)\right|\leqslant C_8(\rho)\frac{E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)}{r}, \tag{22}$$

2) if $\beta > \alpha$ then

$$|\Phi_\sigma(y,x)| \leqslant \widetilde{C}_7(\rho)E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)(\ln\frac{1+r^2}{r^2})\exp(\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial\Phi_\sigma}{\partial y_i}(y,x)\right| \leqslant \widetilde{C}_8(\rho)E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)\frac{1}{2}\exp(\sigma Re\omega_0^\rho). \tag{23}$$

Here all coefficients $C_i(\rho)$ and $\widetilde{C}_i(\rho)$, $\quad i = 1,\dots 8$ depend on ρ.

Proof of Theorem 3.2. From the definition of $\Pi(y,x,\sigma)$ and Lemma 1 we have

$$\Pi(y,x,\sigma) = \Gamma(y,x) + G(y,x,\sigma),$$

where

$$G(y,x,\sigma) = ||G_{kj}(y,x,\sigma)||_{m\times m} = \\ = \left|\left|\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\delta_{kj}g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j-x_j)\frac{\partial}{\partial y_i}g_m(y,x,\sigma)\right|\right|_{m\times m}.$$

Let us prove that $A(\partial_y)G(y,x,\sigma) = 0$. Since $\Delta_y g_m(y,x,\sigma) = 0$, $\Delta_y = \sum\limits_{k=1}^{m}\frac{\partial^2}{\partial y_k^2}$ and taking into account relation for the jth column $G^j(y,x,\sigma)$

$$\div G^j(y,x,\sigma) = \frac{1}{2\mu(\lambda+2\mu)}\cdot\frac{\partial}{\partial y_j}g_m(y,x,\sigma),$$

we obtain relation for the kth components of $A(\partial_y)G^j(y,x,\sigma)$

$$\sum_{i=1}^{m}A(\partial_y)_{ki}G_{ij}(y,x,\sigma) = \mu\Delta_y\left[\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\cdot\delta_{kj}g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j-x_j)\frac{\partial}{\partial y_k}g_m(y,x,\sigma)\right] + \\ + (\lambda+\mu)\frac{\partial}{\partial y_k}divG^j(y,x,\sigma) = \\ = -\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma) + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma) = 0$$

Therefore, each column of matrix $G(y,x,\sigma)$ satisfies system (1) with respect to the variable y everywhere in R^m.

The second condition on the Carleman matrix follows from inequalities (18)–(23). The proof of the theorem is complete. □

For fixed $x \in D_\rho$ we denote the part of S, where $\beta \geqslant \alpha$ by S^*. If $x = x_0 = (0,\dots,0,x_m) \in D_\rho$ then $S = S^*$. Consider the point $(0,\dots,0) \in D_\rho$. Suppose that

$$\frac{\partial U}{\partial n}(0) = \frac{\partial U}{\partial y_m}(0), \quad \frac{\partial\Phi_\sigma(0,x)}{\partial n} = \frac{\partial\Phi_\sigma(0,x)}{\partial y_m}.$$

Let

$$U_\sigma(y) = \int_{S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y, \ \ x \in D_\rho. \tag{24}$$

Theorem 3.3. *Let $U(x)$ be a regular solution of system (1) in D_ρ, such that*

$$|U(y)| + |T(\partial_y,n)U(y)| \leqslant M, \ \ y \in \Sigma. \tag{25}$$

Then
1) if $m = 2n+1,\ \ n \geqslant 1$ *and for* $x \in D_\rho,\ \ \sigma \geqslant \sigma_0 > 0$ *the following estimate is valid:*

$$|U(x) - U_\sigma(x)| \leqslant MC_1(x)\sigma^{m+1}\exp(-\sigma\gamma^\rho),$$

2) if $m = 2n,\ n \geqslant 1,\ x \in D_\rho,\ \sigma \geqslant \sigma_0 > 0$ *the following estimate is valid*

$$|U(x) - U_\sigma(x)| \leqslant MC_2(x)\sigma^m\exp(-\sigma\gamma^\rho),$$

where

$$C_k(x) = C_k(\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m},\ \ k = 1, 2,$$

$C_k(\rho)$ *is a constant that depends on* ρ*.*

Proof. It follows from (5) that

$$U(x) = \int_{S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y +$$
$$+\int\limits_{\partial D_\rho\backslash S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y,\ \ x \in D_\rho.$$

Therefore, we have from (24) that

$$|U(x) - U_\sigma(x)| \leqslant \int\limits_{\partial D_\rho\backslash S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*U(y)]ds_y \leqslant$$
$$\leqslant \int\limits_{\partial D_\rho\backslash S^*}[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\,[|T(\partial_y,n)\Pi(y,x,\sigma)| + |U(y)|]\,ds_y.$$

Taking into accoun inequalities (18)–(23) and condition (25), we obtain for $\beta \leqslant \alpha$ and $m = 2n+1$, $n \geqslant 1$

$$|U(x) - U_\sigma(x)| \leqslant MC_1(\rho)\sigma^{m+1}\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

For $m = 2n,\ \ n \geqslant 1$ we obtain

$$|U(x) - U_\sigma(x)| \leqslant MC_2(\rho)\sigma^m\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

The proof of the theorem is complete. □

One can determine $U(x)$ approximately if, instead of $U(y)$ and $T(\partial_y,n)U(y)$, their continuous approximations $f_\delta(y)$ and $g_\delta(y)$ are given on surface S:

$$\max_S|U(y) - f_\delta(y)| + \max_S|T(\partial_y,n)U(y) - g_\delta(y)| \leqslant \delta,\ \ 0 < \delta < 1. \tag{26}$$

Function $U_{\sigma\delta}(x)$ is defined as follows

$$U_{\sigma\delta}(x) = \int_{s^*}[\Pi(y,x,\sigma)g_\delta(y) - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*f_\delta(y)]ds_y,\ \ x \in D_\rho, \tag{27}$$

where

$$\sigma = \frac{1}{R^\rho}\ln\frac{M}{\delta},\ \ \ R^\rho = \max_{y\in S}Re\omega_0^\rho.$$

Then the following theorem holds.

Theorem 3.4. *Let $U(x)$ be a regular solution of system (1) in D_ρ, such that*

$$|U(y)| + |T(\partial_y, n)U(y)| \leqslant M, \ \ y \in \partial D_\rho.$$

Then,
1) if $m = 2n+1$, $n \geqslant 1$ then the following estimate is valid

$$|U(x) - U_{\sigma\delta}(x)| \leqslant C_1(x)\delta^{(\frac{\gamma}{R})^\rho}\left(ln\frac{M}{\delta}\right)^{m+1},$$

2) if $m = 2n$, $n \geqslant 1$ then the following estimate is valid:

$$|U(x) - U_{\sigma\delta}(x)| \leqslant C_2(x)\delta^{(\frac{\gamma}{R})^\rho}\left(\ln\frac{M}{\delta}\right)^{m},$$

where

$$C_k(x) = C_k(\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}, \ \ k = 1, 2.$$

Proof. It follows from (5) and (27) that

$$U(x) - U_{\sigma\delta}(x) = \int\limits_{\partial D_\rho\setminus S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - \{T(\partial_y,n)\Pi(y,x,\sigma)\}^* U(y)]ds_y +$$

$$+\int_{S^*}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y) - g_\delta(y)\} + \{T(\partial_y,n)\Pi(y,x,\sigma)\}^*(U(y) - f_\delta(y))]ds_y =$$

$$= I_1 + I_2.$$

Taking into account Theorem 3.3, we obtain for $m = 2n+1$, $n \geqslant 1$,

$$|I_1| = MC_1(\rho)\sigma^{m+1}\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m},$$

and for $m = 2n$, $n \geqslant 1$

$$|I_1| = MC_2(\rho)\sigma^{m}\exp(-\sigma\gamma^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

Let us consider $|I_2|$:

$$|I_2| = \int_{S^*}(|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|)\left(|T(\partial_y,n)U(y) - g_\delta(y)| + |U(y) - f_\delta(y)|\right)ds_y.$$

Taking into account Theorem 3.1 and condition (26), we obtain for $m = 2n+1$, $n \geqslant 1$

$$|I_2| = \widetilde{C}_1(\rho)\sigma^{m+1}\delta\exp(-\sigma\gamma^\rho + \sigma Rew_0^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}$$

and for $m = 2n$, $n \geqslant 1$,

$$|I_2| = \widetilde{C}_2(\rho)\sigma^{m}\delta\exp(-\sigma\gamma^\rho + \sigma Rew_0^\rho)\int_{\partial D_\rho}\frac{ds_y}{r^m}.$$

Therefore, from

$$\sigma = \frac{1}{R^\rho}\ln\frac{M}{\delta}, \ \ R^\rho = \max_{y\in S} Re\omega_0^\rho.$$

The theorem is proved. □

Corollary 1. The limits

$$\lim_{\sigma\to\infty}U_\sigma(x) = U(x), \ \ \lim_{\delta\to 0}U_{\sigma\delta}(x) = U(x)$$

hold uniformly on any compact set from D_ρ.

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Задача Коши для уравнения теории упругости

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Аннотация. Рассматривается задача об аналитическом продолжении решения системы теории упругости в область по значениям решения и его напряжений на части границы этой области, т. е. задача Коши.

Ключевые слова: задача Коши, теория упругости, эллиптическая система, некорректно поставленная задача, матрица Карлемана, регуляризация.

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Examples of Computing Power Sums of Roots of Systems of Equations

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Abstract. Examples of computing power sums of roots of systems of equations, including transcendental, are considered. Since the number of roots of such systems is, as a rule, infinite, it is necessary to study the power sums of roots in a negative degree. Formulas for finding residue integrals, their connection with power sums of roots to a negative degree, multidimensional analogues of Waring's formulas are given.

Keywords: transcendental systems of equations, power sums of roots, residue integrals.

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Introduction

Basing on the multidimensional logarithmic residue, L. A. Aizenberg has obtained formulas for power sums of roots of systems of non-linear algebraic equations in $\mathbb{C}^n$ [1, Theorem 2, Corollary 2]. These formulas enable us to find sums of values of holomorphic functions in roots without calculation of roots themselves, and to develop a new method of investigation of systems of equations in $\mathbb{C}^n$. For different types of systems, such formulas have different forms.

This was proposed by L. A. Aizenberg [1], and the development of this idea was continued in monograph [2]. The main idea of the method is to find power sums of roots of a system in positive degrees, and then use either one-dimensional or multidimensional Newton recurrent formulas to recover them. Unlike the classical elimination method, this method is less time consuming and does not increase the multiplicity of roots. The base of the method is a formula [1] obtained by using the multidimensional logarithmic residue for evaluation of sums of values of an arbitrary polynomial in roots of a given system of algebraic equations without calculation of the roots themselves.

As a rule, we cannot obtain formulas for the sums of roots of non-algebraic (transcendent) equations, because the set of the roots can be infinite, and power series of their coordinates can be divergent. However, the non-algebraic systems of equations arise, for instance, in the problems of chemical kinetics [3]. Therefore, such systems demand further investigations.

The power sums of negative degrees of roots of various transcendent systems are studied in papers [4–9]. These sums are calculated by means of residue integral over skeletons of polydisks with center at the origin. Note that this residue integral in general is not a multidimensional

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logarithmic residue, or the Grothendieck residue. There exist formulas of residue integrals for various types of homogeneous systems of lower orders, and established connections with power sums of roots of the system in negative degree.

More complicated systems are investigated in the works [7, 8]. Here the lower homogeneous parts allow expansion into product of linear factors, and the cycles of integration in the residue integrals are determined by these factors.

The subjects of the paper [9] are algebraic and transcendent systems of equations, where the lower homogeneous parts of functions form non-degenerated system of algebraic equations. Formulas were found for the residue integrals, power sums of the roots in negative degree, and multidimensional analogs of the Waring formula, i. e., the relations between the coefficients of the equations with the residue integrals. In the next section we use the results of this article.

1. Principal statements

Consider a system of equations $f(z) = 0$, where $f(z) = (f_1(z),\ f_2(z), \ldots , f_n(z))$ are functions of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \tag{1}$$

where P_j is the lower homogeneous part of the function, i.e. degree of all monomials (in all variables) in Q_j is strictly greater than in P_j.

$$P_j(z) = \sum_{\|\beta^j\|=m_j} b^j_{\beta^j} z^{\beta^j},$$

and functions Q_j develop in Taylor series in a neighborhood of the origin that converge absolutely and uniformly:

$$Q_j(z) = \sum_{\|\alpha^j\|>m_j} a^j_{\alpha^j} z^{\alpha^j}.$$

For non-degenerate systems of polynomials P, i.e. such that there exists only one their common zero – the origin, one can show that ([9, Lemma 1]) the cycle

$$\Gamma_P = \{z \in \mathbb{C}^n : |P_j| = r_j,\ r_j > 0,\ j = \overline{1, n}\}$$

is a compact set that does not intersect with the coordinate axes for almost all r_j.

Denote by J_γ the residue integral

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z^{\gamma+I}} \cdot \frac{df}{f} =$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n},$$

where $\gamma = (\gamma_1, \ldots \gamma_n)$.

Theorem 1.1 ([9], Theorem 1)**.** *Under the assumtions made, for a system of equations of the form (1) we have*

$$J_\gamma = \sum_{\|\alpha\|\leqslant\|\gamma\|+n} (-1)^{\|\alpha\|} \mathfrak{M}\left[\frac{\Delta \cdot Q^\alpha}{z^\gamma \cdot P^{\alpha+I}}\right], \tag{2}$$

where Δ is the Jacobian of the system (1), $Q^\alpha = Q_1^{\alpha_1} \cdot \ldots \cdot Q_n^{\alpha_n}$, and $\mathfrak{M}$ is a linear functional that to a Laurent polynomial assigns its free term.

We shall additionally assume that the system of polynomials P does not have zeroes at the infinity in the space $\overline{\mathbb{C}}^n$ and consider the case $Q_j(z)$ are polynomials of degree s_j with the condition: for each i

$$\deg_{z_i} P_i < \deg_{z_i} Q_i, \quad \deg_{z_j} P_i \geqslant \deg_{z_j} Q_i, \quad i \neq j.$$

We make in the functions $f_j(z) = P_j(z) + Q_j(z)$ the change $z_i = \dfrac{1}{w_i}$ assuming that all $w_i \neq 0$. We get

$$f_j\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \frac{1}{w_1^{m_j^1} \cdot \ldots \cdot w_j^{s_j^j} \cdot \ldots \cdot w_n^{m_j^n}} \cdot \left(\widetilde{P}_j(w) + \widetilde{Q}_j(w)\right),$$

where $\widetilde{P}_j(w)$ and $\widetilde{Q}_j(w)$ are polynomials with the property $\deg \widetilde{P}_j > \deg \widetilde{Q}_j$, and $\widetilde{f}_j(w) = \widetilde{P}_j(w) + \widetilde{Q}_j(w)$.

Denote by $\Gamma_{\widetilde{P}}$ the cycle

$$\Gamma_{\widetilde{P}} = \{w \in \mathbb{C}^n : |\widetilde{P}_j| = \varepsilon_j, \quad \varepsilon_j > 0\}.$$

Then for an arbitrary multi-index γ the integral J_γ is equal to ([9, Lemma 9])

$$J_\gamma = \frac{(-1)^n}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\widetilde{f}_1}{\widetilde{f}_1} \wedge \frac{d\widetilde{f}_2}{\widetilde{f}_2} \wedge \ldots \wedge \frac{d\widetilde{f}_n}{\widetilde{f}_n}.$$

Moreover, if $w_1, \ldots, w_p$ are zeroes of the system $\widetilde{f}(w) = 0$ (counting multiplicities) where $w_k = (w_{k1}, w_{k2}, \ldots, w_{kn})$ then

$$J_\gamma = \sum_{j=1}^{s} w_{j1}^{\gamma_1+1} \cdot w_{j2}^{\gamma_2+1} \cdot \ldots \cdot w_{jn}^{\gamma_n+1}.$$

These zeroes are related with the zeroes $z_1, \ldots, z_p$ of the original system that do not lie on the coordinate axes via $z_{km} = \dfrac{1}{w_{km}}$. Collecting obtained formulas and computing the integral in (2) using the transformation formula for the Grothendieck residue, we get the main result.

Theorem 1.2 ([9], Theorem 6)**.** *Under the assumptions made, the power sum of roots of the system (1) is equal to*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdot \ldots \cdot z_{jn}^{\gamma_n+1}} = J_\gamma =$$

$$= \sum_{\|\alpha\| \leqslant \|\gamma\|+n} (-1)^{||\alpha||} \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_{\widetilde{P}}} w^{\gamma+I} \cdot \frac{\widetilde{\Delta} \cdot \widetilde{Q}^\alpha dw}{\widetilde{P}^{\alpha+I}} =$$

$$= \sum_{||K|| \leqslant ||\gamma||+n} \frac{(-1)^{||K||+n} \prod_{s=1}^n \left(\sum_{j=1}^n k_{sj}\right)!}{\prod_{s,j=1}^n (k_{sj})!} \mathfrak{M}\left[\frac{w^{\gamma+I} \cdot \widetilde{\Delta} \cdot \det A \cdot \widetilde{Q}^\alpha \prod_{s,j=1}^n a_{sj}^{k_{sj}}}{\prod_{j=1}^n w_j^{\beta_j N_j + \beta_j + N_j}}\right],$$

where the summation is performed over all integer non-negative matrices $K = \|k_{sj}\|_{s,j=1}^n$ such that the sum $\sum_{s=1}^n k_{sj} = \alpha_j$, and the sum $\sum_{j=1}^n k_{js}$ is denoted by β_s. The polynomial coefficients a_{sj} are taken from the representation

$$w_j^{N_j+1} = \sum_{k=1}^n a_{jk} \widetilde{P}_k,$$

and $\det A$ *is the determinant of the matrix of coefficients.*

2. Examples

Example 1. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = 1 + a_1 z_1 + a_2 z_2 = 0, \\ f_2(z_1, z_2) = 1 + b_1 z_1 + b_2 z_2 = 0. \end{cases} \tag{3}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 + a_1 w_2 + a_2 w_1 = 0, \\ \widetilde{f}_2 = w_1 w_2 + b_1 w_2 + b_2 w_1 = 0. \end{cases}$$

Subtract the second equation from the first one and pass to the system of the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 + a_1 w_2 + a_2 w_1 = 0, \\ \widetilde{f}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2 = 0. \end{cases} \tag{4}$$

The Jacobian $\widetilde{\Delta}$ of the system (4) is equal to

$$\widetilde{\Delta} = \begin{vmatrix} w_2 + a_2 & w_1 + a_1 \\ a_2 - b_2 & a_1 - b_1 \end{vmatrix} = (-a_2 + b_2) w_1 + (a_1 - b_1) w_2 + (a_1 b_2 - a_2 b_1).$$

Note that

$$\begin{cases} \widetilde{Q}_1 = a_1 w_2 + a_2 w_1, \\ \widetilde{Q}_2 = 0. \end{cases}$$

$$\begin{cases} \widetilde{P}_1 = w_1 w_2, \\ \widetilde{P}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2. \end{cases}$$

Let us calculate $\det A$. Since

$$w_1^2 = a_{11} \widetilde{P}_1 + a_{12} \widetilde{P}_2,$$
$$w_2^2 = a_{21} \widetilde{P}_1 + a_{22} \widetilde{P}_2,$$

where $\widetilde{P}_1 = w_1 w_2$, $\widetilde{P}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2$. Therefore, the elements of a_{ii} are equal

$$a_{11} = -\frac{a_1 - b_1}{a_2 - b_2}, \quad a_{12} = \frac{w_1}{a_2 - b_2},$$

$$a_{21} = -\frac{a_2 - b_2}{a_1 - b_1}, \quad a_{22} = \frac{w_2}{a_1 - b_1}.$$

Therefore,

$$\det A = -\frac{w_2}{a_2 - b_2} + \frac{w_1}{a_1 - b_1} = \frac{(a_2 - b_2) w_1 - (a_1 - b_1) w_2}{(a_1 - b_1)(a_2 - b_2)}.$$

By Theorem 1.1

$$J_{(0,0)} = \sum_{\|K\| = k_{11} + k_{12} + k_{21} + k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11} + k_{12})! \cdot (k_{21} + k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times \mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A\cdot\widetilde{Q}_1^{k_{11}+k_{21}}\cdot\widetilde{Q}_2^{k_{12}+k_{22}}\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{2(k_{11}+k_{12})}\cdot w_2^{2(k_{21}+k_{22})}}\right],$$

$$J_{(0,0)} = \sum_{\|K\|=k_{11}+k_{12}+k_{21}+k_{22}\leqslant 2}\frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times$$

$$\times \mathfrak{M}\left[\frac{(-1)^{k_{11}+k_{21}}((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))}{(a_1-b_1)^{1+k_{21}+k_{22}-k_{11}}\cdot(a_2-b_2)^{1+k_{11}+k_{12}-k_{21}}}\right]\times$$

$$\times\left[\frac{(a_1w_2+a_2w_1)^{k_{11}+k_{21}}\cdot 0^{k_{12}+k_{22}}}{w_1^{2k_{11}+k_{12}}\cdot w_2^{2k_{21}+k_{22}}}\right].$$

Calculate the values of the sums using the fact that $\widetilde{Q}_2=0$.

$(0,0,0,0)$:

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))}{(a_1-b_1)\cdot(a_2-b_2)}\right]=0,$$

$(1,0,0,0)$:

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)}{(a_2-b_2)^2\cdot w_1^2}\right]=$$
$$=\frac{a_2(a_1b_2-a_2b_1)}{a_2-b_2},$$

$(0,0,1,0)$:

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)}{(a_1-b_1)^2\cdot w_2^2}\right]=$$
$$=\frac{-a_1(a_1b_2-a_2b_1)}{a_1-b_1},$$

$(2,0,0,0)$:

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2\cdot(a_1-b_1)}{(a_2-b_2)^3\cdot w_1^4}\right]=$$
$$=\frac{-a_2^2(a_1-b_1)}{a_2-b_2},$$

$(0,0,2,0)$:

$$-\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2\cdot(a_2-b_2)}{(a_1-b_1)^3\cdot w_2^4}\right]=$$
$$=\frac{-a_1^2(a_2-b_2)}{a_1-b_1},$$

$(1,0,1,0)$:

$$-\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2}{(a_1-b_1)\cdot(a_2-b_2)\cdot w_1^2\cdot w_2^2}\right]=$$
$$=\frac{-a_1^2(a_2-b_2)^2}{(a_1-b_1)(a_2-b_2)}+\frac{-a_2^2(a_1-b_1)^2}{(a_1-b_1)(a_2-b_2)}+4a_1a_2.$$

Therefore,

$$J_{(0,0)}=4a_1a_2+\frac{a_2(a_1b_2-a_2b_1)}{a_2-b_2}-\frac{a_1(a_1b_2-a_2b_1)}{a_1-b_1}-\frac{2a_1^2(a_2-b_2)}{a_1-b_1}-\frac{2a_2^2(a_1-b_1)}{a_2-b_2}.$$

Calculate the power sum of the root system (3) directly. We multiply the first equation of the system by b_2, the second by a_2 and subtract one from another. Thus

$$z_1 = \frac{a_2 - b_2}{a_1 b_2 - a_2 b_1}\,,$$

$$z_2 = -\frac{a_1 - b_1}{a_1 b_2 - a_2 b_1}\,.$$

By Theorem 1.2

$$J_{(0,0)} = \sum_{j=1}^{p} \frac{1}{z_{j1} \cdot z_{j2}} = -\frac{(a_1 b_2 - a_2 b_1)^2}{(a_1 - b_1)(a_2 - b_2)}\,,$$

which coincides with the value found above.

Example 2. We shall use the result of the previous example in the case of a non-algebraic system. Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2\pi^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k+1)!}\,,$$

which uniformly and absolutely converge on the complex plane and have the order of growth equal to 1/2.

Consider the system of equations

$$\begin{cases} f_1(z_1, z_2) = \dfrac{\sin\sqrt{a_1 z_1 + a_2 z_2}}{\sqrt{a_1 z_1 + a_2 z_2}} = \prod\limits_{m=1}^{\infty}\left(1 - \dfrac{a_1 z_1 + a_2 z_2}{m^2\pi^2}\right) = 0, \\ f_2(z_1, z_2) = \dfrac{\sin\sqrt{b_1 z_1 + b_2 z_2}}{\sqrt{b_1 z_1 + b_2 z_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{b_1 z_1 + b_2 z_2}{s^2\pi^2}\right) = 0. \end{cases}$$

Using the formula obtained above in Example 1 and the well-known expansion of the series, we obtain that the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = \sum_{m,s=1}^{\infty} \frac{(a_1 b_2 - a_2 b_1)^2}{\pi^4 (s^2 a_1 - m^2 b_1)(m^2 b_2 - s^2 a_2)}.$$

$$J_{(0,0)} = \sum_{m=1}^{\infty} \frac{4 a_1 a_2}{\pi^4 m^2} + \sum_{m,s=1}^{\infty} \frac{a_1 b_2 - a_2 b_1}{\pi^4 m^2}\left[\frac{a_1}{m^2 b_1 - s^2 a_1} - \frac{a_2}{m^2 b_2 - s^2 a_2}\right] -$$

$$- \sum_{m,s=1}^{\infty} \frac{2 a_2^2 (m^2 b_1 - s^2 a_1)}{\pi^4 m^4 (m^2 b_2 - s^2 a_2)} - \sum_{m,s=1}^{\infty} \frac{2 a_1^2 (m^2 b_2 - s^2 a_2)}{\pi^4 m^4 (m^2 b_1 - s^2 a_1)}.$$

The value of this series is found in the articles [5] and [7].

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Примеры вычисления степенных сумм корней систем уравнений

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Аннотация. Рассмотрены примеры вычисления степенных сумм корней систем уравнений, в том числе трансцендентных. Так как число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Приведены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.

EDN: GUXTYF
УДК 550.837.9

Physical and Technical Fundamentals of the Seismoelectric Method of Direct Hydrocarbon Prospecting in the Arctic Using Automatic Underwater Vehicles

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Abstract. The article deals with the problems of the implementation of underwater-surface variants of the seismo-electric method of direct search for hydrocarbons in the conditions of the Arctic waters. An estimate is given of the strength of the secondary electric field when a gas reservoir is excited by the action of a seismic source based on an accompanying geophysical vessel and receiving signals on an automatic underwater vehicle. The article also discusses the issues of hardware implementation and navigation binding of waterborne devices.

Keywords: hydrocarbon search, Arctic waters, seismoelectric method, underwater vehicles.

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Introduction

The seismic-electric method of direct hydrocarbon prospecting is based on the excitation of an electric field in porous rocks under the influence of acoustic radiation in the form of seismic shocks. The first publications on this effect were made in the works [1–4].

In [5], the results of marine prospecting operations by a seismic survey complex based on a geophysical vessel and seismic braids towed behind it with pneumatic guns and hydrophones are presented. The so-called "binary" technology of parallel illumination of the geo-section by an artificial electric field created by a towed flooded cable with a current was used [6], which, according to the authors, provides a significant increase in the sensitivity of the method.

It is extremely difficult to implement this technology on land due to the need to create large illumination currents and install branched earths that require heavy vehicles for transportation.

In addition, due to the influence of the processes of the formation of an electric field in a layered inhomogeneous medium, additional interference uncorrelated with the seismic signal is

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received at the input of the sensors of the electric signal of the SE effect. In addition, there remains the problem of moving the grounding of the electric illumination cable along the observation profile.

Using the example of setting this method at the Minusinsk gas condensate field [7, 8] as a semi-active one, that is, without the use of illumination by a special electric field, and a passive method with extracting information from the seismic and electrical noise of the Earth, the author of this article with colleagues showed the possibility of such work on land to the depth of the gas reservoir more than 2 km.

In publications [9–15] It contains various aspects of underwater-subglacial marine seismic exploration based on the use of an accompanying geophysical vessel and towed or located on the bottom of the sea seismic braids.

There are projects to install seismic stations on the sea ground, which makes it possible to significantly reduce the impact of sea surface waves.

1. Calculation results

We will give a numerical estimate of the magnitude of the electric field strength on the sea surface for the specified search parameters: the depth of the position of the productive hydrocarbon reservoir; its power; the conductivity of the seawater reservoir and the surrounding rock.

The pressure of a seismic wave on the hydrocarbon interface with the surrounding rock leads to the appearance of an additional electric charge in the hydrocarbon medium due to the displacement of the deposit surface in the electrostatic field of the Earth. In this model, hydrocarbon deposits are represented as a capacitor whose potential fluctuates synchronously with the seismic pressure field.

In the design scheme, a pulsed non-explosive source is located in the stern area of the base geophysical vessel below the waterline, and the receivers of seismic and electrical signals are placed in the hull of the AUV moving ahead of the vessel on its course at a distance of 200 m (Fig. 1). The electrical conductivity of seawater corresponds to σ_1=4 S/m, and the depth of the sea h_1=100 m. Productive gas reservoir with an area of S_G=3000×1000 m and power h_4=10 m located at a depth of h_2=1000 m. Electrical conductivity of the gas medium of the formation σ_2=10^{-5} S/m, density ρ_2=100 kg/m^3, velocity of propagation of longitudinal seismic waves in gas is V_2=500 m/s.

The pressure of a seismic impact on the formation surface for the far zone of the source is estimated as [16]:

$$P_3 = \frac{4F_1\eta}{\pi R_U^2} \cdot \frac{\lambda_{S1}}{r_1 + r_2} e^{-\alpha r_{12} B_1} \cos\theta = 1.6 \cdot 10^4 Pa. \tag{1}$$

Here, $\lambda_{S1} = \dfrac{V_1}{f_S}$ is the apparent wavelength corresponding to the first Fresnel zone;

$f_S = \dfrac{1}{2\tau} = 100Hz$ is the average frequency of the pulse spectrum of the source with a duration of $\tau = 5 \cdot 10^{-3}$ s;

θ is the angle of incidence of the wave on the formation;

$r_{12} = r_1 + r_2 = 1100$ m is the distance between the source and the reservoir surface;

α=10^{-4} 1/m is coefficient of absorption of a seismic wave by a rock.

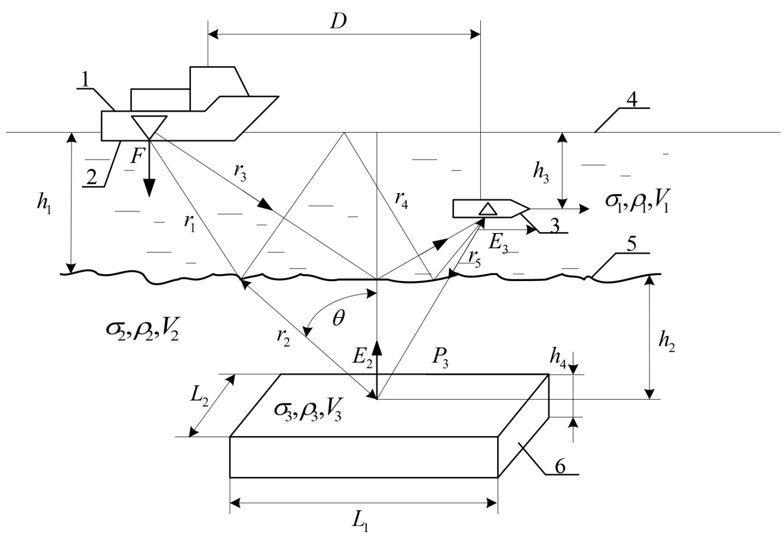

Fig. 1. Calculation scheme for estimating the electric field of the SE effect of a reservoir gas deposit. 1 — a geophysical vessel; 2 — a seismic emitter; 3 — an automatic underwater vehicle; 4 — the surface of the sea; 5 — the seabed; 6 — a productive reservoir of hydrocarbons

The coefficient of passage of a seismic wave into a productive reservoir:

$$B_{PP} = \frac{2\rho_2 V_2}{\rho_2 V_2 + \rho_3 V_3} \approx 1. \tag{2}$$

Absorption in water in the "radiator–bottom" section is not taken into account due to the smallness of its size. The displacement of the upper boundary of the reservoir under the action of a seismic shock is determined through the solution of the Newton differential equation:

$$m \cdot \frac{\delta^2 Z}{\delta t^2} + \frac{F_C}{V_3}\frac{\delta Z}{\delta t} - F_3 = 0. \tag{3}$$

Here, $m = V_3 \tau S_1 \rho_3$ is the mass of the displaced reservoir medium, determined by the depth of impact passage into the reservoir medium during a long pulse τ;
$S_1 = \lambda_{S3}^2 = \left(\frac{V_3}{f_S}\right)^2$ is he area of the Fresnel wave zone on the formation surface;
$F_3 = P_3 S_1$ is impact force on the formation surface;
F_C is resistance force of the formation medium;
Z is displacement of the reservoir surface under the impact of a seismic pulse.

Solution (3), gives:

$$Z = \frac{1}{\left(1 + \dfrac{P_C}{V_3 m}\right)} \frac{P_3 \tau}{V_3 \rho_3}. \tag{4}$$

The intensity of the electric field in the formation created by the impact of a seismic wave can be estimated as:
Solution (3), gives:

$$E_2 = E_0 \frac{Z}{h_4}, \tag{5}$$

Here, E_0=120 V/m is the intensity of the natural electric field of the Earth, causing the full charge of the reservoir.

The current arising through the reservoir element is defined as:

$$I_3 = E_2 \lambda_{S3}^2 \sigma_3 \frac{Z}{h_4}. \tag{6}$$

The total current passing through the entire surface of the formation will be higher by an amount $n = \dfrac{L_1 L_2}{\lambda_{S3}^2}$:

$$I_{3\Sigma} = E_0 \sigma_3 L_1 L_2 \frac{Z}{h_4}. \tag{7}$$

According to our experiments at the Minusinsk gas condensate field, the calculated dependence of the intensity of the secondary electric field over the productive gas reservoir is of a two-humped nature, which corresponds to the field of a vertical electric dipole with a current $I_{3\Sigma}$.

Under the action of this current, the electric charge of the entire formation will be:

$$Q_S = I_{3\Sigma} \tau. \tag{8}$$

Taking into account (6, 7):

$$Q_S = \frac{E_0 \sigma_2 \tau L_1 L_2}{h_4}. \tag{9}$$

The current density caused by this charge at the observation point:

$$j_X = \frac{I_{3\Sigma} K}{4\pi (r_2 + r_6)^2}, \tag{10}$$

Here, $K = e^{-(\beta_1 r_5 + \beta_2 r_2)}$ is the absorption coefficient of the electric field in water and rock.

Because $j_X = \sigma_1 E_3$, then the modulus of the electric field strength at the receiving point:

$$E_3 = \frac{E_0 \sigma_3 L_1 L_2 Z e^{-(\beta_1 r_5 + \beta_2 r_2)}}{4\pi \sigma_2 h_4 (r_2 + r_6)^2}. \tag{11}$$

For the following private parameters: $E3$=120 V/m; σ_3/σ_2=10^{-2}; $L_1 \cdot L_2$=$3 \cdot 10^6$ m^2; Z=$1.6 \cdot 10^{-4}$ m; r_{12}=1100 m; K=0.22, value E_3=3.8 μV/m.

The obtained estimates are in good agreement with the experimental data [7], which allows us to recommend this technique for calculations before the fake search work. Unlike the known approaches, this technique allows to estimate the required impact force of a seismic source at a given depth of search at a simple engineering level.

The graphs of the envelope signals of the secondary electric field along the observation profile are of a two-humped nature, and correspond to the field of a vertical electric dipole created by a pulsating charge of a "capacitor" equivalent to a productive reservoir.

In a spherical coordinate system, the electric field strength of the dipole on the Earth's surface is estimated as [17]:

$$E_r(\theta) = \frac{2P \cos\theta}{4\pi \varepsilon_0 \varepsilon r^3} e^{-\alpha r}, \tag{12}$$

$$E_\theta(\theta) = \frac{2P \sin\theta}{4\pi \varepsilon_0 \varepsilon r^3} e^{-\alpha r}. \tag{13}$$

Here $P = I h_4$ is dipole moment at current I;

$r = r_2 + r_5$;
ε_0 is dielectric constant;
ε is dielectric constant of rock;
h_3 is dipole length (productive reservoir capacity);
$\alpha = \sqrt{\dfrac{\omega\mu\sigma}{2}}$ is the attenuation coefficient of the electric field in the rock;
ω is operating frequency; σ is electrical conductivity of the medium θ is spherical angle.

Fig. 2 shows a graph of the envelope of the horizontal component of the vertical electric field of the dipole along the motion profile of the search engine with the following parameters: I=1 A; f=10 Hz; ε=10; $\sigma=10^{-3}$ S/m; h_2=500, 1000 m.

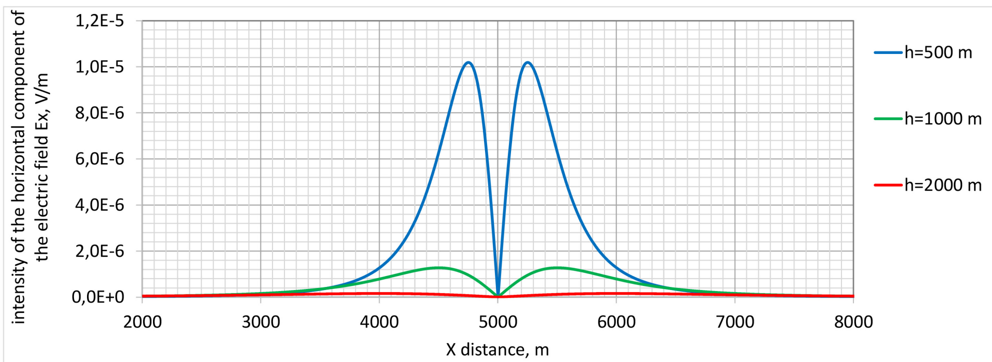

Fig. 2. Distribution of E_x(mV/m) at $h_2 = 1000$ m

In Fig. 3, the finite element method calculates the distribution of the field on the Earth's surface by area, which makes it possible to determine the search area in 3D modification. In this case, you can use several AUVs running a parallel course, capturing subglacial areas.

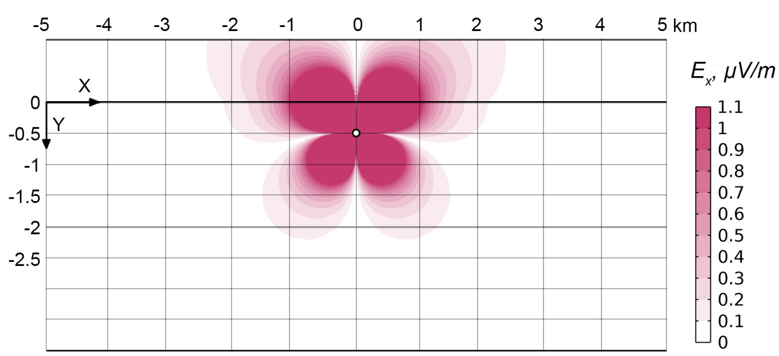

Fig. 3. Distribution of E_x(mV/m) at $h_2 = 1000$ m

The capacitance of a capacitor equivalent to a reservoir can be defined as:

$$C = \frac{\overline{\varepsilon}\varepsilon_0 L_1 L_2}{Z}, \tag{14}$$

where ε_0 is constant. ε is dielectric constant of the «capacitor» medium.

Taking for example $\overline{\varepsilon}$=10; by $L_1 \cdot L_2 = 3{\cdot}10^6$ m^2; Z=1.6 $\cdot$ 10^{-4} m, we get C=0.26 F. In this case, the time constant of the discharge of the formation on the surrounding rock:

$$T_p = C\frac{h_2}{G_2 L_1 L_2}. \tag{15}$$

By $\sigma_2 = 10^{-5}$ S/m, we get $T_p \approx 0.1$ s.

Since the duration of the seismic pulse was assumed as $\tau = 5{\cdot}10^{-3}$ c and relationships $T_p/\tau = 20$, then, with the continuous repetition of shocks, characteristic of the action of a seismic wave, there will be a constant increase in the intensity of the secondary electric field on the surface.

Due to the fact that the length of the anomaly along the motion profile of the search engine approximately corresponds to the double depth of the position of the productive reservoir, then at the speed of movement $V = 1$ m/s, this anomaly will be passed in time $T_n = 2h/V = 2 \cdot 10^3$ c and during this time, 200 seismic source impacts will act on the formation. If we take 20 periods of their repetition for averaging signals, then the signal-to-noise ratio increases in $\sqrt{20} = 4.5$ times, which corresponds to their group processing of classical seismic survey from 20 geophones.

Since in these conditions it is not required to accurately determine the coordinates, it is quite possible to use in the work a navigation reference to the difference-dimensional long-wave systems such as "Laurent" (USA), "Zeus" (Russia) in the frequency range of 100 kHz.

2. Hardware

It is possible to create the following complexes of the seismoelectric method:

1. is a small-sized submarine (seismic source) and a group of automatic underwater vehicles (AUV).

2. is a basic geophysical vessel with a seismic source and an AUV group.

3. is a basic underwater robot with a group of seismic sources and a group of PPR.

4. is a basic geophysical vessel without a seismic source and an AUV group with reception of electrical and seismic noise of the earth in the frequency range 0.1–20 Hz [11].

According to the first variant of the seismic source, it is located on a submarine (GROOVE) of a small class with a displacement of 100–200 tons (Fig. 4). The AUVs are placed in the bow torpedo tubes, through which they are pushed along the course of movement at a distance of 100–200 m. When working with the 3D AUV method, they are positioned orthogonally to the course. The AUV is controlled from the PA via a hydroacoustic channel. Navigation binding is implemented either from the accompanying vessel, or directly by receiving signals from the above-mentioned long-wave navigation systems. It is advisable to use broadband electromechanical emitters with pseudorandom coding of a sequence of seismic signals providing a minimum power level as an on-board seismic source [18, 19].

The second option (Fig. 1) does not require the development of a special submarine. The third option (Fig. 4) differs from the first by placing the seismic source on an autonomous underwater operation.

Finally, the fourth modification of the system does not require illumination of the geo-section by an artificial seismic source. The natural noise fields of the Earth are used with the processing of

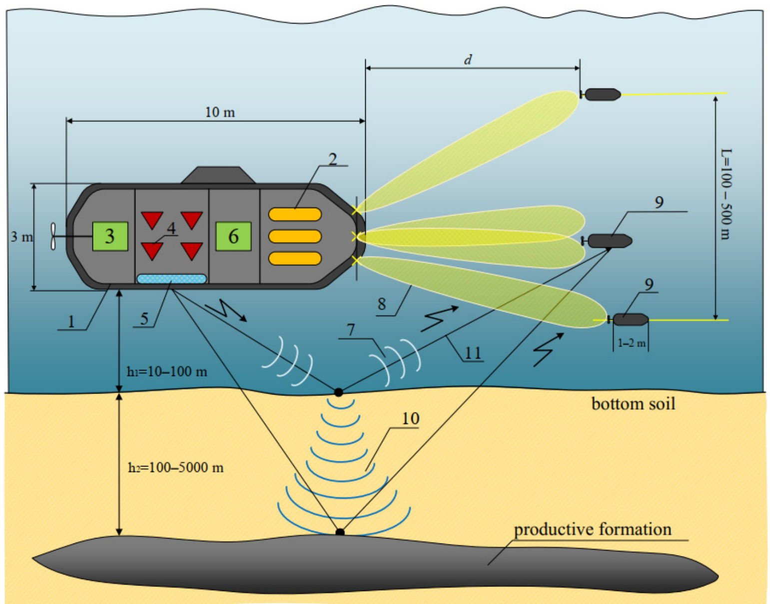

Fig. 4. Underwater seismic survey system USSS-M-1: 1 — submarine hull; 2 — torpedo tubes; 3 — screw drive; 4 — seismic emitters; 5 — shock plate; 6 — deckhouse; 7 — reflected wave from the ground; 8 — hydroacoustic rays of the control channel; 9 — automatic underwater vehicles; 10 — shock wave in the geological environment

electrical and seismic signals by the method of mutual correlation according to the algorithm [8]:

$$R(\tau) \approx \frac{1}{T} \int \overline{E}(t) \cdot \overline{S}(t-\tau) dt. \tag{16}$$

Here $\overline{E}(t)\cdot$ and $\overline{S}(t-\tau)$ is accordingly, signals from sensors of electric and seismic noise fields, normalized by dispersion [8];
T is observation time.

Of course, all four options require the management of the commands of the base geophysical vessel through the sonar channel, which is simultaneously the carrier of all outboard means. Today, there are many developments of underwater robots of the required class all over the world, so it is only necessary to create the hardware and software necessary for conducting search operations [20].

3. Navigation binding of coordinates of underwater vehicles

Next, we will consider the methods of radio navigation anchoring the coordinates of the AUV in an underwater position relative to the accompanying vessel or by signals from long-wave navigation systems such as "Loran" (USA or "Zeus" (Russia) [21]. In any case, it is necessary to ensure the reception of navigation signals under water or under ice. In [22], the authors described a parametric method for receiving electromagnetic signals in seawater based on controlling its conductivity by acoustic radiation. If an electromagnetic signal with vertical polarization comes from a third-party radio station on the surface of the water, then a horizontal component with an electric field strength is formed under the surface (Fig. 5) [23]:

$$E_X = \frac{E_{Z0}}{\sqrt{60\lambda\sigma}}. \tag{17}$$

Here λ is the length of the electromagnetic wave, and the σ is electrical conductivity of water. E_Z is the intensity of the vertical component of the field on the sea surface. It can be seen from (17) that seawater, due to the refraction effect, greatly reduces the signal energy.

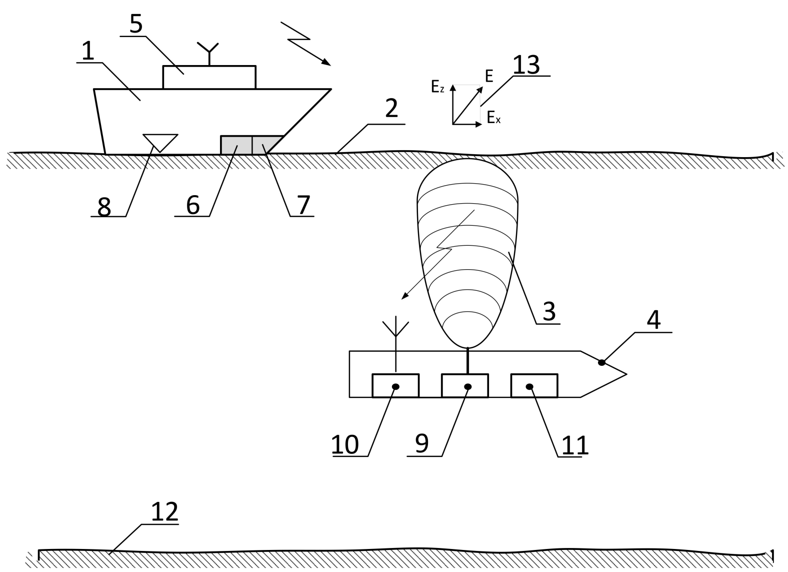

Fig. 5. The scheme of reception of the control signal of underwater vehicles: 1 — the hull of the vessel; 2 — the surface of the sea; 3 — acoustic beam parametric channel (PC); 4 — autonomous underwater vehicle (AUV); 5 — radio control AUV; 6 — equipment hydroacoustic control channel (HCC); 7 — antenna HCC; 8 — seismic emitter; 9 — PC transmitter; 10 — PC navigation signal receiver; 11 — hydrophone and magnetic receiver; 12 — seabed; 13 — PC electromagnetic field vector

For example, when λ= 3000 m (frequency 100 kHz) and σ = 4 S/m, the refractive index is $\sqrt{60\lambda\sigma}$= 848. When taken under ice due to a significant decrease in their conductivity, the refractive index is reduced by 60 times. The parametric effect of controlling the conductivity of seawater additionally reduces the signal level by the modulation coefficient m=10^{-3} at the density of the acoustic radiation power flux I=1 W/m^2.

If the power of the acoustic emitter on the underwater vehicle is P_a [W], then , when the sea surface is irradiated from below , the power flux density will be:

$$\Pi_a = \frac{P_a Q}{4\pi h^2} e^{-\beta h}. \tag{18}$$

Here h is depth of the underwater vehicle position UV, β is absorption coefficient of acoustic radiation of seawater; Q is the directivity coefficient of the acoustic antenna UV.

From (17), the required power of the acoustic emitter will be:

$$P_a = \frac{4\pi h^2 \Pi_a}{Q} e^{\beta h}. \tag{19}$$

For example, let the depth of movement of the AUV, relative to the sea surface, be $h = 10$ m. If an acoustic transmitter is used on the AUV for a parametric navigation channel, highlighting the surface of the sea with a power flow density $\Pi_a = 1$ W/m^2 on the frequency $f_a = 100$ kHz, then at the wavelength λ_a=0.15 m and effective area of the acoustic antenna $S_a = 0.01$ m^2, at the absorption coefficient β=0.36 $f_a^{3/2}$ dB/km, we get $P_a = 2$ W. According to the graphs, Fig. 3, the required accuracy of the navigation reference of the AUV, at the depth of the position of the productive reservoir of hydrocarbons $h_2 = 1000$ m, will make $\Delta x = 1/2,\ h_2 = 500$ m.

Such accuracy can be achieved using signals from long-wave navigation systems, or by receiving signals to the AUV by parametric method by direct reading from the sea surface, or by broadcasting satellite navigation system signals via the onboard radio station of the accompanying vessel.

Conclusion

A quantitative assessment of the intensity of the secondary electric field of the seismoelectric effect of a productive gas reservoir of hydrocarbons is given for the specified search parameters: the depth of the reservoir position; its size; the electrical conductivity of the host rock and the hydrocarbon medium; the position of the carriers of the field sensors in the marine environment and the impact force of the seismic source.

To work in Arctic conditions of difficult ice conditions, it is recommended to use automatic underwater vehicles that allow the implementation of the seismic-electric method, including under ice.

The problems of navigation binding of automatic underwater vehicles are discussed.

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Физико-технические основы сейсмоэлектрического метода прямых поисков углеводородов в условиях Арктики с использованием автоматических подводных аппаратов

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Аннотация. В статье рассматриваются проблемы реализации подводно-надводных вариантов сейсмоэлектрического метода прямых поисков углеводородов в условиях арктических морей. Дается оценка напряженности вторичного электрического поля при возбуждении газового пласта ударами сейсмического источника, базирующегося на сопровождающем геофизическом судне, и приема сигналов на автоматическом подводном аппарате. Обсуждаются вопросы аппаратурной реализации и навигационной привязки роботов в подводном положении.

Ключевые слова: поиск углеводородов, арктические воды, сейсмоэлектрический метод, подводные аппараты.

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On the Ill-posed Cauchy Problem for the Polyharmonic Heat Equation

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Abstract. We consider the ill-posed Cauchy problem for the polyharmonic heat equation on recovering a function, satisfying the equation $(\partial_t+(-\Delta)^m)u=0$ in a cylindrical domain in the half-space $\mathbb{R}^n\times[0,+\infty)$, where $n\geqslant 1$, $m\geqslant 1$ and Δ is the Laplace operator, via its values and the values of its normal derivatives up to order $(2m-1)$ on a given part of the lateral surface of the cylinder. We obtain a Uniqueness Theorem for the problem and a criterion of its solvability in terms of the real-analytic continuation of parabolic potentials, associated with the Cauchy data.

Keywords: the polyharmonic heat equation, ill-posed problems, integral representation method.

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In this short note we continue to investigate the ill-posed Cauchy problem for parabolic operators in various function spaces, see [1, 2] for the second order operators in the Hölder spaces, or [3–5] for the second order operators in the anisotropic Sobolev spaces. Actually the general schemes related to investigation of the ill-posed Cauchy problem for elliptic operators (see [6–8] for the second order operators or [9, 10] for the Cauchy–Riemann system in one and many complex variables or [12, 13] for general elliptic operators with the unique continuation property) are still applicable in this new situation.

In the present paper we concentrated our efforts on the solvability criterion of the ill-posed Cauchy problem for a simple class of Petrovsky $2m$-parabolic partial differential operators

$$(\partial_t+(-\Delta)^m), \tag{1}$$

where $m\geqslant 1$ and Δ is the Laplace operator in $\mathbb{R}^n$, $n\geqslant 1$, that are often called polyharmonic heat operators, see [14, Ch.2, Sec. 1], [15]. Namely the problem consists of the recovering a function, satisfying the equation $(\partial_t+(-\Delta)^m)u=0$ in a cylindrical domain in the half-space $\mathbb{R}^n\times[0,+\infty)$, via its values and the values of its normal derivatives up to order $(2m-1)$ on a given part of the lateral surface of the cylinder. The crucial difference between the heat equation (or the parabolic Lamé system) and the polyharmonic heat equation is the fact that the fundamental solution of the polyharmonic heat operator is given by a non-elementary function. The situation resembles somehow the matter with the fundamental solutions to the Helmholtz operator $\Delta+c_0^2$: for $n=3$ it is given by $\dfrac{-e^{\pm\iota c_0|x|}}{4\pi|x|}$ (here ι is the imaginary unit) while for $n=2$ it is represented by the Hankel functions of the second kind (actually, some versions of the Bessel functions), see, [16, Ch. III, Sec. 11]. Of course, it is not a surprise, because after an application of the Laplace transform L with respect to the variable t (if applicable) to (1), one arrives at the parameter depending elliptic equation

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$$(\iota\tau + (-\Delta)^m)L(u) = 0, \tag{2}$$

coinciding with the Helmholtz equation for $m = 1$ regarding the generalized function $L(u)$ as an unknown and τ as a real parameter. Actually, this seemingly simple approach, reducing the parabolic equations to elliptic ones, is known for decades, see [17]. It gives a lot of qualitative information on the connection between the corresponding solutions of the differential equations of different kinds. However one needs very delicate properties of the Laplace transform in order to obtain really useful formulas solving the parabolic problems with the use of elliptic theory, see for instance, [3] for the heat equation and the related remark on properties of the Laplace transform [18]. Thus, we will act in the framework of mentioned above scheme invented by L.Aizenberg and developed in [12].

1. Preliminaries

Let Ω be a bounded domain in n-dimensional linear space $\mathbb{R}^n$ with the coordinates $x = (x_1, \dots, x_n)$. As usual we denote by $\overline{\Omega}$ the closure of Ω, and we denote by $\partial\Omega$ its boundary. In the sequel we assume that $\partial\Omega$ is piece-wise smooth. We denote by Ω_T the bounded open cylinder $\Omega \times (0, T)$ in $\mathbb{R}^{n+1}$ with a positive altitude T. Let also $\Gamma \subset \partial\Omega$ be a non empty connected relatively open subset of $\partial\Omega$. Then $\Gamma_T = \Gamma \times (0,\ T)$ and $\overline{\Gamma_T} = \overline{\Gamma} \times [0,\ T]$.

We consider the functions over subsets in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. As usual, for $s \in \mathbb{Z}_+$ we denote by $C^s(\Omega)$ the space of all s times continuously differentiable functions in Ω. Next, for a (relatively open) set $S \subset \partial\Omega$ denote by $C^s(\Omega \cup S)$ the set of such functions from the space $C^s(\Omega)$ that all their derivatives up to order s can be extended continuously onto $\Omega \cup S$. The standard topology of these metrizable spaces induces the uniform convergence on compact subsets in $\Omega \cup S$ together with all partial derivatives up to order s. We will also use the standard Banach Hölder spaces $C^s(\overline{\Omega})$ and $C^{s,\lambda}(\overline{\Omega})$ (cf. [19], [20, Ch.1, Sec. 1], [21]), and the related metrizable spaces $C^{s,\lambda}(\Omega \cup S)$.

Let also $L^p(\Omega)$, $p \geqslant 1$, be the Lebesgue spaces, $H^s(\Omega)$, $s \geqslant 0$, stand for the Sobolev spaces if $s \in \mathbb{N}$ and for the Sobolev-Slobodetskii spaces if $s > 0$, $s \notin \mathbb{N}$.

To investigate the polyharmonic heat equation we need also the anisotropic ($2m$-parabolic) spaces, see [20, Ch. 1], [21, Ch. 8] for $m = 1$ and [14] for $m \geqslant 1$. With this aim, let $C^{2ms,s}(\Omega_T)$, $m \in \mathbb{N}$, stand for the set of all the continuous functions u in Ω_T, having in Ω_T the continuous partial derivatives $\partial_t^j \partial_x^\alpha u$ with all the multi-indexes $(\alpha, j) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ satisfying $|\alpha| + 2mj \leqslant 2ms$ where, as usual, $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$. Similarly, we denote by $C^{2ms+k,s}(\Omega_T)$ the set of continuous functions in Ω_T, such that all partial derivatives $\partial^\beta u$ belong to $C^{2ms,s}(\Omega_T)$ if $\beta \in \mathbb{Z}_+^n$ satisfies $|\beta| \leqslant k$, $k \in \mathbb{Z}_+$. Of course, it is natural to agree that $C^{2ms+0,s}(\Omega_T) = C^{2ms,s}(\Omega_T)$, $C^{0,0}(\Omega_T) = C(\Omega_T)$ and $C^0(\Omega) = C(\Omega)$. We also denote by $C^{2ms+k,s}((\Omega \cup S)_T)$ the set of such functions u from the space $C^{2ms+k,s}(\Omega_T)$ that their partial derivatives $\partial_t^j \partial_x^{\alpha+\beta} u$, $2mj + |\alpha| \leqslant 2ms$, $|\beta| \leqslant k$, can be extended continuously onto $(\Omega \cup S)_T$. The standard topology of these metrizable spaces induces the uniform convergence on compact subsets of $(\Omega \cup S)_T$ together with all partial derivatives used in its definition (the cases $S = \emptyset$ and $S = \partial D$ are included).

We use also the anisotropic Hölder spaces (cf., [20, Ch. 1], [21, Ch. 8]) for $m = 1$ and [14] for $m \geqslant 1$. Let $C^{2ms+k,s,\lambda,\lambda/2}((\Omega \cup S)_T)$ stand for the set of anisotropic Hölder continuous functions with a power λ over each compact subset of $(\Omega \cup S)_T$ together with all partial derivatives $\partial_x^{\alpha+\beta} \partial_t^j u$ where $|\beta| \leqslant k$, $|\alpha| + 2mj \leqslant 2ms$. Clearly, $C^{2ms+k,s,\lambda,\lambda/2}(\overline{\Omega_T})$ is a Banach space with the natural norm, see, for instance, [21, Ch. 8] for $m = 1$ and [14] for $m \geqslant 1$. In general, the space $C^{2ms+k,s,\lambda,\lambda/2}((\Omega \cup S)_T)$ can be treated again as a metrizable space, generated by a system of seminorms associated with a suitable exhaustion $\{\Omega_i\}_{i \in \mathbb{N}}$ of the set $\Omega \cup S$.

In order to invoke the Hilbert space approach, we need anisotropic ($2m$-parabolic) Sobolev spaces $H^{2ms,s}(\Omega_T)$, $s \in \mathbb{Z}_+$, see, [20, 22] for $m = 1$ or [14] for $m \geqslant 1$, i.e. the set of all the measurable functions u in Ω_T such that all the generalized partial derivatives $\partial_t^j \partial_x^\alpha u$ with all the multi-indexes $(\alpha, j) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ satisfying $|\alpha| + 2mj \leqslant 2ms$, belong to the Lebesgue class $L^2(\Omega_T)$. This is the Hilbert space with the natural inner product $(u, v)_{H^{2ms,s}(\Omega_T)}$. We also may define $H^{2ms,s}(\Omega_T)$ as the completion of the space $C^{2ms,s}(\overline{\Omega_T})$ with respect to the norm $\| \cdot \|_{H^{2ms,s}(\Omega_T)}$ generated by the inner product $(u, v)_{H^{2ms,s}(\Omega_T)}$. For $s = 0$ we have $H^{0,0}(\Omega_T) = L^2(\Omega_T)$.

We also will use the so-called Bochner spaces of functions depending on (x, t) from the strip $\mathbb{R}^n \times [T_1, T_2]$. Namely, for a Banach space $\mathcal{B}$ (for example, on a subdomain of $\mathbb{R}^n$) and $p \geqslant 1$, we denote by $L^p([T_1, T_2], \mathcal{B})$ the Banach space of all the measurable mappings $u : [T_1, T_2] \to \mathcal{B}$ with the finite norm $\|u\|_{L^p([T_1,T_2],\mathcal{B})} := \|\|u(\cdot, t)\|_{\mathcal{B}}\|_{L^p([T_1,T_2])}$, see, for instance, [23, ch. Sec. 1.2]. The space $C([T_1, T_2], \mathcal{B})$ is introduced with the use of the same scheme; this is the Banach space of all the continuous mappings $u : [T_1, T_2] \to \mathcal{B}$ with the finite norm $\|u\|_{C([T_1,T_2],\mathcal{B})} := \sup_{t\in[T_1,T_2]} \|u(\cdot, t)\|_{\mathcal{B}}$.

Let now $\Delta = \sum\limits_{j=1}^{n} \partial^2_{x_j,x_j}$ be the Laplace operator in $\mathbb{R}^n$ and let $\mathcal{L}_m = \partial_t + (-\Delta)^m$ stand for the polyharmonic heat operator in $\mathbb{R}^{n+1}$. Of course, for $m = 1$ it coincides with the usual heat operator.

Now let $\partial_\nu = \sum\limits_{j=1}^{n} \nu_j \partial_{x_j}$ denote the derivative at the direction of the exterior unit normal vector $\nu = (\nu_1, \dots, \nu_n)$ to the surface $\partial\Omega$. If $\partial\Omega \in C^{2m-1}$ then the higher order normal derivatives ∂_ν^j are defined near $\partial\Omega$. We fix also a Dirichlet system $\{B_j\}_{j=0}^{2m-1}$ of order $(2m-1)$ consisting of boundary differential operators with smooth coefficients near $\partial\Omega$, i.e. $\mathrm{ord} B_j = j$ and for each $x \in \partial\Omega$ the characteristic polynomials $\sigma(B_j)(x, \zeta)$ related to the operators B_j do not vanish for $\zeta = \nu(x)$. The sets $(1, \partial_\nu, \partial_\nu^2, \dots \partial_\nu^{2m-1})$ and $(1, \partial_\nu, \Delta, \partial_\nu\Delta, \Delta^2, \dots \Delta^{m-1}, \partial_\nu\Delta^{m-1})$ are precisely the Dirichlet systems because $\sigma(\partial_\nu^j)(x, \nu(x)) = \sigma(\partial_\nu\Delta^j)(x, \nu(x)) = \sigma(\Delta^j)(x, \nu(x)) = 1$ for each $j \in \mathbb{N}$.

We consider the Cauchy problem for the polyharmonic heat equation in the cylinder Ω_T in the sense of the Cauchy–Kowalevski Theorem with respect to the space variables, cf. [24].

Problem 1. *Given $m \geqslant 1$, functions $u_j \in C^{2m-j+1,0}(\overline{\Gamma_T})$, $1 \leqslant j \leqslant 2m$, and $f \in C(\overline{\Omega}_T)$ find a function $u \in C^{2m,1}(\Omega_T) \cap C^{2m-1,0}((\Omega \cup \overline{\Gamma})_T)$ satisfying*

$$\mathcal{L}_m u = f \ \text{ in } \Omega_T, \tag{3}$$

$$B_j u(x, t) = u_{j+1}(x, t) \ \text{ on } \overline{\Gamma_T} \text{ for all } \ 0 \leqslant j \leqslant 2m-1. \tag{4}$$

If the hypersurface Γ and the data of the problem are real analytic then the Cauchy–Kowalevski theorem implies that problem (3), (4) has one and only one solution in the class of (even formal) power series. However the theorem does not imply the existence of solutions to Problem 1 because it grants the solution in a small neighbourhood of the hypersurface Γ_T only (but not in a given domain Ω_T!). We emphasize that, unlike the classical case, we do not ask for the hypersurface Γ or/and the coefficients of the operators B_j or/and the data f or/and u_j to be real analytic.

Of course, the above trick with the Laplace transform suggests us that the problem is equivalent to an ill-posed problem for the strongly elliptic operator $(-\Delta)^m$ in Ω with the Cauchy data on Γ, i.e. Problem 1 is ill-posed itself, too.

2. Solvability conditions

We begin this section proving that Problem 1 can not have more than one solution in the spaces of differentiable (non-analytic) functions.

To investigate Problem 1, we use an integral representation constructed with the use the fundamental solution $\Phi_m(x, t)$ to polyharmonic heat operator $\mathcal{L}_m$. If $m = 1$ then

$$\Phi_1(x,t) = \begin{cases} \dfrac{e^{-\frac{|x|^2}{4\mu t}}}{\left(2\sqrt{\pi\mu t}\right)^n} & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases} \tag{5}$$

see, for instance, [19, 25]. Unfortunately, if $m > 1$ then the fundamental solution can not be represented as an elementary function, see, for instance, [14, Ch. 2, Sec. 1], [15],

$$\Phi_m(x,t) = \begin{cases} k_{n,m} t^{-n/2m} \displaystyle\int_0^{+\infty} \rho^{n-1} e^{-\rho^{2m}} \Big(\frac{|x|\rho}{t^{1/2m}}\Big)^{1-n/2} J_{n/2-1}\Big(\frac{|x|\rho}{t^{1/2m}}\Big) d\rho & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases} \tag{6}$$

where $k_{n,m}$ is a normalization constant and J_p is the Bessel function of the first kind and of order p (see, for example, [16, Ch. 5, Sec. 23]).

The fundamental solution allows to construct a useful integral Green formula for the operator $\mathcal{L}_m$. With this purpose, Denote by $\{C_0, \dots C_{2m-1}\}$ the Dirichlet system associated with the Dirichlet system $\{B_0, \dots B_{2m-1}\}$ via (first) Green formula for the operator Δ^m, i.e.

$$\int_{\partial\Omega} \Big(\sum_{j=0}^{2m-1} C_{2m-1-j} v B_j u \Big) ds = (\Delta^m u, v)_{L^2(\Omega)} - (u, \Delta^m v)_{L^2(\Omega)}$$

for all $u, v \in C^\infty(\overline{\Omega})$. For instance, if $\{B_0, \dots B_{2m-1}\} = (1, \partial_\nu, \Delta, \partial_\nu \Delta, \dots \Delta^{m-1}, \partial_\nu \Delta^{m-1})$ then $\{C_0, \dots C_{2m-1}\} = (1, -\partial_\nu, \Delta, -\partial_\nu \Delta, , \dots \Delta^{m-1}, -\partial_\nu \Delta^{m-1})$.

Consider the cylinder type domain $\Omega_{T_1,T_2} = \Omega_{T_2} \setminus \overline{\Omega_{T_1}}$ with $0 \leqslant T_1 < T_2$ and a closed measurable set $S \subset \partial\Omega$. For functions $f \in L^2(\Omega_{T_1,T_2})$, $v_j \in L^2([0,T], H^{2m-j-1/2}(S_T))$, $h \in H^{1/2}(\Omega)$ we introduce the following potentials:

$$I_{\Omega,T_1}(h)(x,t) = \int\limits_{\Omega} \Phi(x-y,t) h(y) dy, \quad G_{\Omega,T_1}(f)(x,t) = \int\limits_{T_1}^{t} \int\limits_{\Omega} \Phi(x-y,\ t-\tau) f(y,\tau) dy d\tau,$$

$$V^{(j)}_{S,T_1}(v_j)(x,t) = \int\limits_{T_1}^{t} \int\limits_{S} C_j \Phi_m(x-y,t-\tau) v_j(y,\tau) ds(y) d\tau,\ 0 \leqslant j \leqslant 2m-1$$

(see, for instance, [19, Ch. 1, Sec. 3 and Ch. 5, Sec. 2], [20, Ch. 4, Sec. 1], [26, Ch. 3, Sec. 10] for $m = 1$). The potential $I_{\Omega,T_1}(h)$ is an analogue of the *Poisson integral* and the function $G_{\Omega,T_1}(f)$ is an analogue of the volume heat potential related to $m = 1$. The functions $V^{(0)}_{S,T_1}(v)$ and $V^{(1)}_{S,T_1}(v)$ are often referred to as *single layer heat potential* and *double layer heat potential*, respectively, if $m = 1$. By the construction, all these potentials are (improper) integrals depending on the parameters (x,t).

Next, we need the so-called Green formula for the polyharmonic heat operator.

Lemma 1. *For all* $0 \leqslant T_1 < T_2$ *and all* $u \in {}^{2m,1}(\overline{\Omega_{T_1,T_2}})$ *the following formula holds:*

$$\left.\begin{array}{l} u(x,t) \ \textit{in}\ \Omega_{T_1,T_2} \\ 0 \ \textit{outside}\ \overline{\Omega_{T_1,T_2}} \end{array}\right\} = I_{\Omega,T_1}(u) + G_{\Omega,T_1}(\mathcal{L}_m u) + \sum_{j=0}^{2m-1} V^{(j)}_{\partial\Omega,T_1}(B_j u). \tag{7}$$

Proof. See, for instance, [27, ch. 6, Sec. 12] for $m = 1$ and [28, theorem 2.4.8] for more general operators, admitting fundamental solutions/parametreces. □

Formulas (5), (6) mean that the kernels $\Phi_m(x-y,t-\tau)$ are smooth outside the diagonal $\{(x,t) = (y,\tau)\}$ and real analytic with respect to the space variables. In particular, this means that the $2m$-parabolic operator $\mathcal{L}_m$ is hypoelliptic. Moreover, any $C^{2m,1}(\Omega_{T_1,T_2})$-solution v to the polyharmonic heat equation $\mathcal{L}_m v = 0$ in the cylinder domain Ω_{T_1,T_2} belongs to $C^\infty(\Omega_{T_1,T_2})$ and, actually $v(x,t)$ is real analytic with respect to the space variable $x \in \Omega$ for each $t \in (T_1, T_2)$ (for $m = 1$, see, for instance, [25, Ch. VI, Sec. 1, Theorem 1] and for $m > 1$ see [14, Ch. Sec. 2, Sec. 1, Theorem 2.1]). Then Green formula (7) and the information on the kernel Φ_m provide us with a Uniqueness Theorem for Problem 1.

Theorem 1 (A Uniqueness Theorem)**.** *If Γ has at least one interior point in the relative topology of $\partial\Omega$ then Problem 1 has no more than one solution.*

Proof. For $m = 1$ see [1, Theorem 1, Corollary 1]. For $m > 1$ the proof can be done in the same way with natural modifications. Indeed, under the hypothesis of the theorem there is an interior (in the relative topology of Γ!) point x_0 on Γ. Then there is such a number $r > 0$ that $B(x_0,\ r) \cap \partial\Omega \subset \Gamma$ where $B(x_0,\ r)$ is ball in $\mathbb{R}^n$ with center at x_0 and radius r. Fix an arbitrary point $(x', t') \in \Omega_T$. Clearly, there is a domain $\Omega' \ni x'$ satisfying $\Omega' \subset \Omega$ and $\Omega' \cap \partial\Omega \subset \Gamma \cap B(x_0,\ r)$. Then $(x', t') \in \Omega'_{T_1,T_2}$ with some $0 < T_1 < T_2 < T$.

But $u \in C^{2m,1}(\Omega'_{T_1,T_2}) \cap C^{2m-1,0}(\overline{\Omega'_{T_1,T_2}})$ (for $m = 1$ see, for instance, [19, Ch. 1, Sec. 3 and Ch. 5, Sec. 2] and for $m > 1$ it follows from [14, Ch. 2, Sec. 1, Theorem 2.2]) and $\mathcal{L}_m u = 0$ in Ω'_{T_1,T_2} under the hypothesis of the theorem. Hence formula (7) implies:

$$\left.\begin{array}{r} u(x,t),\ (x,t) \in \Omega'_{T_1,T_2} \\ 0,\ (x,t) \notin \overline{\Omega'_{T_1,T_2}} \end{array}\right\} = I_{\Omega',T_1}(u)(x,t) + \sum_{j=0}^{2m-1} V^{(j)}_{\partial\Omega'\setminus\Gamma,T_1}\left(B_j u\right)(x,t), \tag{8}$$

because $B_j u \equiv 0$ on Γ_T for all $0 \leqslant j \leqslant 2m-1$.

Taking into account the character of the singularity of the kernel $\Phi_m(x-y, t-\tau)$ we conclude that the following properties are fulfilled for the integrals, depending on parameter, from the right hand side of identity (8):

$$I_{\Omega',T_1}(u) \in C^{2m,1}(\{x \in \mathbb{R}^n, T_1 < t < T_2\}),$$

$$V^{(j)}_{\partial\Omega'\setminus\Gamma,T_1<t<T_2}\left(B_j u\right) \in C^{2m,1}(\{x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma), T_1 < t < T_2\})$$

(see, for instance, [19, Ch. 1, Sec. 3 and Ch. 5, Sec. 2], [20, Ch. 4, Sec. 1] or [26, Ch. 3, Sec. 10] for m=1). Moreover, as Φ_m is a fundamental solution to the polyharmonic heat operator then

$$\mathcal{L}_m(x,t)\Phi_m(x-y, t-\tau) = 0 \text{ for } (x,t) \neq (y,\tau),$$

and therefore, using Leibniz rule for differentiation of integrals depending on parameter we obtain:

$$\mathcal{L}_m I_{\Omega',T_1}(u) = 0 \text{ in the domain } \{x \in \mathbb{R}^n, T_1 < t < T_2\},$$

$$\mathcal{L}_m V^{(j)}_{\partial\Omega'\setminus\Gamma,T_1}\left(B_j u\right) = 0 \text{ in } \Omega''_{T_1,T_2} = \{x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma), T_1 < t < T_2\} \text{ for all } 0 \leqslant j \leqslant 2m-1.$$

Hence the function

$$v(x,t) = I_{\Omega',T_1}(u)(x,t) + V^{(j)}_{\partial\Omega'\setminus\Gamma,T_1}\left(B_j u\right)(x,t),$$

satisfies the polyharmonic heat equation $(\mathcal{L}_m v)(x,t) = 0$ in Ω''_{T_1,T_2}. As we mentioned above, this implies that the function $v(x,t)$ is real analytic with respect to the space variable $x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma)$ for any $T_1 < t < T_2$. By the construction the function $v(x,t)$ is real analytic with respect to x in the ball $B(x_0, r)$ and it equals to zero for $x \in B(x_0, r) \setminus \overline{\Omega}$ for all $T_1 < t < T_2$. Therefore, the Uniqueness Theorem for real analytic functions yields $v(x,t) \equiv 0$ in Ω''_{T_1,T_2}, and in the cylinder Ω'_{T_1,T_2}, containing point (x', t'). Now it follows from (8) that $u(x', t') = v(x', t') = 0$ and then, since the point $(x', t') \in \Omega_T$ is arbitrary we conclude that $u \equiv 0$ in Ω_T. □

Now we are ready to formulate a solvabilty criterion for Problem 1. As before, we assume that Γ is a relatively open connected subset of $\partial\Omega$. Then we may find a set $\Omega^+ \subset \mathbb{R}^n$ in such a way that the set $D = \Omega \cup \Gamma \cup \Omega^+$ would be a bounded domain with piece-wise smooth boundary.

It is convenient to set $\Omega^- = \Omega$. For a function v on D_T we denote by v^+ its restriction to Ω^+_T and, similarly, we denote by v^- its restriction to Ω_T. It is natural to denote limit values of $v^\pm$ on Γ_T, when they are defined, by $v^\pm_{|\Gamma_T}$. Actually, for $m = 1$ similar solvability criterions for Problem 1 were obtained in [1] and [4].

Theorem 2 (Solvability criterion)**.** *Let $\lambda \in (0,1)$, $\partial\Omega$ belong to $C^{2m-1+\lambda}$ and let Γ be a relatively open connected subset of $\partial\Omega$. If $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$, $u_j \in C^{2m-j,0,\lambda,\lambda/2}(\overline{\Gamma_T})$, $1 \leqslant j \leqslant 2m$, then Problem (3), (4) is solvable in the space $C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T)$ if and only if there is a function $F \in C^\infty(D_T)$ satisfying the following two conditions: 1) $\mathcal{L}_m F = 0$ in D_T, 2) $F = G_{\Omega,0}(f) + \sum\limits_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})$ in Ω^+_T.*

Proof. Necessity. Let a function $u(x,t) \in C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T)$ satisfy (3), (4). Clearly, the function $u(x,t)$ belongs to the space $C^{2m,1,\lambda,\lambda/2}(\Omega'_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\overline{\Omega'_T})$ for each cylindrical domain Ω'_T with such a base Ω' that $\Omega' \subset \Omega$ and $\overline{\Omega'} \cap \partial\Omega \subset \Gamma$. Besides, $\mathcal{L}u = f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega'_T})$. Without loss of the generality we may assume that the interior part Γ' of the set $\overline{\Omega'} \cap \partial\Omega$ is non-empty. Consider in the domain D_T the functions

$$\mathcal{F} = G_{\Omega,0}(f) + \sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1}) \text{ and } F = \mathcal{F} - \chi_{\Omega_T} u, \tag{9}$$

where χ_M is a characteristic function of the set $M \subset \mathbb{R}^{n+1}$. By the very construction condition 2) is fulfilled for it. Note that $\chi_{\Omega_T} u = \chi_{\Omega'_T} u$ in D'_T, where $D' = \Omega' \cup \Gamma' \cup \Omega^+$. Then Lemma 1 yields

$$F = G_{\Omega\setminus\overline{\Omega'},0}(f) + \sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1}) - I_{\Omega',0}(u) \text{ in } D'_T. \tag{10}$$

Arguing as in the proof of Theorem 1 we conclude that each of the integrals in the right hand side of (10) is smooth outside the corresponding integration set and each satisfies homogeneous polyharmonic heat equation there. In particular, we see that $F \in C^\infty(D'_T)$ and $\mathcal{L}F = 0$ in D'_T because of [25, Ch. VI, Sec. 1, Theorem 1]. Obviously, for any point $(x,t) \in D_T$ there is a domain D'_T containing $(x,\ t)$. That is why $\mathcal{L}_m F = 0$ in D_T, and hence F belongs to the space $C^\infty(D_T)$. Thus, this function satisfies condition 1), too.

Sufficiency. Let there be a function $F \in C^\infty(D_T)$, satisfying conditions 1) and 2) of the theorem. Consider on the set D_T the function

$$U = \mathcal{F} - F. \tag{11}$$

As $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$ then the results of [19, Ch. 1, Sec. 3], [20, Ch. 4, Secs. 11–14] for $m = 1$ and [14, Ch. 2, Sec. 1, Theorem 2.2] for $m > 1$ imply

$$G_{\Omega,0}(f) \in C^{2m,1,\lambda,\lambda/2}(\overline{\Omega^\pm_T}) \cap C^{2m-1,0,\lambda,\lambda/2}(D_T) \tag{12}$$

and, moreover,

$$\mathcal{L}_m G^-_{\Omega,0}(f) = f \text{ in } \Omega_T, \quad \mathcal{L}_m G^+_{\Omega,0}(f) = 0 \text{ in } \Omega^+_T. \tag{13}$$

Since $u_j \in C^{2m-j,0,\lambda,\lambda/2}(\overline{\Gamma_T})$ then the results of [20, Ch. 4, Secs. 11-14], [19, Ch. 5, Sec. 2] for $m = 1$ and [14, Ch. 2, Sec. 1, Theorem 2.2] for $m > 1$ yield

$$V^{(j)}_{\overline{\Gamma},0}(u_j) \in C^\infty(\Omega^\pm_T) \cap C^{2m-1,0,\lambda,\lambda/2}((\Omega^\pm \cup \Gamma)_T), \quad \mathcal{L}^{(j)} V_{\overline{\Gamma},0}(u_j) = 0 \text{ in } \Omega_T \cup \Omega^+_T. \tag{14}$$

Since $F \in C^\infty(D_T) \subset C^{1,0,\lambda,\lambda/2}((\Omega^+ \cup \Gamma)_T)$ then formulas (11)–(14) imply that U belongs $C^{2m,1,\lambda,\lambda/2}(\Omega^\pm_T) \cap C^{2m-1,0,\lambda,\lambda/2}((\Omega^\pm \cup \Gamma)_T)$ and $\mathcal{L}U = \chi_{D_T} f$ in $\Omega_T \cup \Omega^+_T$. In particular, (3) is fulfilled for U^-. Let us show that the function U^- satisfies (4). Since $F \in C^\infty(D_T)$ we see that $\partial^\alpha F^- = \partial^\alpha F^+$ on Γ_T for $\alpha \in \mathbb{Z}_+$ with $|\alpha| \leqslant 2m-1$ and

$$\partial^\alpha F^+_{|\Gamma_T} = \Big(\partial^\alpha G_{\Omega,0}(f) + \sum_{j=0}^{2m-1} \partial^\alpha \big(V^{(j)}_{\overline{\Gamma},0}(u_{j+1})\big)\Big)^+_{|\Gamma_T}.$$

Thus, it follows from formula (12) that for all $0 \leqslant i \leqslant 2m-1$ we have

$$B_i U^-_{|\Gamma_T} = \Big(B_i\Big(\sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})\Big)\Big)^-_{|\Gamma_T} - \Big(B_i\Big(\sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})\Big)\Big)^+_{|\Gamma_T}. \tag{15}$$

Hence, in order to finish the proof we need the following lemma.

Lemma 2. *Let $\Gamma \in C^{2m-1+\lambda}$ and $u_j \in C^{2m-j,0,\lambda,\lambda/2}(\overline{\Gamma_T})$, $1 \leqslant j \leqslant 2m$. Then*

$$\Big(B_i\Big(\sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})\Big)\Big)^-_{|\Gamma_T} - \Big(B_i\Big(\sum_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})\Big)\Big)^+_{|\Gamma_T} = u_{i+1},\ 0 \leqslant i \leqslant 2m-1. \tag{16}$$

Proof. It is similar to the proof of the analogous lemmas for the heat Single and Double Layer Potentials (see, for instance, [1, Lemma 3], [26, Ch. 3, Sec. 10, Theorem 10.1] for $m = 1$ and a different function class or [12, Lemma 2.7] for elliptic potentials). □

Using Lemma 2 and formulas (12), (15), we conclude that $B_j U^-_{|\Gamma_T} = u_{j+1}$ for all $0 \leqslant j \leqslant 2m-1$, i.e. the second equation in (4) is fulfilled for U^-. Thus, function $u(x,t) = U^-(x,t)$ satisfies conditions (3), (4). The proof is complete. □

We note that Theorem 2 is also an analogue of Theorem by Aizenberg and Kytmanov [10] describing solvability conditions of the Cauchy problem for the Cauchy–Riemann system (cf. also [11] in the Cauchy Problem for Laplace Equation or [13] in the Cauchy problem for general elliptic systems).

We note also that formula (11), obtained in the proof of Theorem 2, gives the unique solution to Problem 1. Clearly, if we will be able to write the extension F of the sum of potentials $G_{\Omega,0}(f) + \sum\limits_{j=0}^{2m-1} V^{(j)}_{\overline{\Gamma},0}(u_{j+1})$ from Ω^+_T onto D_T as a series with respect to special functions or a limit of parameter depending integrals then we will get Carleman's type formula for solutions to Problem 1 (cf. [10]). However, for the best way for this purpose is to use the Fourier series in the framework of the Hilbert space theory, see [5]. Unfortunately, this is not a short story because one needs approximation theorems in spaces of solutions to the homogeneous polyharmonic heat equation that we are not ready to prove right now. Thus we finish our paper with a statement extending Theorem 2 to the anisotropic Sobolev spaces, leaving the construction of the Carleman's type formulae for the next article.

First of all, we need the following lemma.

Lemma 3. *Let $\partial\Omega \in C^{2m+1}$ and let Γ be a relatively open connected subset of $\partial\Omega$ with boundary $\partial\Gamma \in C^{2m+\lambda}$. If $u_j \in C^{2m+1-j,0,\lambda,\lambda/2}(\overline{\Gamma_T})$, $1 \leqslant j \leqslant 2m$, then there exist functions $\tilde u_j \in C^{2m+1-j,0,\lambda,\lambda/2}(\partial\Omega_T)$ such that $\tilde u_j = u_j$ on $\overline{\Gamma_T}$, $1 \leqslant j \leqslant 2m$, and a function $\tilde u \in C^{2m,1,\lambda,\lambda/2}(\overline{\Omega_T})$ such that $B_j \tilde u = \tilde u_{j+1}$ on $(\partial\Omega)_T$ for all $0 \leqslant j \leqslant 2m-1$.*

Proof. We may adopt the standard arguments from [29, Lemma 6.37] related to isotropic spaces. Indeed, according to it, under our assumptions, for any $s \leqslant 2m$ and any $v \in C^{s,\lambda}(\overline{\Gamma})$ there is $\tilde v_j \in C^{s,\lambda}(\partial\Omega)$ such that $v = v_0$ on $\overline{\Gamma}$. The construction of the extension involves the rectifying diffeomorphism of $\partial\Gamma$ and a suitable partition of unity of a neighbourhood of $\partial\Gamma$, only. Thus, we conclude there are functions $\tilde u_j \in C^{2m-j+1,0,\lambda,\lambda/2}(\partial\Omega_T)$ such that $\tilde u_j = u_j$ on $\overline{\Gamma_T}$, $1 \leqslant j \leqslant 2m$.

Next, we use the existence of the Poisson kernel $P_{\Delta^{2m},\Omega}(x,y)$ for the Dirichlet problem related to the operator Δ^{2m}, see [30]. It is known that the problem is well-posed over the scale of Hölder spaces in Ω. Namely, if $\partial\Omega \in C^{s+1,\lambda}$, $s \geqslant 2m-1$, then for each $\oplus_{j=0}^{2m-1} v_j \in C^{s-j,\lambda}(\partial\Omega)$ the integral

$$v(x) = \mathcal{P}_{\Delta^{2m},\Omega}(\oplus_{j=0}^{2m-1} v_j)(x) = \int_{\partial\Omega} \left(\sum_{j=0}^{2m-1} (B_j(y) P_{\Delta^{2m},\Omega})(x,y) v_j(y) \right) ds(y)$$

belongs to $C^{s,\lambda}(\overline{\Omega})$ and satisfies $\Delta^{2m} v = 0$ in Ω and $B_j v = v_j$ on $\partial\Omega$ for all $0 \leqslant j \leqslant 2m-1$.

Now, we set

$$\tilde{u}_0(x) = \mathcal{P}_{\Delta^{2m},\Omega}(\oplus_{j=0}^{2m-1} \tilde{u}_{j+1})(\cdot,0)(x) \in C^{2m-1,\lambda}(\overline{\Omega}) \cap C^{2m,\lambda}(\Omega).$$

Now, we may take as $\tilde{u}(x,t) \in C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\overline{\Omega_T})$ the unique solution to the parabolic initial boundary problem

$$\begin{cases} \partial_t \tilde{u}(x,t) + \Delta^{2m} \tilde{u}(x,t) = 0 & \text{in} \quad \Omega_T, \\ \oplus_{j=0}^{2m-1} B_j \tilde{u}(x,t) = \oplus_{j=0}^{2m-1} \tilde{u}_{j+1}(x,t) & \text{on} \quad (\partial\Omega)_T, \\ \tilde{u}(x,0) = \tilde{u}_0(x) & \text{on} \quad \overline{\Omega}, \end{cases}$$

see, for instance, [20, Ch. 5, Sec. 6] for $m=1$ or [14, Ch. 3, Sec. 1] for $m \geqslant 1$. But of course, there are other possibilities to choose a function $\tilde{u}$ with the desired properties. □

Under the assumptions of Lemma 3, we set

$$\tilde{\mathcal{F}} = G_{\Omega,0}(f) + \sum_{j=0}^{2m-1} V^{(j)}_{\partial\Omega,0}(\tilde{u}_{j+1}) + I_{\Omega,0}(\tilde{u}). \tag{17}$$

Corollary 1. *Let $\lambda \in (0,1)$, $\partial\Omega$ belong to $C^{2m+1+\lambda}$ and let Γ be a relatively open connected subset of $\partial\Omega$ with boundary $\partial\Gamma \in C^{2m+\lambda}$. If $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$, $u_j \in C^{2m-j+1,0,\lambda,\lambda/2}(\overline{\Gamma_T})$, then Problem (3), (4) is solvable in the space $C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2m,1}(\Omega_T)$ if and only if there is a function $\tilde{F} \in C^\infty(D_T) \cap H^{2m,1}(D_T)$ satisfying the following two conditions: 1') $\mathcal{L}\tilde{F} = 0$ in D_T, 2') $\tilde{F} = \tilde{\mathcal{F}}$ in Ω_T^+.*

Proof. First of all, we note that, by Green formula (7), we have $\tilde{\mathcal{F}} = G_{\Omega,0}(f - \mathcal{L}\tilde{u}) + \chi_{\Omega_T}\tilde{u}$ and then $\tilde{\mathcal{F}} \in C^{2m,1,\lambda,\lambda/2}(\overline{\Omega_T^{\pm}})$ because of (12). On the other hand,

$$\tilde{\mathcal{F}} - \mathcal{F} = \sum_{j=0}^{2m-1} V^{(j)}_{\partial\Omega\setminus\Gamma,0}(\tilde{u}_{j+1}) + I_{\Omega,0}(\tilde{u}). \tag{18}$$

This means that the function $\tilde{\mathcal{F}} - \mathcal{F}$ satisfies the $\mathcal{L}(\tilde{\mathcal{F}} - \mathcal{F}) = 0$ in D_T and hence the function $\mathcal{F}$ extends to D_T as a solution of the heat equation if and only if function $\tilde{\mathcal{F}}$ extends to D_T as a solution of the polyharmonic heat equation, too.

Let Problem (3), (4) be solvable in the space $C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2m,1}(\Omega_T)$. Then formulas (9) and (18) imply

$$\tilde{F} = \tilde{\mathcal{F}} - \chi_{\Omega_T} u \in H^{2m,1}(\Omega_T^{\pm}) \text{ and } \mathcal{L}\tilde{F} = 0 \text{ in } D_T.$$

Now, as $\tilde{F} \in H^{2m,1}(\Omega_T^{\pm}) \cap C^\infty(D_T)$ (see [25, Ch. VI, Sec. 1, Theorem 1]) we conclude that $\tilde{F} \in H^{2m,1}(D_T)$, i.e. conditions 1'), 2') of the corollary are fulfilled.

If conditions 1'), 2') of the corollary hold true then conditions 1), 2) of Theorem 2 are fulfilled, too. Moreover, formulas (11) and (18) imply that in D_T we have

$$U = \mathcal{F} - F = \tilde{\mathcal{F}} - \tilde{F} \in H^{2m,1}(\Omega_T^{\pm}) \tag{19}$$

and the U^- is the solution to Problem 1 in the space $C^{2m,1,\lambda,\lambda/2}(\Omega_T) \cap C^{2m-1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2m,1}(\Omega_T^{\pm})$ by Theorem 2. □

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О некорректной задаче Коши для решений полигармонического уравнения теплопроводности

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Аннотация. Мы рассматриваем некорректную задачу Коши для полигармонического оператора теплопроводности о восстановлении функции, удовлетворяющей уравнению $(\partial_t+(-\Delta)^m)u = 0$ в цилиндрической области в полупространстве $\mathbb{R}^n \times [0, +\infty)$, где $n \geqslant 1$, $m \geqslant 1$, а Δ – оператор Лапласа, по заданным ее значениям и значениям ее нормальных производных до порядка $(2m-1)$ включительно на части боковой поверхности цилиндра. Нами получены теорема единственности для этой задачи Коши в анизотропных пространствах Соболева, а также необходимые и достаточные условия ее разрешимости в терминах вещественно-аналитического продолжения параболических потенциалов, ассоциированных с данными Коши.

Ключевые слова: полигармоническое уравнение теплопроводности, некорректные задачи, метод интегральных представлений.

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Incomplete Least Squared Regression Function Estimator Based on Wavelets

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Abstract. In this paper, we introduce an estimator of the least squares regression function, for Y right censored by R and $\min(Y, R)$ left censored by L. It is based on ideas derived from the context of wavelet estimates and is constructed by rigid thresholding of the coefficient estimates of a series development of the regression function. We establish convergence in norm L_2. We give enough criteria for the consistency of this estimator. The result shows that our estimator is able to adapt to the local regularity of the related regression function and distribution.

Keywords: non-parametric regression, L_2 error, least squares estimators, orthogonal series estimates, convergence in the L_2-norm, twice censored data, regression estimation, hard thresholding.

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Regression is defined as being the set of statistical methods widely used to analyse the relationship between a variable and one or more others. For a long time, the regression of a random variable Y on a vector X of random variables designated the conditional mean of Y given X. Nowadays, the term regression designates any element of the conditional distribution of Y given X, as a function of X. We can for example be interested in the conditional mean, the conditional median, or the conditional variance. In presence of functional data, which are doubly infinite dimensional problems, the appeal to non parametric estimation is unavoidable. The starting point in this regards is a prediction problem that leads to the regression function due to the minimization of the mean squared error i.e., L_2 risk. In this setting, one can usually consider the model $Y = m(X) + \varepsilon$ where ε is centred and is independent of X with the explained variable fully observed. In the case of complete observation of $(X,\ Y)$, an abundant literature in this field can be found for instance in Györfi and al (2002) and references there in. However, in several situations the variable of interest X may be subject to randomly right and left censoring in the same sample. The lifetime Y is right censored by a variable R (which itself represents a survival

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time) and the minimum between Y and R is censored by a censorship variable on the left. A symbolical example of this model is the one given in Morales and al. (1991) that investigates the cause of death of trees on a farm. This kind of censoring model is exactly the Model one studied in Patilea and Rolin, for which local averaging estimates of $m(x) = \mathbf{E}(Y|X = x)$ has been introduced by Messaci (2010). In Kebabi and Messaci (2012), least squares estimator of $m(x)$ has been proposed and its L_2-norm convergence has been established. In this paper, we are mainly interested in least squares estimation approaches of the regression function for the Model I of Patilea and Rolin. Particularly, we investigate a least squares method based on wavelets. The use of a wavelets based approach is motivated by the possibility to achieve optimal convergence rates despite the high dimensionality of the problem. Moreover, wavelets are excellent approximators for signals with rapid local changes such as cusps, discontinuities, sharp spikes, etc. On the other hand, accurate wavelet decomposition, using only a few wavelet coefficient, can represent signals allowing dimensionality reduction and sparsity. So explicitly, the purpose of this paper is the construction of non-linear orthogonal series estimates by rigid transformation (thresholding) of the coefficients estimates of a regression function series development. The first part of our study is devoted to the introduction of the least squares estimators of the regression function for censored data and to some convergence properties. An important idea is introduced which consists in the estimation of orthogonal series of the regression function. Then, we present the estimation of the coefficients of these series, based on a wavelet system, is presented. In the second part, we list the proofs.

1. Model and recalls

Let (X, Y) be a random vector with values in $\mathbb{R}^d \times \mathbb{R}$ with $\mathbf{E}(Y)^2 < \infty$ and the dependence of Y on the value of X is of interest. Let R and L be censoring positive random variables. More specifically, the objective is to find a function $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(X)$ is a "good approximation" of Y.

1.1. Model

We introduce orthogonal series estimates of $m(x) = \mathbf{E}(Y|X = x)$ with respect to sample of iid $\mathcal{D}_n = \{X_i,\ Z_i = \max(\min(Y_i, R_i), L_i),\ A_i\}$ from the same distribution as (X, Z, A) or $Z = \max(\min(Y, R), L)$ and

$$A = \begin{cases} 0 & \text{if} \quad L < Y < R, \\ 1 & \text{if} \quad L < R \leqslant Y, \\ 2 & \text{if} \quad \min(Y, R) \leqslant L. \end{cases}$$

Indeed, let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary (measurable) function and denote X distribution par μ then

$$\begin{aligned} \mathbf{E}|f(X) - Y|^2 &= \mathbf{E}|f(X) - m(X) + m(X) - Y|^2 = \\ &= \mathbf{E}|f(X) - m(X)|^2 + \mathbf{E}|m(X) - Y|^2 = \\ &= \mathbf{E}|m(X) - Y|^2 + \int |f(x) - m(x)|^2 \mu(dx). \end{aligned}$$

In the sequel we will denote by F_V the distribution function of the random variable V and by $S_V = 1 - F_V$ its survival function and $T_V = \sup\{t : F_V(t) < 1\}$ and $I_V = \inf\{t : F_V(t) \neq 0\}$ the end points of the support of the variable V. Assume that the variables X, Y, R et L satisfies the following hypotheses

H_1 : $\quad Y, R$ and L are independent.

H_2 : (L, R) is independent of (X, Y).
H_3 : $\exists T < T_R$ and $I > I_L$ such that, $\forall n \in \mathbb{N}, \forall i (1 \leqslant i \leqslant n) : A_i = 0 \Rightarrow I \leqslant Z_i \leqslant T$ a.s.
H_4 : F_L is continuous on $]0, \infty[$.
H_5 : $T_R \leqslant T_Y \leqslant T_L < \infty$ and $I_Y \leqslant I_L < I_R$.

H_1 is an inherent hypothesis of Patilea's et al . H_3 seems to be acceptable because $I \leqslant Z_i \leqslant T$ when $A_i = 0$. H_5 guarantees in particular that the model is identifiable.
Let h a mapping on $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, we introduce as unbiased estimator of $\mathbf{E}(h(X, Y))$ the amount

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{h(X_i, Z_i)}{S_R(Z_i)F_L(Z_i)}. \tag{1}$$

Indeed, under hypothesis H_1, H_2 and H_4. The problem is that functions S_R and F_L are generally unknown, we will replace them respectively with their estimators. Let $(Z'_j)_{1 \leqslant j \leqslant M}$, $(M \leqslant n)$ be the distinct values of Z_i listed in ascending order.

1.2. Estimation and proprieties

Set

$$D_{kj} = \sum_{i=1}^{n} 1_{\{Z_i = Z'_j, A_i = k\}}, \text{ and } N_j = \sum_{i=1}^{n} 1_{\{Z_i \leqslant Z'_j\}},$$

thus, [22] suggest estimating S_R by

$$\hat{S}_n(t) = \prod_{j/Z'_j \leqslant t} \left\{1 - \frac{D_{1j}}{U_{j-1} - N_{j-1}}\right\} \text{ and } U_{j-1} = n \prod_{j \leqslant l \leqslant M} \left\{1 - \frac{D_{2l}}{N_l}\right\}, \tag{2}$$

and by inverting time in the Kaplan et al estimator, we can deduce the estimator $\hat{F}_n$ from F_L (left censoring case) witch is

$$\hat{F}_n(t) = \prod_{j/Z'_j > t} \left\{1 - \frac{1_{\{A_j=2\}}}{j}\right\}. \tag{3}$$

Recall that under hypothesis H_1 and H_5, [22] have proven that

$$\sup_{t \in \mathbb{R}^+} \left|\hat{S}_n(t) - S_R(t)\right| \underset{n \longrightarrow \infty}{\longrightarrow} 0 \text{ a.s.} \tag{4}$$

And

$$\sup_{t \in \mathbb{R}^+} \left|\hat{F}_n(t) - F_L(t)\right| \underset{n \longrightarrow \infty}{\longrightarrow} 0 \text{ a.s.} \tag{5}$$

Note that hypothesis H_3 implies that

$$S_R(T) > 0 \text{ and } F_L(I) > 0. \tag{6}$$

In view of equations $(4) - (6)$, we deduce that for n sufficiently large

$$\hat{S}_n(T) > 0 \text{ and } \hat{F}_n(I) > 0 \text{ a.s.}$$

If Y is uncensored , the regression function estimator of the least squares , obtained by minimizing the empirical risk L_2, is $\arg\min_{f \in \mathcal{F}_n} \frac{1}{n}\sum_{i=1}^{n} |f(X_i) - Y_i|^2$, where $\mathcal{F}_n$ is a class of functions that is depending on the sample size n. Thus, in our context, according to the relation h and after having estimated S_R and F_L, the least squares estimator of $m(x)$ is given by

$$\widetilde{m}_n = \arg\min_{f\in\mathcal{F}_n} \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i)-Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} \left(\frac{0}{0} := 0\right). \tag{7}$$

$\mathcal{F}_n$ is a certain family of functions which will be clarified in the theorem. We see that $\hat{S}_n(Z_i)$ does not vanish in the expression of $\widetilde{m}_n$ if $A_i = 0$. It is easy to check that $\hat{F}_n(Z_i)$ does not vanish either if $A_i = 0$ but since Y is bounded, we are going to make some assumptions on our estimator. For that reintroduce the notation of the next use of truncation.

For $0 \leqslant t < \infty$ and $x \in \mathbb{R}$, define

$$\mathbf{T}_{[0,t]}(x) = \begin{cases} t & \text{if } x > t, \\ x & \text{if } 0 \leqslant x \leqslant t, \\ 0 & \text{if } x < 0, \end{cases}$$

and for $f : \mathbb{R}^d \to \mathbb{R}$, define $(\mathbf{T}_{[0,t]}f)(x) = \mathbf{T}_{[0,t]}(f(x))$. We can also use again the fact that this mapping verifies the following relation.

$$\forall b > a, \quad |\mathbf{T}_{[0,b]}(x) - \mathbf{T}_{[0,a]}(x)| \leqslant (b-a). \tag{8}$$

Y being limited and due to $M_n = \max(Z_1, \ldots, Z_n)$ with $M_n \underset{n\to+\infty}{\rightarrow} T_L$ a.s, we finally propose as an estimator of $m(x)$

$$m_n(x) = \mathbf{T}_{[0,M_n]}(\tilde{m}_n(x)). \tag{9}$$

1.3. Wavelet bases

Let $\mathcal{F}_n$ be the set of all piecewise polynomials of degree M (or less) with respect to some partition of $[0,1]$ consisting of $4n^{1-\alpha}$ intervals (or less). Let G_M be set of polynomials of degree M (or less), let P_n be an equidistant partition of $[0,1]$ in $\lceil \log(n) \rceil$ intervals. Denote $G_M \circ P_n$ the set of all piecewise polynomials of degree M (or less) with respect to P_n. We will also need the following notations

$$\mathcal{L}_n^{**} = \mathrm{T}_{\log n}(\mathcal{F}_n).$$

$$\mathcal{F}_n^{**} = \{\forall f \in G_M \circ P_n, \|f\|_\infty \leqslant \log(n)\}.$$

Now adapting the proofs given in Kohler et al [17], We get the following result concerning the convergence of the introduced estimators. We refer, for example to Györfi et al [7] for some definitions and results of the Vapnik et al [23] theory, used in this work.

We introduce orthogonal series estimates in the context of regression estimation with fixed, equidistant design, which is the field where they have been applied most successfully. Let $(x_1, Y_1), \ldots, (x_n, Y_n)$ be data according to the model $Y_i = m(x_i) + \varepsilon_i$ where x_i are fixed (non-random) equidistant points in $[0,1]$, ε_i are i.i.d. random variables with $\varepsilon_i = 0$ and $\mathbf{E}(\epsilon_i) < \infty$ and m is a regression function $f : [0,1] \to \mathbb{R}$.

Assume that $m \in L_2(\mu)$ where μ is Lebesgue measure on $[0,1]$; and $(f_j)_{j\in\mathbb{N}}$ is an orthonormal basis in $L_2(\mu)$, ie

$$\langle f_j, f_k \rangle_{L_2(\mu)} = \int f_j(x) f_k(x) \mu(dx) = \left\{ \begin{array}{lll} 1 & \text{if} & j = k \\ 0 & \text{if} & j \neq k \end{array} \right. .$$

Each function in $L_2(\mu)$ can be arbitrarily approximated by linear combinations of $(f_j)_{j\in\mathbb{N}}$. Then m can be represented by its Fourier series with respect to $(f_j)_{j\in\mathbb{N}}$,

$$m = \sum_{j=1}^{\infty} c_j f_j \quad \text{where} \quad c_j = \langle m, f_j\rangle_{L_2(\mu)} = \int m(x) f_j(x)\mu(dx). \tag{10}$$

Orthogonal series estimates use the estimates of coefficients of a series expansion $\mathbf{E}|f(X)-Y|^2 = \mathbf{E}|m(X)-Y|^2 + \int |f(x)-m(x)|^2\mu(dx)$ to reconstruct the regression function and in the model $Y_i = m(x_i) + \varepsilon_i$, where $x_1, \dots, x_n$ are equidistant in $[0, 1]$; coefficients c_j can be estimated by

$$\hat{c}_j = \frac{1}{n}\sum_{i=1}^{n} Y_i f_j(x_i), j \in \mathbb{N}. \tag{11}$$

The traditional way to deal with these estimated coefficients to construct an estimate

$$m^1{}_n = \sum_{j=1}^{\tilde{K}} \hat{c}_j f_j,$$

m is to truncate the series expansion to an index $\tilde{K}$ and to inject the estimated coefficients.

Here, we try to choose $\tilde{K}$ such that the set of functions$\{f_1, \dots, f_{\tilde{K}}\}$ is the "best" among all the sub-sets $\{f_1\}, \{f_1, f_2\}, \{f_1, f_2, \dots\}$ of $\{f_j\}_{j\in\mathbb{N}}$ in view of the estimation error (7). This implicitly assumes that the most important information m is in the first coefficients $\tilde{K}$ of the series expansion $\mathbf{E}|f(X)-Y|^2 = \mathbf{E}|m(X)-Y|^2 + \int |f(x)-m(x)|^2\mu(dx)$.

[5] have proposed a way to overcome this hypothesis. This consists in contaminating the estimated coefficients, for example, we use all the coefficients whose absolute value is greater than a threshold δ_n (called hard thresholding). This leads to estimates of the form

$$m^2{}_n = \sum_{j=1}^{K} \eta_{\delta_n}(\hat{c}_j) f_j,$$

where K is generally much larger than $\tilde{K}$ in (7), $\delta_n > 0$ is a threshold, and

$$\eta_{\delta_n}(\hat{c}_j) = \begin{cases} \hat{c}_j & \text{if} \quad |\hat{c}_j| > \delta_n \\ o & \text{if} \quad |\hat{c}_j| \geqslant \delta_n \end{cases},$$

in the series expansion, we truncate the estimate at some data-independent height B_n, in other words, we define

$$\bar{m}_n(x) = (T_{B_n}\tilde{m}_n)(x) = \begin{cases} B_n & \text{if } \tilde{m}_n(x) > B_n, \\ \tilde{m}_n(x) & \text{if } -B_n \leqslant \tilde{m}_n(x) \leqslant B_n, \\ 0 & \text{if } \tilde{m}_n(x) < -B_n, \end{cases} \tag{12}$$

where $B_n > 0$ and $B_n \to \infty$ $(n \to \infty)$.

In this paper, we study the consistency of our estimator of orthogonal series. for simplicity we will consider the case where $X \in [0; 1]$ a.s. It is easy to modify the definition of our estimator so that we obtain a weakly and strongly universally consistent estimator for the univariate X. To prove the strong consistency of our estimator we need to make somme changes to its definition. Consider $\alpha \in (0; \frac{1}{2})$. Let functions f_j and coefficients $\hat{c}_j$ be as defined in (10) and (11). Write $(\hat{c}_{(1)}; f_{(1)}), \dots, (\hat{c}_{(K)}; f_{(K)})$

switching $(\hat{c_1}, f_1), \dots, (\hat{c_K}, f_K)$ and

$$|\hat{c}_1| \geqslant |\hat{c}_2| \geqslant \cdots\cdots \geqslant |\hat{c}_k| \tag{13}$$

let's define the estimator ${m^3}_n$ as

$$ {m^3}_n = \sum_{j=1}^{\min\{K, n^{\lfloor 1-\alpha \rfloor}\}} \eta_{\delta_n}(\hat{c}_j) f_j \tag{14} $$

This ensures that ${m^3}_n$ and a linear combination of no more than $n^{1-\alpha}$ functions f_j. And as in $\mathbf{E}|f(X) - Y|^2 = \mathbf{E}|m(X) - Y|^2 + \int |f(x) - m(x)|^2 \mu(dx)$ we can show that

$$ {m^3}_n = {m^3}_{n,J^*} \text{ with } J^* \subseteq \{1, \ldots\ldots, K\} \text{ where } J^* \text{ satisfies } |J^*| \leqslant n^{1-\alpha}. $$

finally we combine the notation of the two estimates to obtain as an estimate of $\tilde{m}_n$ the following formulas m_n^3 and m_n whith $T_L \leqslant B_n = \log(n)$. We will also need the following notations

$$ \mathcal{L}_n^* = \mathrm{T}_{T_L}(\mathcal{F}_n).\ \mathcal{F}_n^* = \{g : \exists f \in G_M \circ P_n, g = \mathrm{T}_{[0,T_L]} f\}. $$

2. Results

Theorem 2.1. *Under hypotheses H_1 – H_5, let $M \in \mathbb{N}$ be fixed, and m_n the m estimator defined by 9, 14 , with $T_L \leqslant B_n = \log(n)$ and $\delta_n \leqslant \dfrac{1}{(\log(n)+1)^2}$. Then*

$$ \int_{R^d} |m_n(x) - m(x)|^2 \mu(dx) \underset{n\to\infty}{\to} 0 \ a.s\ . $$

The following lemma will be used to establish our main result.

Lemma 2.2. *We set the quantity $\bar{m}_n(x) = \mathbb{T}_{[0,T_L]}(\tilde{m}_n(x))$ and with equations (2), (3), we have*

$$ \begin{aligned} &\int_{R^d} |\bar{m}_n(x) - m(x)|^2 \mu(dx) \leqslant \\ &\leqslant 2 \sup_{f\in\mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \ E\,|f(X) - Y|^2 \right| + \\ &+ n\delta_n^2 2(M+1)\frac{(log(n)+1)^2}{n} + \inf_{f\in\mathcal{F}_n^*} \int_{R^d} |f(x) - m(x)|^2 \mu(dx). \end{aligned} \tag{15} $$

3. Proofs

We set the quantity $\bar{m}_n(x) = \mathbb{T}_{[0,T_L]}(\tilde{m}_n(x))$. We first show that the theorem is proved

$$ \int_{R^d} |m_n(x) - m(x)|^2 \mu(dx) \underset{n\to\infty}{\to} 0 \text{ a.s }. \Longleftrightarrow \int_{R^d} |\bar{m}_n(x) - m(x)|^2 \mu(dx) \underset{n\to\infty}{\to} 0 \text{ a.s }. $$

Indeed, according to equation (8), we have $|m_n(x) - \bar{m}_n(x)|^2 \leqslant |T_L - M_n|$, which implies that

$$ \int_{\mathbb{R}^d} |m_n(x) - \bar{m}_n(x)|^2 \leqslant (T_L - M_n)^2 \to 0 \text{ a.s.}. $$

Since by H_5 we have $\lim\limits_{n\to+\infty} M_n = T_L$ a.s Kebabi et al [12]. First, we prove the Lemma 2.2, and finally, we prove the theorem.

Proof Lemma 2.2. We start by proving, first we have

$$\begin{aligned}
&\int_{R^d} |\bar{m}_n(x) - m(x)|^2 \mu(dx) = \\
&= \left\{ \mathbf{E}(|\bar{m}_n(X) - Y|^2 |\mathcal{D}_n) - \inf_{f \in \mathcal{F}_n^*} \mathbf{E} |f(X) - Y|^2 \right\} + \\
&+ \left\{ \inf_{f \in \mathcal{F}_n^*} \mathbf{E} |f(X) - Y|^2 - \mathbf{E} |m(X) - Y|^2 \right\}.
\end{aligned}$$

In addition, the regression function satisfies

$$\inf_{f \in \mathcal{F}_n^*} E |f(X) - Y|^2 - E |m(x) - Y|^2 = \inf_{f \in \mathcal{F}_n^*} \int_{R^d} |f(x) - m(x)|^2 \mu(dx). \tag{16}$$

furthermore

$$\begin{aligned}
& E\left(|\bar{m}_n(X) - Y|^2 |\mathcal{D}_n\right) - \inf_{f \in \mathcal{F}_n^*} E |f(X) - Y|^2 = \\
= \;& \sup_{f \in \mathcal{F}_n^*} \left\{ E\left(|\bar{m}_n(X) - Y|^2 |\mathcal{D}_n\right) - E\left(|f(X) - Y|^2 |\mathcal{D}_n\right) \right\} = \\
= \;& \sup_{f \in \mathcal{F}_n^*} \left\{ E\left(|\bar{m}_n(X) - Y|^2 |\mathcal{D}_n\right) - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\bar{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} + \right. \\
& + \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\bar{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} + \\
& + \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} + \\
& \left. + \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - E |f(X) - Y|^2 \right\} \leqslant \sum_{i=1}^{4} Q_{n,i},
\end{aligned}$$

where the $Q_{n,i}$ are explained below for all i, $1 \leqslant i \leqslant 4$.

- Since $\tilde{m} \in \mathcal{F}_n$,$\bar{m}_n \in \mathcal{F}_n^*$ and $\mathcal{F}_n^* \subset \mathcal{L}_n^*$, it is obvious that

$$\begin{aligned}
Q_{n,1} \;&=\; \sup_{f \in \mathcal{F}_n^*} \left\{ E\left(|\bar{m}_n(X) - Y|^2 |\mathcal{D}_n\right) - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\bar{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} \right\} \leqslant \\
&\leqslant\; \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - E |f(X) - Y|^2 \right|,
\end{aligned}$$

  and

$$\begin{aligned}
Q_{n,4} \;&=\; \sup_{f \in \mathcal{F}_n^*} \left\{ \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - E |f(X) - Y|^2 \right| \right\} \leqslant \\
&\leqslant\; \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - E |f(X) - Y|^2 \right|.
\end{aligned}$$

- Since $\bar{m}_n(X_i) \leqslant T_L$ and $Z_i \leqslant T_L$ a.s., we obtain $1_{\{A_i=0\}}|\tilde{m}_n(X_i) - Z_i| \geqslant 1_{\{A_i=0\}}|\bar{m}_n(X_i) - Z_i|$, which implies

$$Q_{n,2} = \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\bar{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} \leqslant 0$$

- As $\mathcal{F}_n^* \subset \mathcal{F}_n^{**}$ because of $T_L \leqslant \log(n)$ and fix $f \in G_M \circ P_n$. In view of P_n definition, Lemma 18.1 in Györfi et al [7] exist $\bar{J} \subset \{1, \ldots, n\}$ and $\bar{f} \in \mathcal{F}_{n,\bar{J}}$, such that $f(X_i) = \bar{f}(X_i)$ and $|\bar{J}| \leqslant 2(M+1)(log(n)+1)^2$ which implies that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} = \\
&= \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\bar{f}(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} \leqslant \\
&\leqslant n\delta_n^2 2(M+1)\frac{(log(n)+1)^2}{n}.
\end{aligned}$$

From $\tilde{m}$ definition, it is obvious that

$$\begin{aligned}
Q_{n,3} \quad &= \sup_{f \in \mathcal{F}_n^*} \left\{ \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|\tilde{m}_n(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} \right\} \leqslant \\
&\leqslant n\delta_n^2 2(M+1)\frac{(log(n)+1)^2}{n}.
\end{aligned}$$

Inequality (15) is therefore proven. □

Proof Theorem 2.1. It remains to be proven that the three terms of Lemma 2.2 tend to zero almost surely when $n \to \infty$. To do this, we will proceed in three steps. In the first step, we show that

$$\lim_{n\to\infty} \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \ E\,|f(X) - Y|^2 \right| = 0 \text{ a.s.}$$

To do this, we use the following inequalities

$$\begin{aligned}
&\sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \ E\,|f(X) - Y|^2 \right| \leqslant \\
&\leqslant \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i)\hat{F}_n(Z_i)} \right| + \\
&+ \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i)F_L(Z_i)} \right| + \\
&+ \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i)F_L(Z_i)} - E\,|f(X) - Y|^2 \right| \leqslant \sum_{i=1}^{3} Q_{n,i}^*.
\end{aligned}$$

Since $f \in \mathcal{L}_n^*$ implies that $0 \leqslant f(x) \leqslant T_L$, we get – in view of – formulas (4)–(6)

$$Q_{n,1}^* = \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{\hat{S}_n(Z_i)\hat{F}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n} 1_{\{A_i=0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i)\hat{F}_n(Z_i)} \right| \leqslant$$

$$\leqslant \frac{T_L^2}{\hat{S}_n(T) S_R(T) \hat{F}_n(I)} \sup_{t \in \mathbb{R}^+} \left| \hat{S}_n(t) - S_R(t) \right| \underset{n \to \infty}{\to} 0, \text{ a.s.}$$

and

$$Q_{n,2}^* = \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n} \sum_{i=1}^n 1_{\{A_i = 0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i) \hat{F}_n(Z_i)} - \frac{1}{n} \sum_{i=1}^n 1_{\{A_i = 0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i) F_L(Z_i)} \right| \leqslant$$

$$\leqslant \frac{T_L^2}{F_L(I) S_R(T) \hat{F}_n(I)} \sup_{t \in \mathbb{R}^+} \left| \hat{F}_n(t) - F_L(t) \right| \underset{n \to \infty}{\to} 0 \text{ a.s.}$$

Let's introduce the following notations $V = (X, Z, 1_A), V_1 = (X_1, Z_1, 1_{A_1}), \ldots, V_n = (X_n, Z_n, 1_{A_n})$ n i.i.d random vectors with the same distribution as V.

Define

$$\mathcal{H}_n = \Big\{ h : \mathbb{R}^d \times [0, T_L] \times \{0,1\} \to \mathbb{R}^+ : \exists f \in \mathcal{L}_n^* \text{ such as,}$$
$$h(x, z, 1_A) = \frac{1_A |f(x) - z|^2}{S_R(z) F_L(z)}$$
$$\forall (x, z, 1_A) \in \mathbb{R}^d \times [0, T_L] \times \{0,1\} \Big\}.$$

Functions of $\mathcal{H}_n$ are positive and bounded by $\dfrac{T_L^2}{S_R(T) F_L(I)}$, and

$$\mathbf{E} h(V) = \mathbf{E} \left(\frac{1_A |f(X) - Z|^2}{S_R(Z) F_L(Z)} \right) = \mathbf{E} \left[\mathbf{E} \left(\frac{1_A |f(X) - Z|^2}{S_R(Z) F_L(Z)} \mid X, Y \right) \right] = \mathbf{E} \left(|f(X) - Z|^2 \right).$$

under H_1,H_2 et H_4. In addition we have

$$Q_{n,3}^* = \sup_{f \in \mathcal{L}_n^*} \left| \frac{1}{n} \sum_{i=1}^n 1_{\{A_i = 0\}} \frac{|f(X_i) - Z_i|^2}{S_R(Z_i) F_L(Z_i)} - E |f(X) - Y|^2 \right| =$$
$$= \sup_{f \in \mathcal{H}_n} \left| \frac{1}{n} \sum_{i=1}^n h(V) - \mathbf{E} h(V) \right|.$$

For all h_1 and $h_2 \in \mathcal{H}_n$, let f_1 and f_2 be their corresponding functions in $\mathcal{L}_n^*$ then

$$\frac{1}{n} \sum_{i=1}^n |h_1(V_i) - h_2(V_i)|$$
$$= \frac{1}{n} \sum_{i=1}^n \left| 1_{\{A_i = 0\}} \frac{|f_1(X_i) - Z_i|^2}{S_R(Z_i) F_L(Z_i)} - 1_{\{A_i = 0\}} \frac{|f_2(X_i) - Z_i|^2}{S_R(Z_i) F_L(Z_i)} \right| \leqslant$$
$$\leqslant \frac{1}{S_R(T) F_L(I)} \frac{1}{n} \sum_{i=1}^n |(f_1(X_i) + f_2(X_i) - 2 Z_i)(f_1(X_i) - f_2(X_i))| \leqslant$$
$$\leqslant \frac{2 T_L}{S_R(T) F_L(I)} \frac{1}{n} \sum_{i=1}^n |f_1(X_i) - f_2(X_i)|,$$

which implies $\mathcal{N}(\varepsilon, \mathcal{H}_n, V_1^n) \leqslant \mathcal{N}\left(\varepsilon \frac{S_R(T) F_L(I)}{2 T_L}, \mathcal{L}_n^*, X_1^n\right)$, where $\mathcal{N}(\varepsilon, \mathcal{F}_n, Z_1^n)$ denotes the overlapping number. Theorem 9.1 in Györfi et al [7] gives, for all $\delta > 0$

$$p\left\{ \sup_{f \in \mathcal{H}_n} \left| \frac{1}{n} \sum_{i=1}^n h(V_i) - \mathbf{E} h(V) \right| > \delta \right\} \leqslant 8\, E \left\{ \mathcal{N}\left(\delta \frac{S_R(T) F_L(I)}{16 T_L}, \mathcal{L}_n^*, X_1^n \right) \right\} \exp\left(- \frac{n \delta^2 S_R^2(T) F_L^2(I)}{128 T_L^4} \right),$$

which is, in view of Theorem 9.4, Theorem 9.5 and Lemma 13.1 in Györfi et al [7], we get

$$p\left\{\sup_{f\in\mathcal{H}_n}\left|\frac{1}{n}\sum_{i=1}^{n}h(V_i)-\mathbf{E}h(V)\right|>\delta\right\}\leqslant$$

$$\leqslant \quad 8(5n)^{4n^{1-\alpha}}\left(-\frac{288eT_L^2}{\delta\left(S_R(T)F_L(I)\right)^4}\right)^{2(M+2)n^{1-\alpha}}\exp\left(-\frac{n\delta^2S_R^2(T)F_L^2(I)}{128T_L^4}\right).$$

The formula combined with the $V_{\mathrm{T}_{\log n}G_M^+}\leqslant V_{G_M^+}$ of the theorem where $V_{\mathrm{T}_{\log n}G_M^+}$ stands for the VC dimension of the set of graphs of function in G_M , allows to apply Borel Cantelli lemma, to get

$$\sup_{f\in\mathcal{L}_n^*}\left|\frac{1}{n}\sum_{i=1}^{n}1_{\{A_i=0\}}\frac{|f(X_i)-Z_i|^2}{S_R(Z_i)F_L(Z_i)}-\mathbf{E}\left|f(X)-Y\right|^2\right|\underset{n\to\infty}{\to}0 \text{ a.s.}$$

In the second step, we get

$$n\delta_n^2 2(M+1)\frac{(log(n)+1)^2}{n}\underset{n\to\infty}{\to}0 \text{ a.s because } \delta_n\leqslant\frac{1}{(\log(n)+1)^2}.$$

In the third step, we prove that

$$\inf_{f\in\mathcal{F}_n^*}\int_{R^d}\left|f(x)-m(x)\right|^2\mu(dx)\underset{n\to\infty}{\to}0 \text{ a.s.}$$

Since m can be approximated arbitrarily closely by continuously differentiable functions, we may assume without loss of generality that m is continuously differentiable.For each $A\in P_n$ choose some $x_A\in A$ and set $f^*=\sum_{A\in P_n}m(x_A)I_A$.Then $f^*\in G_M\circ P_n$ and for n such that $\|m\|_\infty\ \leqslant T_L\leqslant\log(n)$ we get

$$\inf_{\forall f\in G_M\circ P_n,\|f\|_\infty\ \leqslant T_L}\int_{R^d}\left|f(x)-m(x)\right|^2\mu(dx)\leqslant\sup_{x\in[0,1]}\left|f^*(X)-m(x)\right|^2\leqslant\frac{c}{(\log(n))^2}\underset{n\to\infty}{\to}0.$$

where c is constant as a function of the first derivative of m. □

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Неполная оценка функции регрессии методом наименьших квадратов на основе вейвлетов

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Аннотация. В этой статье мы вводим оценку функции регрессии методом наименьших квадратов для Y, цензурированного справа R, и $\min(Y, R)$, цензурированного слева L. Он основан на идеях, полученных из контекста вейвлет-оценок, и построен путем жесткой пороговой обработки оценок коэффициентов развития ряда функции регрессии. Устанавливаем сходимость по норме L_2. Мы даем достаточно критериев для непротиворечивости этой оценки. Результат показывает, что наша оценка способна адаптироваться к локальной регулярности соответствующей функции регрессии и распределения.

Ключевые слова: непараметрическая регрессия, ошибка L_2, оценки методом наименьших квадратов, оценки ортогональными рядами, сходимость в норме L_2, дважды цензурированные данные, оценка регрессии, жесткий порог.

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Theoretical Analysis for a System of Nonlinear ϕ-Hilfer Fractional Volterra-Fredholm Integro-differential Equations

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Abstract. We investigate the existence of solutions for a system of nonlinear ϕ-Hilfer fractional Volterra–Fredholm integro-differential equations with fractional integral conditions, by using the Krasnoselskii's fixed point theorem and Arzela–Ascoli theorem. Moreover, applying an alternative fixed point theorem due to Diaz and Margolis, we prove the Kummer stability of the system on the compact domains. An example is also presented to illustrate our results.

Keywords: ϕ-Hilfer fractional Volterra-Fredholm integro-differential equation, Kummer's stability, Arzela–Ascoli theorem, Krasnoselskii fixed point theorem.

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Introduction

Fractional order differential equations have become one of the most popular areas of research in mathematical analysis, engineering, economics, control theory, materials sciences, physics, chemistry, and biology (see [1, 2] and the references therein). Scientists have applied various mathematical approaches through diverse research-oriented aspects of fractional differential systems. For instance, existence, stability, and control theory for fractional differential equations were studied [3, 4]. For the first time, Alsina and Ger [5] studied the Hyers-Ulam stability for differential equations. Recently, mathematicians have paid more attention to the study of stability for a wide range of differential systems [6–9]. Volterra integro-differential equations which are an important class of these equations have arise widely in the mathematical modelling of many physical and biological processes, for example biological species coexisting together with increasing and decreasing rate of growth, electromagnetic theory, Wilson-Cowan model and many

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more [10,11]. Although they have considerably been studied in science and engineering, fractional integro-differential equations with mixed fractional operators have been newly introduced [12–15].

In this article, motivated by the research going on in this direction, we study a new system for nonlinear ϕ-Hilfer fractional Volterra-Fredholm integro-differential equations with fractional integral conditions of the form

$$\begin{cases} {}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^t \mathfrak{h}(\eta, s, w(s))ds \ + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, w(s))ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \eta \in \omega := (0, \mathfrak{p}], \\ I_0^{1-\gamma;\phi}w(0) = \mathfrak{w}_0, \quad \mathfrak{w}_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}(.)$ is a ϕ-Hilfer fractional derivative of order $0 < \varsigma \leqslant 1$ and type $0 \leqslant v < 1$, and $I_0^{1-\gamma;\phi}$ is a ϕ-Riemann-Liouville fractional integral of order $1-\gamma(\gamma = \varsigma + v(1-\varsigma))$ with respect to the mapping ϕ. Furthermore, $\mathfrak{g} : \omega \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{h}, \mathfrak{k} : \omega^2 \times \mathbb{R} \to \mathbb{R}$ are given mappings, and $\mathcal{A}$ is a closed linear operator. In the following, we show the existence of solutions to equation (1) based on the Krasnoselskii fixed point theorem and Arzela–Ascoli theorem. Using Kummer's control function, we introduce a new concept of stability and further deduce that the solution of equation (1) is stable in Kummer's sense.

1. Preliminaries

In this section, we present some important definitions and mathematical concepts on the fractional calculus. For details, please see [2, 16] and the references therein. Let $[\mathrm{n}, \mathrm{m}]$ be a finite and closed interval with $0 \leqslant \mathrm{n} < \mathrm{m} < \infty$ and $\mathfrak{C}[\mathrm{n}, \mathrm{m}]$ be the space of continuous functions $\varrho : [\mathrm{n}, \mathrm{m}] \to \mathbb{R}$ equipped with the following norm

$$\|\varrho\|_{\mathfrak{C}[\mathrm{n},\mathrm{m}]} = \max_{\eta \in [\mathrm{n},\mathrm{m}]} |\varrho(\eta)|.$$

Furthermore, the weighted space $\mathfrak{C}_{\gamma,\phi}(\mathrm{n}, \mathrm{m}]$ is defined as

$$\mathfrak{C}_{1-\gamma,\phi}[\mathrm{n}, \mathrm{m}] = \left\{\varrho : (\mathrm{n}, \mathrm{m}] \to \mathbb{R}; (\phi(\eta) - \phi(\mathrm{n}))^{1-\gamma}\varrho(\eta) \in \mathfrak{C}[\mathrm{n}, \mathrm{m}]\right\} \quad \text{where} \quad 0 < \gamma < 1.$$

with norm

$$\|\varrho\|_{C_{1-\gamma,\phi}[\mathrm{n},\mathrm{m}]} = \max_{\eta \in [\mathrm{n},\mathrm{m}]} \left|(\phi(\eta) - \phi(\mathrm{n}))^{1-\gamma}\varrho(\eta)\right|,$$

where $\phi : [\mathrm{n}, \mathrm{m}] \longrightarrow \mathbb{R}$ is an arbitrary function, and $\eta \in [\mathrm{n}, \mathrm{m}]$.

Definition 1.1. *Let* $(\mathrm{n}, \mathrm{m}), -\infty \leqslant \mathrm{n} < \mathrm{m} \leqslant +\infty$ *be a finite or infinite interval of the line* $\mathbb{R}$, Γ *be the gamma function, and* $\varsigma > 0$. *Additionally, let* $\phi(\eta)$ *be a positive function defined on* $[\mathrm{n}, \mathrm{m}]$ *so that* $\phi'(\eta) \geqslant 0$ *on* $(\mathrm{n}, \mathrm{m}]$ *and* $\phi'(\eta)$ *is a continuous function on* (n, m). *The left-and right-sided fractional integrals of a function* ϱ *with respect to the function* ϕ *on* $[\mathrm{n}, \mathrm{m}]$ *are defined by*

$$I_{\mathrm{n}^+}^{\varsigma;\phi}\varrho(x) = \frac{1}{\Gamma(\varsigma)} \int_{\mathrm{n}}^{x} \phi'(\eta)(\phi(x) - \phi(\eta))^{\varsigma-1}\varrho(\eta)d\eta$$

and

$$I_{\mathrm{m}^-}^{\varsigma;\phi}\varrho(x) = \frac{1}{\Gamma(\varsigma)} \int_{x}^{\mathrm{m}} \phi'(\eta)(\phi(\eta) - \phi(x))^{\varsigma-1}\varrho(t)d\eta$$

respectively.

The fractional integrals with the above definition have a semi-group property given by

$$I_{\mathrm{n}^+}^{\varsigma;\phi} I_{\mathrm{n}^+}^{v;\phi} \varrho(x) = I_{\mathrm{n}^+}^{\varsigma+v;\phi} \varrho(x) \quad \text{and} \quad I_{\mathrm{m}^-}^{\varsigma;\phi} I_{\mathrm{m}^-}^{v;\phi} \varrho(x) = I_{\mathrm{m}^-}^{\varsigma+v;\phi} \varrho(x).$$

Additionally, for $\varsigma, \sigma > 0$, we have [17]:

(i) if $\varrho(x) = (\phi(x) - \phi(\mathrm{n}))^{\sigma-1}$ then $I_{\mathrm{n}^+}^{\varsigma;\phi} \varrho(x) = \dfrac{\Gamma(\sigma)}{\Gamma(\varsigma+\sigma)} (\phi(x) - \phi(\mathrm{n}))^{\varsigma+\sigma-1}$, and

(ii) if $\varrho(x) = (\phi(\mathrm{m}) - \phi(x))^{\sigma-1}$ then $I_{\mathrm{m}^-}^{\varsigma;\phi} \varrho(x) = \dfrac{\Gamma(\sigma)}{\Gamma(\varsigma+\delta)} (\phi(\mathrm{m}) - \phi(x))^{\varsigma+\sigma-1}$.

Definition 1.2. *Let $(\mathrm{n}, \mathrm{m}), -\infty \leqslant \mathrm{n} < \mathrm{m} \leqslant +\infty$ be a finite or infinite interval of the line $\mathbb{R}$, $\phi'(\eta) \neq 0$ for all $\eta \in (\mathrm{n}, \mathrm{m})$, and $\varsigma > 0$, $n \in \mathbb{N}$. The left-sided Riemann–Liouville derivative of a function ϱ with respect to ϕ of order ς correspondent to the Riemann–Liouville is defined by*

$$\begin{aligned} D^{\varsigma,\phi} \varrho(\eta) &= \left(\frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n I^{n-\varsigma,\phi} \varrho(\eta) \\ &= \frac{1}{\Gamma(n-\varsigma)} \left(\frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n \times \int_{\mathrm{n}}^{\eta} \phi'(t)(\phi(\eta) - \phi(t))^{n-\varsigma-1} \varrho(t) dt. \end{aligned}$$

Definition 1.3. *Let $n - 1 < \varsigma < n$ with $n \in \mathbb{N}$, $I = [\mathfrak{n}, \mathfrak{m}] (-\infty \leqslant \mathfrak{n} < \mathfrak{m} \leqslant \infty)$ and $\varrho, \phi \in \mathfrak{C}^n([\mathfrak{n}, \mathfrak{m}], \mathbb{R})$ be two mappings such that $\phi'(x) > 0$ for all $x \in I$. The left-and right-sided ϕ-Hilfer fractional derivatives $\mathcal{H}\mathbb{D}_{0^+}^{\varsigma,v;\phi}(.)$ of the arbitrary function ϱ of order ς and type $0 \leqslant v < 1$ are defined by*

$$^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} \varrho(x) = I_{\mathfrak{n}^+}^{v(n-\varsigma);\phi} \left(\frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{\mathfrak{n}^+}^{(1-v)(n-\varsigma);\phi} \varrho(x)$$

and

$$^{\mathcal{H}}\mathbb{D}_{\mathfrak{m}^-}^{\varsigma,v;\phi} \varrho(x) = I_{\mathfrak{m}^-}^{v(n-\varsigma);\phi} \left(-\frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{\mathfrak{m}^-}^{(1-v)(n-\varsigma);\phi} \varrho(x)$$

respectively.

Theorem 1.4. *If $\varrho \in \mathfrak{C}^1[\mathfrak{n}, \mathfrak{m}]$, $\varsigma > 0$, $0 \leqslant v < 1$, and $\gamma = \varsigma + v(1-\varsigma)$ then*

$$^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} I_{\mathfrak{n}^+}^{\varsigma;\phi} \varrho(x) = \varrho(x) \quad \text{and} \quad ^{\mathcal{H}}\mathbb{D}_{\mathfrak{m}^-}^{\varsigma,v;\phi} I_{\mathfrak{m}^-}^{\varsigma;\phi} \varrho(x) = \varrho(x).$$

Additionally, we have

$$I_{\mathfrak{n}^+}^{\varsigma;\phi} \ ^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} \varrho(x) = \varrho(x) - \frac{(\phi(x) - \phi(\mathfrak{n}))^{\gamma-1}}{\Gamma(\gamma)} I_{\mathfrak{n}^+}^{(1-v)(1-\varsigma);\phi} \varrho(\mathfrak{n})$$

and

$$I_{\mathrm{m}^-}^{\varsigma;\phi} \ ^{\mathcal{H}}\mathbb{D}_{\mathrm{m}^-}^{\varsigma,v;\phi} \varrho(x) = \varrho(x) - \frac{(\phi(\mathfrak{m}) - \phi(x))^{\gamma-1}}{\Gamma(\gamma)} I_{\mathrm{m}^-}^{(1-v)(1-\varsigma);\phi} \varrho(\mathfrak{m}).$$

Proof. Ref. [17].

The solution of a hypergeometric differential equation is called a confluent hypergeometric function [18]. There exist different standard forms of confluent hypergeometric functions, such as Kummer's functions, Tricomi's functions, Whittaker's functions, and Coulomb's wave functions. In this paper, we apply the following Kummer (confluent hypergeometric) function to study our stability:

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_2)}{\Gamma(\mathfrak{P}_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_2 + k)} \frac{\mathfrak{z}^k}{k!} \tag{2}$$

which is the solution of the differential equation

$$\mathfrak{z}\frac{d^2\mathfrak{u}}{dz} + (\mathfrak{P}_2 - \mathfrak{z})\frac{d\mathfrak{u}}{d\mathfrak{z}} - \mathfrak{P}_1\mathfrak{u}(\mathfrak{z}) = 0,$$

where $\mathfrak{z}, \mathfrak{P}_1 \in \mathbb{C}$ and $\mathfrak{P}_2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Kummer's function was introduced by Kummer in 1837. The series (2) is also known as the confluent hyper-geometric function of the first kind, and is convergent for any $\mathfrak{z} \in \mathbb{C}$. In this article, we apply it on the real line $\mathbb{R}$ as our control function. Clearly, for $\mathfrak{P}_1 = \mathfrak{P}_2$, we have

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_1; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_1)}{\Gamma(\mathfrak{P}_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_1 + k)}\frac{\mathfrak{z}^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{k!} = e^{\mathfrak{z}}.$$

Letting $\varsigma, v \in \omega$, we consider the following inequality for $\epsilon > 0$

$$\begin{aligned}&\left| {}^{\mathcal{H}}\mathbb{D}_{0+}^{\varsigma,v;\phi}v(\eta) - \mathcal{A}(v(\eta)) + \mathfrak{g}(\eta, v(\eta)) - \int_0^t \mathfrak{h}(\eta, s, v(s))ds - \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, v(s))ds \right| \leqslant \\ &\leqslant \epsilon\Phi(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma})\end{aligned} \tag{3}$$

where Φ is the Kummer's function (see [18]), to define a new stability concept called Kummer's stability.

Definition 1.5. *For a positive constant C_Φ, for all $\epsilon > 0$, and every solution $v \in (\mathfrak{C}[0, \mathfrak{p}], \mathbb{R})$ to inequality* (3)*, if we can find a solution $w \in (\mathfrak{C}[0, \mathfrak{p}], \mathbb{R})$ to Equation* (1)*, with the following property:*

$$|w(\eta) - v(\eta)| \leqslant C_\Phi \in \Phi(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}) \text{ for all } \eta \in [0, \mathfrak{p}]$$

then we say that equation (1) *has Kummer's stability with respect to $\Phi(\varsigma, v, (\phi(\eta) - \phi(0))^{\varsigma})$.*

Our approach is motivated by the fact that inversion of a perturbed differential operator may result from the sum of a compact operator and a contraction mapping (see [19–21] and the references therein). We begin by stating the following Krasnoselskii fixed point theorem, which has many applications in studying the existence of solutions to differential equations:

Theorem 1.6 (Krasnoselskii fixed point theorem)**.** *Let X be a Banach space and $\mathfrak{M} \subseteq X$ be a closed, convex, and non-empty set. Additionally, let $\mathfrak{T}, \mathfrak{S}$ be mappings so that:*

- $\mathfrak{T}u + \mathfrak{S}v \in \mathfrak{M}$ *whenever* $u, v \in \mathfrak{M}$,
- *The operator $\mathfrak{T}$ is continuous and compact, and*
- *Mapping $\mathfrak{T}$ is a contraction.*
  Then, there exists a $w \in \mathfrak{M}$ so that $w = \mathfrak{T}w + \mathfrak{S}w$.

In addition, we mention an alternative fixed point theorem presented by Diaz and Margolis in 1967, and it plays a crucial role in proving our stability result [22].

Theorem 1.7. *Consider the generalized complete metric space $(\mathfrak{X}, \mathrm{Y})$ and let Θ be a self-map operator which is a strictly contraction mapping with the Lipschitz constant $\kappa < 1$. Then, we have two options:*
(i) either for every $\mathrm{n} \in \mathbb{N}, \mathrm{Y}\left(\Theta^{\mathrm{n}+1}\mathfrak{z}, \Theta^{\mathrm{n}}\mathfrak{z}\right) = +\infty$; or
(ii) if there exists $\mathrm{n} \in \mathbb{N}$ so that the operator Θ satisfies $Y\left(\Theta^{n+1}\mathfrak{z}, \Theta^n\mathfrak{z}\right) < \infty$ for some $\mathfrak{z} \in \mathfrak{X}$, then the sequence $\{\Theta^n\mathfrak{z}\}$ tends to a unique fixed point $\mathfrak{z}^*$ of Θ in the set $\mathfrak{X}^* = \{\mathfrak{v} \in \mathfrak{X} : Y(\Theta^n\mathfrak{v}, \Theta^n\mathfrak{v}) < \infty\}$. Furthermore, for all $\hat{z} \in \mathfrak{X}$

$$Y(\mathfrak{z}, \mathfrak{z}^*) \leqslant \frac{1}{1-\kappa}Y(\mathfrak{z}, \Theta_{\mathfrak{z}}).$$

Now, we are ready to prove that equation (1) is equivalent to an integral equation. Then, by the above theorem, we infer that a fixed point exists for the integral equation, so equation (1) has at least one solution.

Proposition 1.8. *Assume that $\mathfrak{g} : \omega \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{h} : \omega^2 \times \mathbb{R} \to \mathbb{R}$ are real-valued continuous mappings, and $\mathcal{A}$ is a closed operator, then the following integral equation is equivalent to equation* (1)*:*

$$\begin{aligned} w(\eta) &= \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi}\Big[\mathfrak{g}(\eta, w(\eta), w(\eta)) + \int_0^s \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \\ &\quad + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,u(\tau))d\tau + \mathcal{A}(w(\eta))\Big] \end{aligned} \tag{4}$$

where $\gamma \geqslant 0$ and we obtain from $\gamma = \varsigma + v(1-\varsigma)$ for $0 < \varsigma \leqslant 1$ and $0 \leqslant v < 1$ in (1)*.*

Proof. Using the properties of the ϕ-Hilfer fractional derivative outlined in the preliminaries, we have where $\gamma = \varsigma + v(1-\varsigma)$. So, by the above equality, we have

$${}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\boldsymbol{w}(\eta) = I^{v(n-\varsigma);\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta) = I^{\gamma-\varsigma;\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta),$$

where $\gamma = \varsigma + v(1-\varsigma)$. So, by the above equality, we have

$$I^{\gamma-\varsigma;\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(\eta,\mathrm{s},w(s))ds + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\mathrm{s},w(s))ds.$$

Now, applying $I^{\gamma;\phi}$ to both sides of the above equation and using Theorem 1.4 , we obtain

$$I^{\varsigma,\phi}I^{\gamma-\varsigma;\phi}D^{\Upsilon\phi}w(\eta) = I^{\varsigma;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^s \mathfrak{h}(\eta,\tau,w(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right)$$

and

$$I^{\Upsilon\phi}D^{\gamma\phi}w(\eta) = I^{\varsigma;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^s h(\eta,\tau,w(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right).$$

Then,

$$w(\eta) = \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I^{\varsigma;\phi}\Big(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^{\mathrm{s}} \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\Big).$$

Conversely, assuming that $w \in \mathcal{C}[0,p]$ satisfies equation (4), we claim that the fractional integro-differential equation (1) holds. We apply ${}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}$ to the equation (4) and imply by Theorem 1.4 that

$$\begin{aligned} &{}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}w(\eta) = \\ &= {}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\Big(\frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I^{\varsigma;\phi}\mathcal{A}(w(\eta)) + I^{\varsigma;\phi}\mathfrak{g}(\eta,w(\eta)) + I^{\varsigma;\phi}\int_0^{\mathrm{s}} \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \\ &\qquad + I^{\varsigma;\phi}\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\Big). \end{aligned}$$

From ${}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\mathfrak{w}_0 = 0$, we obtain

$${}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^\eta \mathfrak{h}(t,s,u(s))ds + \int_0^{\mathfrak{p}} \mathfrak{k}(t,s,u(s))ds.$$

This completes the proof. □

Remark 1. Let $\mathfrak{w} \in \mathfrak{C}(\omega, \mathbb{R})$ satisfy inequality (3). Then the following integral inequality holds

$$\Big|\mathfrak{w}(t) - \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 -$$

$$-I_{0+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta))\Big) - I_{0+}^{\varsigma;\phi}\left[\int_0^s \mathfrak{h}(\eta,\tau,w(\tau))d\tau\right] - I_{0+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right] - I_{0+}^{\varsigma;\phi}(\mathcal{A}(w(\eta)))\Big| \leqslant$$

$$\leqslant \frac{\epsilon}{\Gamma(\varsigma)}\int_0^\eta \phi'(t)(\phi(x)-\phi(\eta))^{\varsigma-1}\Phi\left(\varsigma, v; (\phi(\eta)-\phi(0))^{\varsigma}\right) ds =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\int_0^\eta \phi'(\eta)(\phi(x)-\phi(\eta))^{\varsigma-1}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{(\phi(\eta)-\phi(0))^{t\varsigma}}{t!}ds =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^\eta \phi'(t)(\phi(x)-\phi(\eta))^{\varsigma-1}(\phi(\eta)-\phi(0))^{t\varsigma}ds =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^\eta (\phi(x)-\phi(\eta))^{\varsigma-1}(\phi(\eta)-\phi(0))^{t\varsigma}d\phi(s) =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^{\phi(\eta)-\phi(0)} (\phi(\eta)-\phi(0)-w)^{\varsigma-1}w^{t\varsigma}dw =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma-1}\int_0^{\phi(\eta)-\phi(0)}\left(1-\frac{w}{\phi(\eta)-\phi(0)}\right)^{\varsigma-1}w^{t\varsigma}dw =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma(t+1)}\int_0^1 (1-v)^{\varsigma-1}v^{t\varsigma}dv =$$

$$= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma(t+1)}\frac{\Gamma(t\varsigma+1)\Gamma(\varsigma)}{\Gamma((t+1)\varsigma+1)} \leqslant$$

$$\leqslant \epsilon\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^{\infty}\frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{(\phi(\eta)-\phi(0))^{\varsigma(t+1)}}{t!} \leqslant$$

$$\leqslant \epsilon\Phi\left(\varsigma, v; (\phi(\eta)-\phi(0))^{\varsigma}\right).$$

2. Existence results

In this section, we study equation (1) under the following hypotheses:

- (H1). $\mathfrak{g} \in \mathbb{C}(\omega \times \mathbb{R}, \mathbb{R})$. Moreover, there exists $\mathfrak{q}_1$ such that

$$|\mathfrak{g}(\eta, w)| \leqslant \mathfrak{q}_1\mathfrak{M}_1$$

  where $\eta \in \omega, w \in \mathbb{C}([0,\mathfrak{p}], \mathbb{R})$ and $\mathfrak{M}_1 = \|w\|_{\mathfrak{C}[0,p]}$.

- (H2). There exist $\mathfrak{q}_2^h, \mathfrak{q}_2^k > 0$ such that $|\mathfrak{h}(\eta, s, w)| \leqslant \mathfrak{q}_2^h|w(\eta)|$, $|\mathfrak{k}(\eta, s, w)| \leqslant \mathfrak{q}_2^k|w(\eta)|$ for all $\eta \in \omega$ and $w \in \mathfrak{C}([0,p], \mathbb{R})$.

- (H3). The operator A is bounded and $\|\mathcal{A}\| < \dfrac{\Gamma(\varsigma+1)}{\Gamma(\gamma)(\phi(p)-\phi(0))^{\gamma}}$.

- (H4). The function $\phi(\eta)$ is uniformly continuous for all $\eta \in \omega$.

Lemma 2.1. *Let the operator $\mathcal{T} : \mathbb{C}[0,\mathfrak{p}] \to \mathcal{C}[0,\mathfrak{p}]$ given as*

$$\begin{aligned}(\mathcal{T}w)(\eta) =& \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta)) + \\ &+ I_{0^+}^{\varsigma,\phi}\left[\int_0^s \mathrm{h}(\eta,\tau,w(\tau))d\tau\right] + I_{0^+}^{\varsigma,\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right] + I_{0^+}^{\varsigma;\phi}(\mathcal{A}(w(\eta)))\end{aligned}$$

and assume that the hypotheses (H1)–(H3) *are satisfied. Then, the operator $\mathcal{T}$ maps from the closed ball $\mathfrak{B}_\tau = \{w \in \mathcal{C}([0,\mathfrak{p}] : \|w\| \leqslant \tau\}$ into itself, if*

$$\tau \geqslant \frac{\Gamma(\varsigma+\gamma)\,|r_0|}{\Gamma(\varsigma+\gamma) - \frac{\Gamma(\gamma)\mathfrak{C}_\phi^\varsigma}{\Gamma(\varsigma+\gamma)}\left[\mathfrak{q}_2^h\mathfrak{p} + \mathfrak{q}_2^k\mathfrak{p} + \mathfrak{q}_1\mathfrak{M}_1 + \frac{\Gamma(\varsigma+1)}{\Gamma(\gamma)\mathfrak{C}_\phi^\gamma}\right]} \tag{5}$$

where $\mathfrak{C}_\phi := (\phi(\mathfrak{p}) - \phi(0))$.

Proof. Clearly, we need to prove that if $w(\eta) \in \mathfrak{B}_\tau$ then $(\mathcal{T}w)(\eta) \in \mathfrak{B}_\tau$. For all $\eta \in [0,\mathfrak{p}]$, we have

$$\begin{aligned}|(\mathcal{T}w)(\eta)| \;\leqslant\; & |\mathfrak{w}_0| + \frac{1}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(s)-\phi(0))^{1-\varsigma}} \max_{s\in[0,\eta]} |\,(\phi(s)-\phi(0))^{1-\gamma}\mathfrak{g}(s,w(s)\,|\,ds + \\ &+\frac{1}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(s))^{1-\varsigma}} \max_{s\in[0,\eta]}(\phi(s)-\phi(0))^{1-\gamma} \times \\ &\times \left[\int_0^s |h(\eta,\tau,w(\tau))|d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right] ds + \\ &+\frac{\|\mathcal{A}\|}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}\max_{s\in[0,\eta]}\left|(\phi(s)-\phi(0))^{1-\gamma}w(s)\right|ds}{(\phi(\eta)-\phi(s))^{1-\varsigma}} \leqslant \\ &\leqslant |\mathfrak{w}_0| + \frac{\mathfrak{q}_1\mathfrak{M}_1\tau}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(s))^{1-\varsigma}}ds + \frac{\mathfrak{q}_2^h\tau\mathfrak{p}}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(s))^{1-\varsigma}}ds + \\ &+\frac{\mathfrak{q}_2^k\tau\mathfrak{p}}{\Gamma(\varsigma)}\int_0^{\mathfrak{p}} \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(s))^{1-\varsigma}}ds + \frac{\|\mathcal{A}\|\tau}{\Gamma(\varsigma)}\int_0^\eta \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}ds}{(\phi(\eta)-\phi(s))^{1-\varsigma}} \leqslant \\ &\leqslant |\mathfrak{w}_0| + \frac{1}{\Gamma(\zeta)}\left[\|\mathcal{A}\|\tau + \mathfrak{q}_2^h\tau\mathfrak{p} + \mathfrak{q}_2^k\tau\mathfrak{p} + \mathfrak{q}_1\mathfrak{N}_1 r\right]\int_0^t \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(s))^{1-\varsigma}}ds \leqslant \\ &\leqslant |\mathfrak{w}_0| + \frac{(\phi(\eta)-\phi(0))^{\varsigma+\gamma-1}}{\Gamma(\varsigma)}\left[\|\mathcal{A}\|^\tau + \mathfrak{q}_2^h\tau\mathfrak{p} + \mathfrak{q}_2^k\tau\mathfrak{p} + \mathfrak{q}_1\mathfrak{M}_1\tau\right]B(\varsigma,\gamma),\end{aligned}$$

where B is the beta function. From the formula

$$B(\varsigma,\gamma) = \frac{\Gamma(\varsigma)\Gamma(\gamma)}{\Gamma(\varsigma+\gamma)}$$

we have

$$|(\mathcal{T}w)(\eta)| \leqslant |\mathfrak{w}_0| + \frac{\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^\varsigma}{\Gamma(\varsigma+\gamma)}\left[\|\mathcal{A}\|\tau + \mathfrak{q}_2^h\tau\mathfrak{p} + \mathfrak{q}_2^k\tau\mathfrak{p} + \mathfrak{q}_1\mathfrak{M}_1\tau\right].$$

Applying $H3$ and condition (5), we have

$$|(\mathcal{T}w)(\eta)| \leqslant |\mathfrak{w}_0| + \frac{\Gamma(\gamma)\mathfrak{C}_\phi^\varsigma}{\Gamma(\varsigma+\gamma)}\left[\mathfrak{q}_2^h\tau\mathfrak{p} + \mathfrak{q}_2^k\tau\mathfrak{p} + \mathfrak{q}_1\mathfrak{M}_1\tau + \frac{\Gamma(\varsigma+1)\tau}{\Gamma(\gamma)\mathfrak{C}_\phi^\gamma}\right] \leqslant \tau.$$

This completes the proof. □

The following theorem shows the existence of solutions to the fractional differential equation (1) using Krasnoselskii's fixed point theorem listed above.

Theorem 2.2. *Assume that hypotheses (H1)–(H4) are satisfied. Then, equation* (1) *has a solution.*

Proof. Define $\Im : \mathcal{C}[0,\mathfrak{p}] \to \mathcal{C}[0,\mathfrak{p}]$ as

$$(\Im w)(\eta) = (\Im_1 w)(\eta) + (\Im_2 w)(\eta)$$

where

$$\begin{aligned}(\Im_1 w)(\eta) \quad := \quad & \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta)) + I_{0^+}^{\varsigma,\phi}\left[\int_0^s \mathrm{h}(\eta,\tau,w(\tau))d\tau\right] + \\ & + I_{0^+}^{\varsigma,\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right]\end{aligned}$$

and

$$(\Im_2 w)(\eta) := I_{0^+}^{\varsigma;\phi}(\mathcal{A}(w(\eta))).$$

From Proposition 1.8, solving Equation (1) is equivalent to finding a fixed point for the operator $\Im$ defined on the space $\mathfrak{C}[0,\mathfrak{p}]$.

Suppose that τ satisfies condition (5) and $\mathfrak{B}_\tau = \{w \in \mathfrak{C}([0,p] : \|w\| \leqslant \tau\}$. Due to Lemma 2.1, the operator $\Im$ maps $\mathfrak{B}_\tau$ into itself. Now, we use Krasnoselskii fixed point theorem to show that $\Im$ has a fixed point. □

Claim 1. The operator $\Im_1$ is continuous on $\mathfrak{B}_\tau$. Let $\{w_n\}$ be a sequence in $\mathfrak{B}_\tau$ that converges to w. We need to prove that $\Im_1 w_n \longrightarrow \Im_1 w$. For each $\eta \in [0,\mathfrak{p}]$, we have

$$\begin{aligned}& \big|(\Im_1 w_n)(\eta) - (\Im_1 w)(\eta)\big| \leqslant \\ \leqslant & \frac{1}{\Gamma(\varsigma)}\int_0^{\eta\eta} \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(0))^{1-\varsigma}}\Big|\max_{s\in[0,\eta]}(\phi(s)-\phi(0))^{1-\gamma} g(s, w_n(s), (w_n(s))) - \\ & - g(s, w(s), (w(s)))\Big|ds + \frac{1}{\Gamma(\varsigma)}\int_0^{\eta} \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(0))^{1-\varsigma}}\max_{s\in[0,\eta]}(\phi(s)-\phi(0))^{1-\gamma}\times \\ & \times\left[\int_0^s \Big|\mathfrak{h}(\eta,\tau,w_n(\tau)) - h(\eta,\tau,w(\tau))\Big|d\tau + \int_0^{\mathfrak{p}}\Big|\mathfrak{k}(\eta,\tau,w_n(\tau)) - \mathfrak{k}(\eta,\tau,w(\tau))\Big|d\tau\right]ds.\end{aligned}$$

Since $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{k}$ are continuous, and $w_n \to w$ as $n \to +\infty$ in $\mathfrak{B}_\tau$, we can conclude that $|(\Im_1 w_n)(\eta) - (\Im_1 w)(\eta)| \to 0$ as $n \to +\infty$ by Lebesgue dominated conoergence theorem.

Claim 2. $\Im_1$ is an equicontinuous operator. To prove our second claim, we let $\eta_1, \eta_2 \in \mathtt{w}$ with $\eta_2 < \eta_1$ and $w \in \mathfrak{B}_\tau$,

$$\begin{aligned}& |(\Im_1 w)(\eta_1) - (\Im_1 w)(\eta_2)| \leqslant \\ & \leqslant \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\gamma}(\phi(s)-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}|\mathfrak{w}_0| + \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_1\mathfrak{M}_1\mathfrak{r}}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds + \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_2^h\mathfrak{r}p}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds + \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_2^k\mathfrak{r}p}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds \leqslant\end{aligned}$$

$$\leqslant \frac{\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\gamma}(\phi(s)-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}\left|\mathfrak{w}_0\right|+$$
$$+\frac{\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\varsigma}}{\Gamma(\varsigma)}\left(\mathfrak{q}_1 \mathfrak{M}_1 \mathfrak{r}+\mathfrak{q}_2^h \mathfrak{r}+\mathfrak{q}_2^k \mathfrak{r}\right) \int_0^{\eta_1} \phi^{\prime}(s)(\phi(s)-\phi(0))^{1-\gamma} d s \leqslant$$
$$\leqslant \frac{\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\gamma}(\phi(\mathfrak{p})-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}\left|r_0\right|+$$
$$+\frac{\Gamma(\gamma)\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\varsigma+\gamma}}{\Gamma(\varsigma+1)}\left(\mathfrak{q}_2^h \mathfrak{r}+\mathfrak{q}_2^k \mathfrak{r}+\mathfrak{q}_1 \mathfrak{M}_1 \mathfrak{r}\right) .$$

Hence, we have

$$\left|\left(\Im_1 w\right)\left(\eta_1\right)-\left(\Im_1 w\right)\left(\eta_2\right)\right| \leqslant$$
$$\leqslant \frac{\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\gamma}}{\Gamma(\gamma)}\left|\mathfrak{w}_0\right|+\frac{\Gamma(\gamma)\left(\phi\left(\eta_1\right)-\phi\left(\eta_2\right)\right)^{1-\varsigma+\gamma}}{\Gamma(\varsigma+1)(\phi(b)-\phi(0))^{1-\gamma}}\left(\mathfrak{q}_2^h \mathfrak{r}+\mathfrak{q}_2^k \mathfrak{r}+\mathfrak{q}_1 \mathfrak{M}_1 \mathfrak{r}\right)$$

regarding (H4), the right-hand side of the above inequality tends to zero whenever $\eta_1 \rightarrow \eta_2$ so it clearly claims that $\Im_1$ is equicontinuous. Furthemnore, using the previous lemma, it is uniformly bounded. Therefore, by Arzela-Ascoli Theorem, $\Im_1$ is compact on $\mathfrak{B}_{\mathrm{r}}$.

Claim 3. The operator $\Im_2$ is a contraction. Let $w_1, w_2 \in \mathbb{C}_{1-\gamma, \varsigma}([0, \mathfrak{p}])$, then, we have

$$\left|\left(\Im_2 w_1\right)(t)-\left(\Im_2 w_2\right)(t)\right| \leqslant$$
$$\leqslant \frac{\|A\|}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi^{\prime}(s)}{(\phi(\eta)-\phi(s))^{1-\varsigma}}\left|w_1(s)-w_2(s)\right| d s$$
$$\leqslant \frac{\|A\| \Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^\gamma}{\Gamma(\varsigma+1)}\left|w_1(\eta)-w_2(\eta)\right| .$$

By (H3), we infer that $\|A\| \Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^\gamma<\Gamma(\varsigma+1)$. Thus, $\Im_2$ is a contraction mapping. By Theorem 1.6, the mapping $\mathcal{T}$ has at least a fixed point, which directly implies that equation (1) has a solution. This completes the proof.

3. Stability analysis

In this section, we present the Kummer stability with respect to $\Phi\left(\varsigma, v ;(\phi(\eta)-\phi(0))^\varsigma\right)$ for equation (1) based on Theorem 1.7. We begin by assuming the following hypotheses:
(K1) $\mathfrak{g} \in \mathbb{C}(\omega \times \mathbb{R}, \mathbb{R})$. Moreover, there exists $\mathcal{L}_{\mathfrak{g}}>0$ such that

$$\left|\mathfrak{g}\left(\eta, \mathfrak{w}_1\right)-\mathfrak{g}\left(\eta, \mathfrak{w}_2\right)\right| \leqslant \mathcal{L}_{\mathfrak{g}}\left|\mathfrak{w}_1-\mathfrak{w}_2\right| \tag{6}$$

for all $\eta \in[0, \mathfrak{p}]$.
(K2) $\mathfrak{h}, \mathfrak{k}: \omega^2 \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions which satisfies a Lipschitz condition in the third argument, i.e., there exist $\mathcal{L}_{\mathfrak{h}}, \mathcal{L}_{\mathfrak{k}}>0$ such that

$$|\mathfrak{h}(\eta, s, \mathfrak{w})-\mathfrak{h}(\eta, s, \mathfrak{v})| \leqslant \mathcal{L}_{\mathfrak{h}}|\mathfrak{w}-\mathfrak{v}|, \tag{7}$$

$$|\mathfrak{k}(\eta, s, \mathfrak{w})-\mathfrak{k}(\eta, s, \mathfrak{v})| \leqslant \mathcal{L}_{\mathfrak{k}}|\mathfrak{w}-\mathfrak{v}| \tag{8}$$

for all $s, \eta \in \omega$ and $\mathfrak{w}, \mathfrak{v} \in \mathbb{R}$.

Theorem 3.1. *Suppose that* $\mathfrak{g}, \mathfrak{h}$ *and* $\mathfrak{k}$ *satisfy* $K1$ *and* $K2$. *Additionally, let*

$$\|\mathcal{A}\| < \frac{\Gamma(\varsigma+1) - (2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}})\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}{\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}. \tag{9}$$

If a continuously differentiable function $w : \omega \to \mathbb{R}$ *for* $\epsilon \geqslant 0$ *satisfies*

$$\left| {}^{\mathcal{H}}\mathbb{D}_{0+}^{\varsigma,v;\phi} w(\eta) - \mathcal{A}(w(\eta)) - \mathfrak{g}(\eta, w(\eta)) - \int_0^{\eta} \mathfrak{h}(\eta, s, w(s))ds - \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, w(s))ds \right| \leqslant$$
$$\leqslant \epsilon \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right)$$

for all $\eta \in \omega$, *then there exists a unique continuous function* $v_0 : \omega \to \mathbb{R}$ *that satisfies equation* (1) *and*

$$|w(\eta) - v_0(\eta)| \leqslant$$
$$\leqslant \frac{\Gamma(\varsigma+1)\epsilon}{\Gamma(\varsigma+1) - (2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathfrak{L}_{\mathfrak{h}} + \mathfrak{p}\mathfrak{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}} \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \tag{10}$$

for all $\eta \in \omega$.

Proof. Let $\mathfrak{Y} := \mathcal{C}_{1-\gamma,\phi}(0, \mathfrak{p}]$ be endowed with the following generalized metric, defined by

$$d^*(\mathfrak{w}, \mathfrak{v}) = \inf\left\{\mathcal{C} \geqslant 0 : |\mathfrak{w}(\eta) - \mathfrak{v}(\eta)| \leqslant \mathcal{C}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \quad \text{for all} \quad \eta \in \omega\right\} \tag{11}$$

for all $\mathfrak{w}, \mathfrak{v} \in \mathfrak{Y}$. It is not difficult to see that $(\mathfrak{Y}, d^*)$ is a complete generalized metric space [6]. Define the operator $\mathcal{S} : \mathfrak{Y} \to \mathfrak{Y}$ by

$$(\mathcal{S}w)(\eta) := \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} + \mathfrak{w}_0 + I_{0+}^{\varsigma;\phi}(\mathcal{A}(w(\eta))) + I_{0+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta)) +$$
$$+ I_{0+}^{\varsigma;\phi}\left[\int_0^s \mathfrak{h}(\eta, \tau, w(\tau))d\tau\right] + I_{0+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta, \tau, w(\tau))d\tau\right]$$

for all $\eta \in \omega$ and $w \in \mathfrak{Y}$. For any $\mathfrak{w}, \mathfrak{v} \in \mathfrak{Y}$), choose a constant $\mathcal{K}$ so that $d^*(\mathfrak{w}, \mathfrak{v}) \leqslant \mathcal{K}$, i.e,

$$|\mathfrak{w}(t) - \mathfrak{v}(t)| \leqslant \mathcal{K}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \tag{12}$$

for all $\eta \in \omega$. So, using Remark 1, we have

$$|(\mathcal{S}\mathfrak{w})(\eta) - (\mathcal{S}\mathfrak{v})(\eta)| \leqslant$$
$$\leqslant \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} |\mathfrak{g}(\eta, \mathfrak{w}(\eta)) - \mathfrak{g}(t, \mathfrak{v}(\eta))| ds +$$
$$+ \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \left[\int_0^s |\mathfrak{h}(\eta, \tau, \mathfrak{w}(\tau)) - \mathfrak{h}(\eta, \tau, v(\tau))| d\tau\right] ds +$$
$$+ \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \left[\int_0^{\mathfrak{p}} |\mathfrak{k}(\eta, \tau, \mathfrak{w}(\tau)) - \mathfrak{k}(\eta, \tau, \mathfrak{v}(\tau))| d\tau\right] ds +$$
$$+ \frac{\|\mathcal{A}\|}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} |\mathfrak{w}(\eta) - \mathfrak{v}(\eta)| \leqslant$$
$$\leqslant \frac{(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\mathcal{K}\epsilon}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \leqslant$$
$$\leqslant \frac{(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}{\Gamma(\zeta+1)} \mathcal{K}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right),$$

which indicates that

$$d^*((\mathcal{S}\mathfrak{w}),(\mathcal{S}\mathfrak{v})) \leqslant \frac{(2\mathcal{L}_{\mathfrak{g}}+\mathfrak{p}\mathcal{L}_{\mathfrak{h}}+\mathfrak{p}\mathcal{L}_{\mathfrak{k}}+\|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^{\varsigma}}{\Gamma(\varsigma+1)} d^*(\mathfrak{w},\mathfrak{v})$$

for all $\mathfrak{w},\mathfrak{v} \in (\mathfrak{Y}, d^*)$. From (8), we have $(2\mathcal{L}_{\mathfrak{g}}+\mathfrak{p}\mathcal{L}_{\mathfrak{h}}+\mathfrak{p}\mathcal{L}_{\mathfrak{k}}+\|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^{\varsigma} < \Gamma(\varsigma+1)$. Hence, the operator $\mathcal{S}$ is a strict contraction. Moreover, for element $\mathfrak{v}_0 \in (\mathfrak{Y}, d^*)$, we have

$$\begin{aligned}
&\left|(\mathcal{S}\mathfrak{v}_0)(\eta)-\mathfrak{v}_0(\eta)\right| \leqslant \\
\leqslant &\left|\mathfrak{v}_0(\eta)-\frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{v}_0 - I_{0^+}^{S\phi\phi}\mathfrak{g}(\eta,\mathfrak{v}_0(\eta)) - I_{0^+}^{S/\phi}\left[\int_0^{\varsigma}\mathfrak{h}(\eta,\tau,v_0(\tau))\,d\tau\right] - I_{0^+}^{S_0}(\mathcal{A}(\mathfrak{v}_0(\eta)))\right| \\
\leqslant &\epsilon\Phi(\alpha,\beta;(\phi(\eta)-\phi(0))^{\alpha})
\end{aligned}$$

for all $\eta \in \omega$. In summary, $d^*(\mathcal{S}\mathfrak{v}_0,\mathfrak{v}_0) \leqslant 1$ and $d^*\left(\mathcal{S}^{\mathrm{n}}\mathfrak{v}_0,\mathcal{S}^{\mathrm{n}+1}\mathfrak{v}_0\right) < +\infty$ for all $\mathrm{n} \in \mathbb{N}$. According to Theorem 1.7, there exists a unique continuous function $\mathfrak{w}:\omega \to \mathbb{R}$ such that $\mathcal{S}\mathfrak{w}=\mathfrak{w},\mathfrak{w}$ satisfies Equation (1) for all $\eta \in \omega$ and

$$\begin{aligned}
\mathfrak{w}(\eta) = &\frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + \\
+I_{0^+}^{\varsigma;\phi}\mathfrak{g}(\eta,\mathfrak{v}_0(\eta)) &+ I_{0^+}^{\varsigma;\phi}\left[\int_0^{s}\mathfrak{h}(\eta,\tau,\mathfrak{v}_0(\tau))\,d\tau\right] + I_{0^+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}}\mathfrak{k}(\eta,\tau,\mathfrak{v}_0(\tau))\,d\tau\right] + I_{0^+}^{\varsigma;\phi}(\mathcal{A}(\mathfrak{v}_0(\eta)))
\end{aligned}$$

for every $\eta \in \omega$. In addition, it follows from the above calculations that

$$d^*(\mathfrak{w},\mathfrak{v}_0) \leqslant \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+1)-(2\mathcal{L}_{\mathfrak{g}}+\mathfrak{p}\mathcal{L}_{\mathfrak{h}}+\mathfrak{p}\mathcal{L}_{\mathfrak{k}}+\|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^{\varsigma}}$$

which justifies inequality (10). □

4. Application

This section presents an example which illustrate the validity and applicability of our main results.

Example 1. Let $K:[0,1]\times[0,1] \to \mathbb{R}$ be a continuous function and $w(\eta)$ be a continuous function on $[0,1]$ so that $|K(\eta,\lambda)w(\eta)| < \dfrac{\Gamma(1/3)}{3\Gamma(2/9)}(e-1)^{-\frac{2}{9}}$. Consider the following fractional Volterra-Fredholm integro-differential system

$$\begin{cases}
{}^{\mathcal{H}}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{2}{3};e^{\eta}}w(\eta) = \displaystyle\int_0^1 K(\eta,\lambda)w(\eta)d\eta + \frac{1}{5}\sin(w(\eta)) + \int_0^{\eta}\sin\left(\frac{3}{5}\eta w(s)\right)ds + \\
\qquad\qquad + \displaystyle\int_0^1\cos\left(\frac{2}{5}\eta w(s)\right)ds \\
4I_0^{\frac{2}{9};\phi}w(0) = \mathfrak{w}_0, \quad \mathfrak{w}_0 \in \mathbb{R}
\end{cases} \tag{13}$$

for all $\eta \in [0,1]$ and continuous real-valued functions w on $[0,1]$, we have $\left|\dfrac{1}{5}\sin(w(\eta))\right| \leqslant \dfrac{1}{5}|w(\eta)|$. Moreover,

$$\left|\int_0^t \sin\left(\frac{3}{5}\eta w(s)\right)ds\right| \leqslant \frac{3}{5}|(w(\eta))|, \quad \left|\int_0^1\cos\left(\frac{2}{5}\eta w(s)\right)ds\right| \leqslant \frac{2}{5}|(w(\eta))|$$

for all $\eta \in [0,1]$. Furthermore, by assumption, the operator $3\int\limits_0^1 K(\eta,\lambda)w(\eta)d\eta$ is bounded and we have $\left|\int\limits_0^1 K(\eta,\lambda)w(\eta)d\eta\right| \leqslant \dfrac{\Gamma(4/3)}{\Gamma(2/9)(e-1)^{2/9}}$, for all $\eta,\lambda \in [0,1]$ and continuous functions w. So (H1)–(H3) are satisfied for $\mathfrak{q}_1 = 1/5$ and $\mathfrak{q}_2^h = 3/5$, $\mathfrak{q}_2^k = 2/5$. Therefore, Theorem 2.2 proved that equation (13) has at least one solution.

Conclusions

In this paper, we considered a class of fractional Volterra–Fredholm integro-differential equations including a closed linear operator. Next, we used the Krasnoselskii fixed-point theorem to investigate the existing result under some mild conditions. Moreover, we introduced and then proved the Kummer stability of ϕ-Hilfer fractional Volterra–Fredholm integro-differential equations on the compact domains.

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Теоретический анализ системы нелинейных ϕ-хильферовских дробных интегро-дифференциальных уравнений Вольтерра-Фредгольма

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Аннотация. Существование решений системы нелинейных ϕ-хильферовских дробных интегро-дифференциальных уравнений Вольтерра–Фредгольма с дробно-интегральными условиями исследуется с помощью теорем Красносельского о неподвижной точке и теорем Арцела–Асколи. Более того, применяя альтернативную теорему о неподвижной точке Диаса и Марголиса, мы доказываем куммеровскую устойчивость системы на компактных областях. Также представлен пример, иллюстрирующий наши результаты.

Ключевые слова: ϕ-хильферовское дробное интегро-дифференциальное уравнение Вольтера-Фредгольма, устойчивость Куммера, теорема Арцела–Асколи, теорема Красносельского о фиксированной точке.

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Fantappiè G-transform of Analytic Functionals

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Abstract. To each analytic functional on the space $O'(\mathbb{C}^n)$, a function $f(z)$ holomorphic in a neighborhood of the origin and an entire function of exponential type $F(z)$ are associated so that the coefficients c_α of the power series expansion of $f(z)$ are given by values of $F(\alpha)$. We study the problem of finding a connection between the domain where the function $f(z)$ extends to and the growth of the function $F(z)$.

Keywords: analytic functionals, G-transform, Fantappiè transformation, entire functions, continuation of holomorphic functions.

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Introduction

Let the function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

be holomorphic in a neighborhood of the origin. In many branches of mathematics, the question arises of existence of the *coefficient function*, i.e. such a function $F(z)$ that

$$F(n) = c_n, \quad n = 0, 1, 2, \dots \tag{2}$$

and the relationship between the properties of functions $f(z)$ and $F(z)$.

There always exists an entire coefficient function $F(z)$ of exponential type. Indeed, if γ is a contour around the point $z = 0$ in the positive direction, then by the well-known formula

$$c_n = \frac{1}{2\pi i} \int_\gamma f(z) z^{-n-1}\, dz.$$

After the change $z = e^{-\zeta}$ the contour γ turns to a contour Γ connecting two points P and Q such that $\Im(Q - P) = 2\pi$ and we get the integral representation

$$c_n = \frac{1}{2\pi} \int_\Gamma f(e^\zeta) e^{n\zeta}\, d\zeta.$$

As is well known, the function

$$F(z) = \frac{1}{2\pi} \int_\Gamma f(e^\zeta) e^{z\zeta}\, d\zeta$$

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is an entire function of exponential type (see, for example, [1, Sec. 1] for which the equality (2) holds.

Recall that an entire function $F(z)$ is said to be of exponential type if there is an a such that for a sufficiently large z we have

$$|F(z)| < e^{a|z|}. \tag{3}$$

The lower bound H of such numbers a is called the type of the function $F(z)$.

A change of the contour γ leads to another function $F^*(z)$ with similar properties. If Γ and Γ^* are two such contours and Γ^* connects points P^* and Q^* for which $\Im(Q^* - P^*) = 2\pi$, then

$$F^*(z) - F(z) = B(z)\sin\pi z, \quad B(z) = \frac{e^{\pi}}{\pi}\int_P^{P^*} f(e^{\zeta})e^{z\zeta}\,d\zeta.$$

As noted by L. Bieberbach ([1, Sec. 1]), the question arises of finding an entire function of exponential type with the minimal growth, and the problem of finding a connection between the domain where the function $f(z)$ extends to and the growth of the function $F(z)$.

To formulate the result we introduce the indicator function $h(\varphi)$ of the entire function $F(z)$ of exponential type which characterizes the growth along the ray $\arg z = \varphi$:

$$h(\varphi) = \limsup_{r\to\infty}\frac{\ln|F(re^{i\varphi})|}{r}.$$

Let K be the indicator diagram of the function $F(z)$, i.e. a convex compact set for which $h(\varphi)$ is the support function.

Theorem 1. *Let $F(z)$ be an entire function of exponential type with the indicator diagram K. The function $f(z)$, for which $F(n) = c_n$, $n = 1, 2, \dots$ and c_0 is arbitrary, is holomorphic in the connected component of the complement of the set e^{-K} to the whole plane that contains the point $z = 0$. For the convergence radius R of the series* (1) *the following estimate is valid*

$$R \geqslant e^{-h(0)}. \tag{4}$$

The proof of this theorem as well as a number of other results related to the case of entire functions of coefficients $F(z)$ of exponential type can be found in the book [1]. A closer connection between the properties of the function $f(z)$ and the entire function of coefficients $F(z)$ of exponential type can be formulated as conditions on a compact set K.

Theorem 2. *Let K be a closed bounded convex set. In order for the function* (1) *to be holomorphic in the connected component of the complement of the set e^{-K} to the whole plane that contains the point $z = 0$ and not holomorphic in any larger domain of the same kind, it is necessary and sufficient that the width of the set K in the direction of the imaginary axis is less than 2π and there exists an entire function of the coefficients $F(z)$ of exponential type with the indicator diagram K.*

If an entire function of the coefficients $F(z)$ with the specified properties exists, then it is unique,

$$R = e^{-h(0)}, \tag{5}$$

and the function $f(z)$ itself can be analytically continued to an infinite point along one of the radii and in a neighborhood of infinity its series decomposition is

$$f(z) = c_0 - \sum_{n=1}^{\infty} F(-n)z^{-n}.$$

Further application of the theory of analytic functionals and their G-transformation allowed more universal methods to study the question of the continuation of the series (1) depending on the nature of growth of functions $F(z)$ in the one-dimensional case and study the case of several dimensions. The main references are [2–5]

1. Laplace and Avanissan-Gay transforms of analytic functionals

Definition 1. *The elements in the dual space $\mathcal{O}'(\mathbb{C}^n)$ of the space $\mathcal{O}(\mathbb{C}^n)$ of entire functions, equipped with the topology of uniform convergence on compact sets, are called analytic functionals. An analytic functional T is said to be carried by a compact set K if for every neighborhood ω of K there is a constant C_ω such that*

$$|T(\varphi)| \leqslant C_\omega \sup_\omega |\varphi|, \quad \varphi \in \mathcal{O}(\mathbb{C}^n). \tag{6}$$

If K is compact, we denote by $O'(\mathbb{C}^n, K)$ the space of all analytic functionals carried by the set K.

Definition 2. *If $T \in \mathcal{O}'(\mathbb{C}^n)$, we define its Laplace transform by*

$$F_T(\zeta) = T_z(e^{(\zeta,z)}), \quad \zeta \in \mathbb{C}^n, \quad (\zeta, z) = \zeta_1 z_1 + \cdots + \zeta_n z_n.$$

The Laplace transform is an entire analytic function of ζ. From (6) we obtain the estimate

$$|F_T(\zeta)| \leqslant C_\omega \exp(\sup_{z\in\omega} \Re(z,\zeta)).$$

Set

$$H_K(\zeta) = \sup_{z\in K} \Re(z,\zeta).$$

If K is convex we have

$$K = \{z\colon \Re(z,\zeta) \leqslant H_K(\zeta), \zeta \in \mathbb{C}^n\}$$

otherwise

$$K \subset \{z\colon \Re(z,\zeta) \leqslant H_K(\zeta), \zeta \in \mathbb{C}^n\}.$$

The following Ehrenpreis-Martineau theorem characterizes the Laplace transform of analytic functionals (see [6]).

Theorem 3. *If $T \in \mathcal{O}'(\mathbb{C}^n, K)$, then $F_T(\zeta)$ is an entire analytic function and for every $\delta > 0$ there is a constant C_δ such that*

$$|F_T(\zeta)| \leqslant C_\delta \exp(H_K(\zeta) + \delta|\zeta|). \tag{7}$$

Conversely, if K is a convex compact set and $F(\zeta)$ an entire function satisfying (7) for every $\delta > 0$, there exists a unique *analytic functional $T \in \mathcal{O}'(\mathbb{C}^n, K)$ such that the Laplace transform of T is $F(\zeta)$.*

Let $U = \{z \in \mathbb{C}\colon -\pi < \Im z < \pi\}$ and $\Omega = U^n$. If $K \subset \Omega$ is a compact set and K_j its j-projection on $\mathbb{C}$, $(z_1, \ldots, z_n) \to z_j$, then we denote by $e^{-K} = \{e^{-z} : z \in K\}$ and we put

$$\Omega(K) = \prod_{j=1}^{n} [\mathbb{C} \setminus \exp(-K_j)],$$
$$\widetilde{\Omega}(K) = \prod_{j=1}^{n} [\widetilde{\mathbb{C}} \setminus \exp(-K_j)],$$

where $\widetilde{\mathbb{C}}$ is a compactification of $\mathbb{C}$. Let $\mathcal{O}_0(\Omega(K))$ denote the space of functions holomorphic in $\Omega(K)$ and continuous in $\widetilde{\Omega}(K)$ vanishing in $\widetilde{\Omega}(K) \setminus \Omega(K)$.

If $T \in \mathcal{O}'(\mathbb{C}^n, K)$, $K \subset \Omega$ we denote by $G(T)$ its Avanissan–Gay transform (G-transform):

$$G(T)(z) := T_\zeta\left(\prod_{j=1}^{n} \frac{1}{1 - z_j \exp \zeta_j}\right).$$

Proposition 1. *Let $T \in \mathcal{O}'(\mathbb{C}^n, K)$, $K \subset \Omega$. Then*

1. *$G(T)(z) \in \mathcal{O}_0(\Omega(K))$.*

2. *In a neighborhood of the origin, we have*

$$G(T)(z) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} F_T(\alpha) z^\alpha$$

while the following expansion is valid at infinity:

$$G(T)(z) = (-1)^n \sum_{\alpha \in \mathbb{N}^n} F_T(-\alpha) z^{-\alpha}.$$

3. *The map $G : \mathcal{O}'(K) \to H_0\left(\Omega(K)\right)$, given by $T \mapsto G(T)$, is an injection, but in general it is not a surjection. When $K \subset \Omega$ is a direct product, then the G-transform gives an isomorphism between $\mathcal{O}'(K)$ and $H_0(\Omega(K)$.*

As a consequence of the these results on G-transform, we obtain an important uniqueness theorem for entire functions of exponential type.

Proposition 2. *A necessary and sufficient condition for two analytic functionals $T_1, T_2 \in \mathcal{O}'(\Omega)$ to coincide is the identity $F_{T_1}(\alpha) = F_{T_2}(\alpha)$ for every $\alpha \in \mathbb{N}^n$.*

G-transforms of analytic functionals are similar to their Cauchy transforms and proofs of the main results use duality of spaces of functionals and spaces of holomorphic functions on suitable sets. Because of the structure of the Cauchy kernel in several dimensions similar results are possible only in the case of compact sets given as products of plane convex compacts.

In this paper we propose to consider a projective analogue of G-transform associated with the Fantappiè transform of analytic functionals and to use a general form of analytic functionals for convex domains and compacts, which was first established in the works of L. Aizenberg [13] and A. Martineau.

2. Sets and maps in projective space

Suppose, as usual, $\mathbb{C}^n$ is the space of row-vectors of dimension n with elements from the field $\mathbb{C}$. The complex projective space $\mathbb{CP}^n$ is defined as the set of one-dimensional linear subspaces (or what is the same, complex lines passing through 0) in $\mathbb{C}^{n+l}$. We denote by p the map of the set $\mathbb{C}^{n+1} \setminus \{0\}$ into $\mathbb{CP}^n$, which assigns to a point the subspace containing it. Projecting the open sets from $\mathbb{C}^{n+l} \setminus 0$, this map gives a topology in $\mathbb{CP}^n$. As is familiar, p is continuous and $\mathbb{CP}^n$ is compact in this topology.

The complex lines lying in the plane

$$\{z \in \mathbb{C}^{n+1} \colon z_{n+1} = 0\}$$

are called infinite points of $\mathbb{CP}^n$, and the other complex lines are called finite points. The map p maps the plane

$$\{z \in \mathbb{C}^{n+1} \colon z_{n+1} = 1\}$$

homomorphically onto the set of finite points of $\mathbb{CP}^n$. Each point $z = (z_1, \dots, z_n)$ can be identified with the corresponding finite point $p(z_1, \dots, z_n, 1)$, and $\mathbb{C}^n$ is represented as an open, everywhere dense set in the compact $\mathbb{CP}^n$.

As a rule, we shall define continuous functions on open sets in $\mathbb{CP}^n$, defining their values (by a formula) only on everywhere dense subsets.

Let D be an open set in $\mathbb{CP}^n$. The space $\mathcal{O}(D)$ consists of functions which are holomorphic in D, and convergence in $\mathcal{O}(D)$ by definition means uniform convergence on each compact subset of D. We define $\mathcal{O}_0(D)$ as the closed subspace in $\mathcal{O}(D)$, consisting of functions satisfying the following condition: for any $z \in D \setminus \mathbb{C}^n$ one can find a neighborhood U of it in $\mathbb{CP}^n$, in which f is holomorphic and $f(\zeta) = O(|\zeta|^{-n}), \zeta \to \infty$.

By a plane of dimension k in $\mathbb{CP}^n$ is meant the image of a $(k+1)$-dimensional linear subspace of $\mathbb{C}^{n+l}$ under the map p. In particular, the infinite points of $\mathbb{CP}^n$ form a hyperplane in it, and the closure of a complex line from $\mathbb{C}^n$ is a line.

We denote by $M(m,n)$ the set of all matrices of size $m \times n$ over the field $\mathbb{C}$. In particular, $\mathbb{C}^n = M(1, n)$. For any matrix $A \in M(m,n)$ we shall denote by A' the transposed matrix. Then for $z \in \mathbb{C}^n$ and $\zeta \in \mathbb{C}^n$ we have $z\zeta' = z_1\zeta_1 + \dots + z_n\zeta_n$. To each k-dimensional plane $\alpha \subset \mathbb{CP}^n$ there corresponds in a one-to-one fashion its dual $(n-k-1)$-dimensional plane

$$\alpha' = \{\zeta \in \mathbb{CP}^n \colon z\zeta' = 0 \ \text{ for all } \ z \in \alpha\}.$$

In particular, points and hyperplanes are the duals of each other.

For an arbitrary $M \subset \mathbb{CP}^n$ the adjoint set of

$$M^* := \{\zeta \in \mathbb{CP}^n \colon z_1\zeta_1 + \dots + z_{n+1}\zeta_{n+1} \neq 0 \ \text{ for all } \ z \in M\}$$

consists of points dual to hyperplanes not passing through M. One can also consider M^* as the complement of the union of hyperplanes dual to points of M. Obviously if $M_1 \subset M_2$, then $M_1^* \supset M_2^*$. The set $M^{**} = (M^*)^*$ always contains M.

The condition $M^{**} = M$ means that through each point $z \notin M$ there passes a hyperplane not passing through M, i.e., that M is Martineau linearly convex. It is known (cf., e.g., [21]), that if M is open, then M^* is compact, and if M is compact, then M^* is open.

Let D be an open set in $\mathbb{CP}^n$. The space $\mathcal{O}(D)$ consists of functions which are holomorphic in D, and convergence in $\mathcal{O}(D)$ by definition means uniform convergence on each compact subset of D. We define the space $\mathcal{O}(K)$ for K as the inductive limit of the spaces $\mathcal{O}(D)$ over all open $D \supset K$.

We define $\mathcal{O}_0(D)$ as the closed subspace in $\mathcal{O}(D)$, consisting of functions satisfying the following condition: for any $z \in D \setminus \mathbb{C}^n$ one can find a neighborhood U of it in $\mathbb{CP}^n$, in which f is holomorphic and $f(\zeta) = O(|\zeta|), \zeta \to \infty$. We denote by $\mathcal{O}_0^*(D)$ the space of continuous linear functionals on $\mathcal{O}_0(D)$.

Definition 3. *Let E be an open or compact set in $\mathbb{CP}^n$ and assume that E^* is non-empty. We define the Fantappiè transform*

$$\mathcal{F} : \mathcal{O}_0^*(E) \to \mathcal{O}_0(E^*) \quad \text{by} \quad \mathcal{F}\mu(z) = \mu\left(\frac{1}{z_1\zeta_1 + \cdots + z_{n+1}\zeta_{n+1}}\right).$$

We call E strongly linearly convex if this correspondence establishes an isomorphism of the spaces $\mathcal{O}_0^*(E)$ and $\mathcal{O}_0(E^*)$.

It is known ([11, 17]), cf. also [12, p. 237]), that all convex compacta and all convex domains are strongly linearly convex. Aizenberg showed in [13] that all Martineau linearly convex domains with sufficiently smooth boundary, their closures, and also domains and compacta which can be approximated by such sets are strongly linearly convex.

The main references are [7, 8, 10–15]

3. Fantappiè G-transform of analytic functionals

Definition 4. *If $T \in \mathcal{O}'(\mathbb{C}^n, K)$, $K \subset \Omega$ we denote by $FG(T)$ its Fantappiè G-transform (FG-transform):*

$$FG(T)(z) := T_\zeta\left(\frac{1}{1 + z_1e^{\zeta_1} + \cdots + z_ne^{\zeta_n}}\right).$$

Theorem 4. *Let $T \in \mathcal{O}'(\mathbb{C}^n, K)$, $K \subset \Omega$. Then*

1. *$FG(T)(z) \in \mathcal{O}_0((e^K)^*)$.*
2. *In a neighborhood of the origin we have*

$$FG(T)(z) = \sum_{|\alpha|\geqslant 0} (-1)^{|\alpha|}\frac{\alpha!}{|\alpha|!}F_T(\alpha)z^\alpha. \tag{8}$$

3. *The map $FG : \mathcal{O}'(\mathbb{C}^n, K) \to \mathcal{O}_0\left(((e^{-K})^*)\right)$ given by $T \longmapsto FG(T)$ is an injection, but in general it is not a surjection. When e^{-K} is a convex compact set the FG-transform gives an isomorphism between $\mathcal{O}'(\mathbb{C}^n, K)$ and $O_0\left((e^K)^*\right)$.*

Proof.

1. Let $T \in \mathcal{O}'(\mathbb{C}^n, K)$. Since the space $\mathcal{O}(\mathbb{C}^n)$ is dense in $\mathcal{O}(\Omega)$, there is an extension of T to the analytic functional $S \in \mathcal{O}'(\Omega)$ which is carried by the set K.

   Consider a biholomorphic mapping $(\zeta_1, \ldots, \zeta_n) \mapsto (w_1, \ldots, w_n) = \left(e^{\zeta_1}, \ldots, e^{\zeta_n}\right)$ of the set Ω to the set $(\mathbb{C} \setminus (-\infty, 0])^n$. Under this mapping the functional $S \in \mathcal{O}'(\mathbb{C}^n, K)$ corresponds to the functional $S_1 \in \mathcal{O}'((\mathbb{C} \setminus (-\infty, 0])^n)$ carried by the set e^K, and more precisely by the set $(e^K)^{**} \supset e^K$. The restriction of the functional S_1 to the space $\mathcal{O}(((e^K)^{**})$ defines some functional $\mu \in \mathcal{O}'((e^k)^{**})$.

The FG-transform of the functional T takes the form of the Fantappiè transform of the functional μ:

$$G(T)(z) = T_\zeta\left(\frac{1}{1+z_1e^{\zeta_1}+\cdots+z_ne^{\zeta_n}}\right) = \mu\left(\frac{1}{1+z_1w_1+\cdots+z_nw_n}\right). \tag{9}$$

But according to the known properties of the Fantappiè transform, the function $G(T)(z)$ is holomorphic in $\mathcal{O}_0\left((e^K)^*\right)$.

2. Consider a series expansion of the kernel

$$\frac{1}{1+z_1e^{\zeta_1}+\cdots+z_ne^{\zeta_n}} = \sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}z^\alpha(e^\zeta)^\alpha, \tag{10}$$

$z^\alpha(e^\zeta)^\alpha = z_1^{\alpha_1}\dots z_n^{\alpha_n}(e^{\zeta_1})^{\alpha_1}\dots(e^{\zeta_n})^{\alpha_n}$.

For z in a sufficiently small neighborhood of the origin and for all ζ in some neighborhood U of the compact K, each term of the series (10) is majorized by a term of the convergent multiple geometric series

$$\sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}q^\alpha = \frac{1}{1+q_1+\cdots+q_n}.$$

It follows that the series (9) converges uniformly in $\zeta\in U$ and we get

$$G(T)(z) = T_\zeta\left(\frac{1}{1+z_1e^{\zeta_1}+\cdots+z_ne^{\zeta_n}}\right) = T_\zeta\left(\sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}z^\alpha(e^\zeta)^\alpha\right) =$$
$$= \sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}z^\alpha T_\zeta\left(e^{(\zeta,\alpha)}\right) = \sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}F_T(\alpha)z^\alpha.$$

3. The map $FG:\mathcal{O}'(\mathbb{C}^n,K)\to\mathcal{O}_0\left(((e^{-K})^*)\right)$ is an injection.

Let $T\in\mathcal{O}'(\mathbb{C}^n,K)$ and $FG(T)(z)=0$. The expansion (8) implies that $F_T(\alpha)=0$ for all α. According to Proposition 2 we have $T=0$.

When e^{-K} is a convex compact set the FG-transform gives an isomorphism between $\mathcal{O}'(\mathbb{C}^n,K)$ and $O_0\left((e^K)^*\right)$. This follows from the following sequence of isomorphisms

$$\mathcal{O}'(\mathbb{C}^n,K)\simeq\mathcal{O}'(K)\simeq\mathcal{O}'(e^K)\simeq O_0\left((e^K)^*\right).$$

The last isomorphism is valid due to the strong linear convexity of the compact set e^K. $\square$

Example. Consider the function

$$f(z) = \frac{1}{1+z_1+\cdots+z_n} = \sum_{|\alpha|\geqslant 0}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}z^\alpha.$$

Here $F(\alpha)\equiv 1$ for all $\alpha\in\mathbb{N}^n$. Take $T=\delta_0$. This analytic functional is the only one for which the Laplace transform $F_T(\alpha)\equiv 1$. According to Theorem 4 the function $f(z)$ extends to the domain

$$(e^{-K})^* = \{(1,\dots,1)\}^* = \{z: 1+z_1+\cdots+z_n\neq 0\},$$

which coincides with the domain of holomorphicity of the function $f(z)$.

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G-преобразование Фантаппье аналитических функционалов

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Аннотация. С каждым аналитическим функционалом пространства $O'(\mathbb{C}^n)$ ассоциируются голоморфная в окрестности начала координат функция $f(z)$ и целая функция экспоненциального типа $F(z)$ таким образом, что коэффициенты c_α разложения функции $f(z)$ определяются значениями $F(\alpha)$. Изучается задача о нахождении связи между областью, в которую продолжается функция $f(z)$, и ростом функции $F(z)$.

Ключевые слова: аналитические функционалы, G-преобразование, целые функции, продолжение голоморфных функций.

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On Multiple Zeros of Entire Functions of Finite Order of Growth

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Abstract. The article is devoted to determination of the number of multiple zeros of entire functions of finite order of growth.

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Our work is devoted to the problem of multiple zeros of entire functions. For polynomials, this question is a classical problem, and its solution is included in algebra textbooks (see, for example, [1]).

Recall the statement. Consider a polynomial $P(z)$ of degree n. Denote by S_j the power sums of the roots of a polynomial of degree j.

Theorem 1. *In order for the polynomial $P(z)$ to have multiple roots, it is necessary and sufficient that*

$$D(P) = a_0^{2n-2}\begin{vmatrix} n & S_1 & S_2 & \dots & S_{n-1} \\ S_1 & S_2 & S_3 & \dots & S_n \\ \dots & \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & S_{n+1} & \dots & S_{2n-2} \end{vmatrix} = 0.$$

Here a_0 is the leading coefficient of the polynomial $P(z)$.

The determinant of $D(P)$ is called the *discriminant* of the polynomial $P(z)$.

For entire functions, the question of multiple zeros needs to be clarified. An entire function may have no zeros at all, like, for example, the function e^z, or an infinite number of zeros like $\sin z$. Therefore, we have to consider various options here.

1. Let an entire function have the form

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k, \quad f(0) = a_0 = 1. \tag{1}$$

The following statement is true ([2], corollary 1.4.1).

Theorem 2. *In order for the function $f(z)$ to be an entire function of finite order k_0 that has no zeros, it is necessary and sufficient that the determinant*

$$D(P) = a_0^{2n-2}\begin{vmatrix} a_1 & a_0 & 0 & \dots & 0 \\ 2a_2 & a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ka_k & a_{k-1} & a_{k-2} & \dots & a_1 \end{vmatrix} = 0 \quad \text{for all} \quad k > k_0, \tag{2}$$

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where k_0 is the minimal number with this property.

2. Consider an entire function of finite order of growth of the form (1). Find the order ρ of the function f. To do this, we apply the formula ([3], ch. 7)

$$\varliminf_{n\to\infty} \frac{\ln(1/|a_n|)}{n\ln n} = \frac{1}{\rho}.$$

If ρ is a fractional number, then the function $f(z)$ is known to have an infinite number of zeros (see [3]). First we will assume that ρ is an integer.

Let us take a sequence of complex numbers $s_0,\ s_1,\ s_2,\ \dots$. It defines an infinite Hankel matrix

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & s_4 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3}$$

The consecutive main minors of the matrix S are denoted by $D_0,\ D_1,\ D_2,\ \dots$. In addition, we set $D_{-1} = 1$.

If for every $p \in \mathbb{N}$ there exists a minor of the matrix S of order p that is not equal to zero, then the matrix has infinite rank. If, starting from some p, all minors of larger orders are zero, then the matrix S has finite rank. The smallest such p is called the *rank* of the matrix.

We recall a statement regarding matrices S of finite rank p ([4], ch. 16, Sec. 10).

Theorem 3. *If an infinite Hankel matrix has finite rank p, then the minor $D_{p-1} \neq 0$.*

Thanks to the properties of entire functions, the power sums of σ_k

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N}$$

are absolutely convergent series for $k > \rho$. Here, the zeros of the entire function $f(z)$ are denoted by α_n. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha_n| \leqslant \ldots$.

The smallest such k is denoted by k_0 and we denote $s_j = \sigma_{2k_0+j},\ j = 0, 1, \ldots$. Consider an infinite Hankel matrix S of the form (3). In the monograph ([2], Theorem 1.4.5) the following statement is proved

Theorem 4. *In order for the function f to have a finite number of zeros, it is necessary and sufficient that the rank of the matrix S is finite, while the number of different zeros of f is equal to the rank of S.*

In this case, we can write the function f as follows:

$$f(z) = e^{-Q(z)}\, P(z), \tag{4}$$

where $Q(z)$ is a polynomial of degree $p = \rho$, and $P(z)$ is a polynomial of some degree m

$$P(z) = \sum_{k=0}^{m} b_k\, z^k = 1 + b_1\, z + \ldots + b_m\, z^m.$$

The number m is the number of roots of the function $f(z)$ together with their multiplicities. To find the polynomial $P(z)$, one needs to factorize the function $f(z)$ (see Sec. 1.6.5 from [2]).

Take the logarithm of both parts in the formula (4). Let

$$\ln f(z) = \sum_{k=1}^{\infty} \tilde{a}_k\, z^k = \tilde{a}_1\, z + \ldots + \tilde{a}_n\, z^n + \ldots,$$
$$\ln P(z) = \sum_{k=1}^{\infty} \tilde{b}_k\, z^k = \tilde{b}_1\, z + \ldots + \tilde{b}_n\, z^n + \ldots .$$

The coefficients of $\tilde{a}_n$ can be found by the following formula (see [2], Lemma 1.2.1)

$$\tilde{a}_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} a_1 & 1 & 0 & \ldots & 0 \\ 2a_2 & a_1 & 1 & \ldots & 0 \\ 3a_3 & a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ na_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{vmatrix} = 0 \quad \text{for all} \quad n \geqslant 1,$$

The coefficients b_k are found from the theorem ([2], Theorem 1.6.4, [6]).

Theorem 5. *The formulas are valid*

$$b_k = \frac{-\begin{vmatrix} (m+p)\,\tilde{a}_{m+p} & \ldots & (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (p+1)\,\tilde{a}_{p+1} \\ (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (m+p+2)\,\tilde{a}_{m+p+2} & \ldots & (p+2)\,\tilde{a}_{p+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (2m+p-1)\,\tilde{a}_{2m+p-1} & \ldots & (2m+p)\,\tilde{a}_{2m+p} & \ldots & (p+m)\,\tilde{a}_{p+m} \end{vmatrix}}{\begin{vmatrix} (m+p)\,\tilde{a}_{m+p} & \ldots & (p+1)\,\tilde{a}_{p+1} \\ (m+p+1)\,\tilde{a}_{m+p+1} & \ldots & (p+2)\,\tilde{a}_{p+2} \\ \ldots & \ldots & \ldots \\ (2m+p-1)\,\tilde{a}_{2m+p-1} & \ldots & (p+m)\,\tilde{a}_{p+m} \end{vmatrix}},$$

$k = 1, \ldots, m$. *In the numerator k, the th column is replaced by the column*

$$\begin{pmatrix} -(m+p+1)\,\tilde{a}_{m+p+1} \\ -(m+p+2)\,\tilde{a}_{m+p+2} \\ \ldots \\ -(2m+p)\,\tilde{a}_{2m+p} \end{pmatrix}$$

Here m this is the smallest k for which b_k is different from zero.

Corollary 1. *A function $f(z)$ has multiple roots if and only if the polynomial $P(z)$ has multiple roots.*

Let us give an example.
Consider the function

$$f(z) = 1 + 2z + \sum_{k=2}^{\infty} \left(\frac{2^k}{k!} - \frac{2^{k-2}}{(k-2)!} \right) z^k = 1 + 2z + z^2 - \frac{2z^3}{3} - \frac{4z^4}{3} - \frac{16z^5}{15} + \ldots .$$

It is not difficult to calculate that the order of growth of this function is $\rho = 1$.
By Lemma 1.2.1 of [2], the power sums of S_j with even numbers are 2, with odd numbers are 0. Therefore, the rank of the Hankel matrix is S

$$S = \begin{pmatrix} 2 & 0 & 2 & \ldots \\ 0 & 2 & 0 & \ldots \\ 2 & 0 & 2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is equal to 2.

From Theorem 5 and Lemma 1.2.1 from [2] we get

$$\tilde{a}_2 = \frac{-1}{2}\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -1,$$

$$\tilde{a}_3 = \frac{1}{3}\begin{vmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = 0,$$

$$\tilde{a}_4 = \frac{-1}{4}\begin{vmatrix} 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ -\frac{16}{3} & -\frac{2}{3} & 1 & 2 \end{vmatrix} = -\frac{1}{2},$$

$$\tilde{a}_5 = \frac{1}{5}\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ -2 & 1 & 2 & 1 & 0 \\ -\frac{16}{3} & -\frac{2}{3} & 1 & 2 & 1 \\ -\frac{16}{3} & -\frac{4}{3} & -\frac{2}{3} & 1 & 2 \end{vmatrix} = 0.$$

From here we find

$$b_1 = -\frac{\begin{vmatrix} 4\tilde{a}_4 & 2\tilde{a}_2 \\ 5\tilde{a}_5 & 3\tilde{a}_3 \end{vmatrix}}{\begin{vmatrix} 3\tilde{a}_3 & 2\tilde{a}_2 \\ 4\tilde{a}_4 & 3\tilde{a}_3 \end{vmatrix}} = -\frac{\begin{vmatrix} -2 & -2 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}} = 0,$$

$$b_2 = -\frac{\begin{vmatrix} 3\tilde{a}_3 & 4\tilde{a}_4 \\ 4\tilde{a}_4 & 5\tilde{a}_5 \end{vmatrix}}{\begin{vmatrix} 3\tilde{a}_3 & 2\tilde{a}_2 \\ 4\tilde{a}_4 & 3\tilde{a}_3 \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}} = -1.$$

The remaining b_k is zero. Therefore, the polynomial $P(z)$ is equal to

$$P(z) = 1 - z^2.$$

It has two roots ± 1 and has no multiple roots. Therefore, the function $f(z)$ has no multiple roots.

3. Let an function $f(z)$ of the form (1) have an infinite number of zeros, then the rank of the matrix S (3) is infinite. Multiple zeros can only have finite multiplicities. Therefore, if $f(z)$ has an infinite number of zeros, then it has an infinite number of distinct zeros.

Multiple zeros are the common zeros of the function and its derivative, i.e., the zeros of the resultant. So the question is whether the function and its derivative have common zeros.

The approach to determining the resultant of two integer functions is considered in a number of papers [5–7], but for arbitrary entire functions of finite growth order it is not yet known how to find the common zeros of the function and its derivative.

Let an entire function $f(z)$ have the order ρ. Due to the properties of entire functions, power sums σ_k

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

are absolutely convergent series for $k > \rho$. Here, as before, α_n are zeros of the entire function $f(z)$. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha_n| \leqslant \ldots$. The smallest such k is denoted by k_0. We assume that k_0 is an integer.

We will introduce, as in the previous section, power sums $s_j = \sigma_{2k_0+j}$, $j = 0, 1, \dots$ and an infinite Hankel matrix S of the form (3). Its rank is infinite.

Consider its submatrices of the order m:

$$S^m = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_m \\ s_1 & s_2 & s_3 & \dots & s_{m+1} \\ s_2 & s_3 & s_4 & \dots & s_{m+2} \\ \dots & \dots & \dots & \dots & \dots \\ s_m & s_{m+1} & s_{m+2} & \dots & s_{2m+1} \end{pmatrix}. \tag{5}$$

We introduce finite power sums

$$\sigma_k^m = \sum_{n=1}^{m} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

$s_j^m = \sigma_{2k_0+j}^m$ and matrices

$$S_m^m = \begin{pmatrix} s_0^m & s_1^m & s_2^m & \dots & s_m^m \\ s_1^m & s_2^m & s_3^m & \dots & s_{m+1}^m \\ s_2^m & s_3^m & s_4^m & \dots & s_{m+2}^m \\ \dots & \dots & \dots & \dots & \dots \\ s_m^m & s_{m+1}^m & s_{m+2}^m & \dots & s_{2m+1}^m \end{pmatrix}. \tag{6}$$

Consider an infinite matrix

$$A = \begin{pmatrix} \frac{1}{\alpha_1^{k_0}} & \frac{1}{\alpha_2^{k_0}} & \frac{1}{\alpha_3^{k_0}} & \dots \\ \frac{1}{\alpha_1^{k_0+1}} & \frac{1}{\alpha_2^{k_0+1}} & \frac{1}{\alpha_3^{k_0+1}} & \dots \\ \frac{1}{\alpha_1^{k_0+2}} & \frac{1}{\alpha_2^{k_0+2}} & \frac{1}{\alpha_3^{k_0+3}} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{7}$$

Then we have

$$S = A \cdot A',$$

where A' is the transpose of the matrix A. If the function f has multiple zeros, then the matrix A has the same columns.

Denote by A_m the main submatrix of the matrix A of order m. Then $S_m^m = A_m \cdot A_m'$. If the function $f(z)$ has multiple zeros, then $\det A_m = 0$, starting from some m. Since the matrix A_m is a Vandermonde matrix up to a nonzero multiplier, the opposite is also true: if its determinant is 0, at least two of its columns coincide.

Thus, the next statement is true.

Lemma 1. *In order for the function $f(z)$ to have multiple zeros, it is necessary and sufficient that* $\det A_m = 0$ *starting from some m.*

Since $S_m^m = A_m \cdot A_m'$, the following statement is true

Proposition 1. *In order for the function $f(z)$ to have multiple zeros, it is necessary and sufficient that* $\det S_m^m = 0$ *starting from some m.*

In order to find $\det S_m^m$, we first need to factorize the function f (see point 2). Suppose that after factorization, the function $f(z)$ takes the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\alpha_j}\right).$$

Thus, the function $f(z)$ is a function of genus zero (or an entire function of the first order of growth of minimal type ([3], Chapter 7). In this case, the series

$$\sum_{j=0}^{\infty} \frac{1}{\alpha_j}$$

absolutely converges. Then the coefficients a_k of the function $f(z)$ take the form

$$a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\alpha_{j_1} \cdots \alpha_{j_k}}.$$

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Кратные нули целых функций конечного порядка роста

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Аннотация. Статья посвящена определению числа кратных нулей целой функции конечного порядка роста.

Ключевые слова: целая функция, кратный нуль.

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On the Blaschke Factors in Polydisk

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Abstract. The purpose of this work is to construct a multidimensional analogue of the Blaschke factors. The relevance of the construction of this analogue was prompted by a recent joint article by Alpay and Yger devoted to the multidimensional interpolation theory for functional spaces in special Weyl polyhedra. By such a factor we understand a set of special inner rational functions in a unit polydisk. We construct inner rational functions for the case of three complex variables, in particular, using the Lee-Yang polynomial from the theory of phase transitions in statistical mechanics.

Keywords: Blaschke product, Lee-Yang polynomial.

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Introduction

In 1915 Wilhelm Blaschke introduced a very important class of functions of one complex variable, which allowed solving important problems of interpolation theory in a unit disk.

Definition 0.1 ([1]). *The one-dimensional Blaschke product is a function of the form:*

$$B(z) = \prod_{k\geqslant 1} \frac{z - z_k}{1 - \bar{z}_k z}, \tag{1}$$

where $\{z_1, z_2, \ldots, z_n, \ldots\}$ *is a sequence of points in the unit disk* $D \subset \mathbb{C}$.

In the case of a finite number of points $\{z_k\}$ from the disk, no restrictions are imposed on them, however, when moving to a countable set of points, the so-called Blaschke condition is added for the correctness of the definition:

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty. \tag{2}$$

This concept made it possible to solve important problems of interpolation theory in a single disk. For example, Blaschke's theorem states that a sequence $\{z_k\}$ in a disk is a zero set for a holomorphic function bounded in D if and only if the series in (2) converges.

Except for the case of bounded functions, similar descriptions have been obtained for functions from Hardy classes.

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Before proceeding to an analogue of the Blaschke factors in the multidimensional case, let us take a closer look at the Blaschke factors in the one-dimensional case. Note that each factor $b_k = \dfrac{z_k - z}{1 - \bar{z}_k z}$ of the product (1) is a fractional rational function of the form:

$$b_k = \frac{p(z)}{q(z)} = z\frac{\overline{q(1/\bar{z})}}{q(z)}.$$

In the case when q has real coefficients, the functions b_k can be represented as:

$$b_k = z\frac{q(1/z)}{q(z)}.$$

The idea of our generalization of the Blaschke factors is to construct such 'elementary' factors for several complex variables. We would like to note that such a construction was carried out under the influence of the results of Alpay and Yger [2].

1. Multidimensional analogue of the Blaschke factor in the space $\mathbb{C}^3$

By the analogue of the Blaschke factor in $\mathbb{C}^3$ we shall understand the triple of special inner rational functions in the unit polydisk of $\mathbb{C}^3$. We will construct inner rational functions using the Lee-Yang polynomials (see [3]). In order to do this, we fix an arbitrary symmetric $n \times n$ matrix (a_{jk}) with real coefficients satisfying the condition $0 < |a_{jk}| < 1$. The corresponding Lee-Yang polynomial is constructed according to the given matrix as follows:

$$f(z_1, z_2, \ldots, z_n) = \sum_J \prod_{j \in J} \left(z_j \prod_{k \notin J} a_{jk} \right),$$

where J runs over the set of all subsets of $\{1, 2, \ldots, n\}$.

Let us present some important properties of this polynomial. Recall that the amoeba A_f of the polynomial f is defined as the image $\operatorname{Log} V$ of the zero set $V = \{z \in (\mathbb{C}\backslash 0)^n : f(z) = 0\}$ under the map $\operatorname{Log} : (z_1, \ldots, z_n) \to (\ln|z_1|, \ldots, \ln|z_n|)$ (see [4, 5, 6]). Taking into account the following expression:

$$f(z_1, z_2, \ldots, z_n) = z_1 z_2 \ldots z_n f(1/z_1, 1/z_2, \ldots, 1/z_n)$$

the amoeba of the polynomial f is symmetric with respect to the origin. Moreover, the following theorem is valid.

Theorem 1.1 (M. Passare, A. Tsikh [7]). *Let A be the amoeba of the Lee-Yang polynomial, then the closed positive and negative orthants $\pm\mathbb{R}^n_+$ intersect the amoeba A only at the origin:*

$$\mathbb{R}^n_+ \cap A = -\mathbb{R}^n_+ \cap A = \{0\}$$

Consider the Lee–Yang polynomial in three variables associated with the matrix

$$(a_{jk}) = \begin{pmatrix} a_{11} & a & b \\ a & a_{22} & c \\ b & c & a_{33} \end{pmatrix},$$

where $\{a_{11}, a_{22}, a_{33}, a, b, c\} \in (-1, 1)\backslash\{0\}$. The corresponding Lee–Yang polynomial is

$$f = (z_1 z_2 z_3 + bc z_1 z_2 + ab z_2 z_3 + ac z_1 z_3) + (ab z_1 + ac z_2 + bc z_3 + 1).$$

We introduce the following

Notation. *Split the polynomial into two parts and denote*

- $f_1 = z_1z_2z_3 + bcz_1z_2 + abz_2z_3 + acz_1z_3$,
- $f_2 = abz_1 + acz_2 + bcz_3 + 1$.

Next, we fix an arbitrary (z_1^0, z_2^0, z_3^0) from the distinguished boundary

$$\Delta = \{|z_j| = 1, j = 1, 2, 3\}$$

of the polydisk $D^3 \subset \mathbb{C}^3$ and consider the following sequence of functions:

$$\begin{aligned} p_1 &= f_1(z_1^0, z_2, z_3), \quad p_2 = f_1(z_1, z_2^0, z_3), \quad p_3 = f_1(z_1, z_2, z_3^0), \\ q_1 &= f_2(z_1^0, z_2, z_3), \quad q_2 = f_2(z_1, z_2^0, z_3), \quad q_3 = f_2(z_1, z_2, z_3^0). \end{aligned}$$

Definition 1.1. *We call the map* $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ *a three-dimensional analogue of the Blaschke factor if the zeros of the polynomial* $f_2 = abz_1 + acz_2 + bcz_3 + 1$ *do not intersect the open unit polydisk* D^3 *(in this case,* q_i *will be called valid).*

Recall an important definition.

Definition 1.2 (see [8])**.** *A function* $g \in H^\infty(D^n)$ *is called inner if its radial boundary values* $g^*(s)$ *satisfy the condition* $|g^*(w)| = 1$ *almost everywhere on* T^n*.*

In fact, the Blaschke product is an inner function. Our theorem below shows that the definition we introduced corresponds to this property.

Theorem 1.2. *The functions* p_j/q_j *in Definition 1.1 of the Blaschke factors are inner functions in the polydisk* D^3*.*

Proof. Since $q_1 = f_2(z_1^0, z_2, z_3)$, the zeros of the denominator q_1 on the unit distinguished boundary are also zeros of the polynomial f_2. It can be noted that for $|ab|+|ac|+|bc| < 1$, the polynomial f_2 has no zeros in the closure of the unit polydisk $\bar{D^3}$, but then the denominators q_j have no zeros in the same closure. For $|ab| + |ac| + |bc| = 1$, the polynomial f_2 has a single zero on the distinguished boundary $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ and has no zeros inside the polydisk. In this case, the denominator q_i has a single zero on the distinguished boundary if $\hat{z}_i = z_i^0$; otherwise, q_i does not vanish in the closure of a single polydisk. If the inequality $|ab| + |ac| + |bc| > 1$ is satisfied, then the polynomial f_2 has zeros inside the polydisk, so the corresponding denominators q_j are not valid. Thus, the permissible denominators vanish at no more than one point from the distinguished boundary. Therefore, almost everywhere on T^3 we have:

$$p_1 = f_1(z_1^0, z_2, z_3) = z_1^0z_2z_3 + bcz_1^0z_2 + abz_2z_3 + acz_1^0z_3 = \\ = z_1^0z_2z_3\left(\frac{bc}{z_3} + \frac{ab}{z_1^0} + \frac{ac}{z_2} + 1\right) = z_1^0z_2z_3f_2\left(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3}\right),$$

$$p_2 = f_1(z_1, z_2^0, z_3) = z_1z_2^0z_3 + bcz_1z_2^0 + abz_2^0z_3 + acz_1z_3 = \\ = z_1z_2^0z_3\left(\frac{bc}{z_3} + \frac{ab}{z_1} + \frac{ac}{z_2^0} + 1\right) = z_1z_2^0z_3f_2\left(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3}\right),$$

$$p_3 = f_1(z_1, z_2, z_3^0) = z_1z_2z_3^0 + bcz_1z_2 + abz_2z_3^0 + acz_1z_3^0 = \\ = z_1z_2z_3^0\left(\frac{bc}{z_3^0} + \frac{ab}{z_1} + \frac{ac}{z_2} + 1\right) = z_1z_2z_3^0f_2\left(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0}\right).$$

From these equalities we obtain the following chains of equalities for modules of functions $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$, which are valid almost everywhere on the distinguished boundary:

$$\left|\frac{p_1}{q_1}\right| = \left|\frac{z_1^0 z_2 z_3 f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = |z_1^0 z_2 z_3| \left|\frac{f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = \left|\frac{f_2(\bar{z_1^0}, \bar{z_2}, \bar{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = \left|\frac{\bar{f_2}(z_1^0, z_2, z_3)}{f_2(z_1^0, z_2, z_3)}\right| = 1,$$

$$\left|\frac{p_2}{q_2}\right| = \left|\frac{z_1 z_2^0 z_3 f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = |z_1 z_2^0 z_3| \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = \left|\frac{f_2(\bar{z_1}, \bar{z_2^0}, \bar{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = \left|\frac{\bar{f_2}(z_1, z_2^0, z_3)}{f_2(z_1, z_2^0, z_3)}\right| = 1,$$

$$\left|\frac{p_3}{q_3}\right| = \left|\frac{z_1 z_2 z_3^0 f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = |z_1 z_2 z_3^0| \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = \left|\frac{f_2(\bar{z_1}, \bar{z_2}, \bar{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = \left|\frac{\bar{f_2}(z_1, z_2, z_3^0)}{f_2(z_1, z_2, z_3^0)}\right| = 1.$$

That is, moduli of the radial boundary values are almost everywhere on the distinguished boundary equal to 1, so functions

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$$

are inner by definition. □

To describe the valid denominators of q_i, we need the following

Definition 1.3 (see [9], Sec. 14, p. 125)**.** *Let C be a nonempty convex set. Then the closed convex set*

$$\hat{C} = \{x | \forall\, x^* \in C, \langle x, x^* \rangle \leqslant 1\},$$

is called the polar of the set C.

If the set C itself is closed and contains the origin, then it coincides with the polar of its polar set

$$\hat{\hat{C}} = C.$$

For more information about convex sets and other properties of the polar, see [9].

Let us find the polar of the cube $K = [-1, 1]^3$. For this we need to find all points that satisfy the equation

$$xx^* + yy^* + zz^* \leqslant 1 \ \ \forall\, x^* \in [-1, 1],\ y^* \in [-1, 1],\ z^* \in [-1, 1].$$

Since we compare everything with one in this equation, there is no point in checking the fulfillment of this inequality for intermediate values. Because if the inequality holds for boundary values, then it holds automatically for values inside the segment. Therefore, we have the system:

$$xx^* + yy^* + zz^* \leqslant 1 \quad \forall\, x^*, y^*, z^* \in \{-1, 1\},$$

which can be written as a single inequality $|x| + |y| + |z| \leqslant 1$ in the conjugate space.

We need the constructed polar to describe the valid denominators, namely

Theorem 1.3. *The denominators q_i are valid if and only if the pairwise products* $(ab, ac, bc) = (x, y, z)$ *lie in the polar and satisfy the system of inequalities:*

$$\begin{cases} 1 > \dfrac{yz}{x} > 0 \\ 1 > \dfrac{xz}{y} > 0 \\ 1 > \dfrac{xy}{z} > 0 \end{cases}.$$

Proof. If the denominators q_j are valid, then the polynomial $f_2 = abz_1 + acz_2 + bcz_3 + 1$ has no zeros inside the unit polydisk. And this is possible, as we have shown above, if and only if $|ab| + |ac| + |bc| \leqslant 1$, that is, when the pairwise products of (ab, ac, bc) lie in the polar. The system of inequalities arises from the following reasoning:

$$\begin{cases} a \in (-1,1)\backslash\{0\} \\ b \in (-1,1)\backslash\{0\} \\ c \in (-1,1)\backslash\{0\} \end{cases} \sim \begin{cases} 0 < a^2 < 1 \\ 0 < b^2 < 1 \\ 0 < c^2 < 1 \end{cases} \sim \begin{cases} 0 < \frac{bc\cdot a^2}{bc} < 1 \\ 0 < \frac{ac\cdot b^2}{ac} < 1 \\ 0 < \frac{ab\cdot c^2}{ab} < 1 \end{cases} \sim$$

$$\sim \begin{cases} 0 < \frac{ab\cdot ac}{bc} < 1 \\ 0 < \frac{ab\cdot bc}{ac} < 1 \\ 0 < \frac{ac\cdot bc}{ab} < 1 \end{cases} \sim \begin{cases} 0 < \frac{xy}{z} < 1 \\ 0 < \frac{xz}{y} < 1 \\ 0 < \frac{yz}{x} < 1 \end{cases}.$$

□

Let us visualize the resulting set. In Fig. 1, 2, we can see the points from the boundary of the polar satisfying the resulting system of inequalities. In the first figure, the edges of the polar set of the cube (of the regular octahedron) are highlighted in red, and the intersection is green.

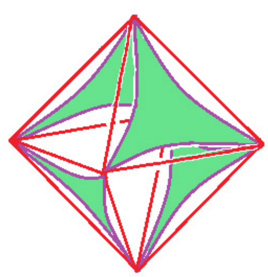

Fig. 1. Visual representation of the intersection of solutions of the system of inequalities and the boundary of the polar

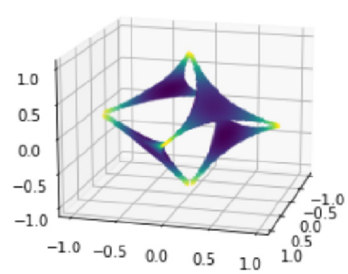

Fig. 2. Computer representation of the intersection of solutions of the system of inequalities and the boundary of the polar

2. The second approach to constructing an analogue of the Blaschke factors

In this section, when constructing a generalization of the Blaschke factors, we start from the form of this factor in the one-dimensional case. As we noted earlier, each factor $\dfrac{z_k - z}{1-\bar{z}_k z}$ of the product (1) is a rational function of the form:

$$b_k = \frac{p(z)}{q(z)} = z\frac{\overline{q(1/\bar{z})}}{q(z)},$$

where $q(z) = 1 - \bar{z}_k z$ is a polynomial of the first degree that has no zeros inside the unit disk D. In this case, when switching to the multidimensional case, we can take as such a polynomial the following one

$$q(z_1, \dots, z_n) = 1 + \zeta_1 z_1 + \ldots + \zeta_n z_n,$$

where $(\zeta_1, \dots, \zeta_n) \in \{(z_1, \dots, z_n) \in D^n : |z_1| + \ldots + |z_n| \leqslant 1\}$. With such restrictions on the coefficients of a given linear polynomial, it will not have zeros inside the unit polydisk D^n. Note that these constraints are consistent with the one-dimensional case. As a numerator $p(z_1, \dots, z_n)$ we can take the following polynomial, which also agrees with the case of one complex variable:

$$p(z_1, \dots, z_n) = z_1 \cdot \ldots \cdot z_n \cdot \overline{q(1/\bar{z}_1, \dots, 1/\bar{z}_n)}.$$

Now fix an arbitrary point $(z_1^0, \dots, z_n^0)$ from the distinguished boundary

$$\Delta = \{|z_j| = 1, j = 1, \dots, n\}$$

of the polydisk D^n and consider the following set of functions:

$$p_1 = p(z_1^0, \dots, z_n), \quad \ldots, \quad p_n = p(z_1, \dots, z_n^0),$$
$$q_1 = q(z_1^0, \dots, z_n), \quad \ldots, \quad q_n = q(z_1, \dots, z_n^0).$$

Definition 2.1. *The map* $\left(\dfrac{p_1}{q_1}, \dots, \dfrac{p_n}{q_n}\right)$ *is called the multidimensional analogue of the Blaschke factor.*

Theorem 2.1. *The functions p_j/q_j in the definition of the analogue of the Blaschke factor are inner rational functions in the polydisk D^n.*

Proof. Since $q_1 = q(z_1^0, \dots, z_n)$, the zeros of the denominator q_1 on the unit distinguished boundary are also zeros of the polynomial q. It can be noted that for $|\zeta_1| + \ldots + |\zeta_n| < 1$ the polynomial q has no zeros in the closure of the unit polydisk $\bar{D}^n$, but then the denominators q_j have no zeros in the same closure. For $|\zeta_1| + \ldots + |\zeta_n| = 1$, the polynomial q has a single zero on the distinguished boundary $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ and has no zeros inside the polydisk. In this case, the denominator q_i has a single zero on the distinguished boundary if $\hat{z}_i = z_i^0$; otherwise, q_i does not vanish in the closure of a single polydisk. Thus, q_i vanish at no more than one point from the distinguished boundary. Therefore, almost everywhere on T^n we have

$$p_1 = p(z_1^0, z_2, \dots, z_n) = z_1^0 z_2 \cdot \ldots \cdot z_n + \bar{\zeta}_1 z_2 \ldots z_n + \ldots + \bar{\zeta}_n z_1^0 z_2 \ldots z_{n-1} =$$
$$= z_1^0 \cdot \ldots \cdot z_n \left(\frac{\bar{\zeta}_1}{z_1^0} + \ldots + \frac{\bar{\zeta}_n}{z_n} + 1\right) = z_1^0 \cdot \ldots \cdot z_n \cdot \bar{q}\left(\frac{1}{\bar{z}_1^0}, \dots, \frac{1}{\bar{z}_n}\right).$$

From this equality we obtain the following chain of equalities for the module of the function $\dfrac{p_1}{q_1}$, which holds almost everywhere on the distinguished boundary:

$$\left|\frac{p_1}{q_1}\right| = \left|\frac{z_1^0 \cdot \ldots \cdot z_n \cdot \bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| = |z_1^0 \cdot \ldots \cdot z_n| \left|\frac{\bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| =$$

$$= \left|\frac{\bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| = \left|\frac{\bar{q}(z_1^0, \ldots, z_n)}{q(z_1^0, \ldots, z_n)}\right| = 1.$$

That is, the module of the radial boundary values are almost everywhere on the distinguished boundary equal to 1, so the function $\dfrac{p_1}{q_1}$ is inner by definition. Having carried out similar reasoning for the remaining rational functions

$$\frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n},$$

we obtain the statement of the theorem. □

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О множителях Бляшке в полидиске

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Аннотация. Цель настоящей работы состоит в построении многомерного аналога множителя Бляшке. На актуальность построения данного аналога нас натолкнула недавняя совместная статья Алпая и Ижера, которая посвящена многомерной интерполяционной теории для функциональных пространств в специальных полиэдрах Вейля. Под таким множителем мы будем понимать набор специальных внутренних рациональных функций в единичном поликруге. Построение внутренних рациональных функций для случая трех комплексных переменных произведем, в частности, с помощью многочлена Ли-Янга из теории фазовых переходов статистической механики.

Ключевые слова: произведение Бляшке, многочлен Ли-Янга.

EDN: RXMMFV
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Green's Function on a Parabolic Analytic Surface

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Abstract. The class of plurisubharmonic functions on a complex parabolic surface is considered in this paper. The concepts of the Green's function and pluripolar set are also introduced and their potential properties are studied.

Keywords: parabolic manifold, parabolic surface, regular parabolic surface, Green's function, pluripolar set.

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1. Introduction and preliminaries

This paper is devoted to plurisubharmonic (psh) functions on a complex parabolic manifold and surfaces embedded in the space $\mathbb{C}^N$. The concepts of Green's function and pluripolar set are introduced and a number of their potential properties are studied.

Parabolic manifolds presumably were considered for the first time by P. Griffiths, J. King [1] and by W. Stoll [2, 3]. They were used in the construction of the multidimensional Nevanlinna theory for holomorphic map $f: X \to P$, where X is a parabolic manifold $\dim X = n$, and P is a compact Hermitian manifold, $\dim P = m$. Various types of parabolicity were classified by A. Aytuna and A. Sadullaev [4–6].

Definition 1. *A Stein manifold $X \subset \mathbb{C}^N$, $\dim X = n$ is called parabolic if it does not contain different from a constant plurisubharmonic function bounded from above, i.e., if $u(z)$ is plurisubharmonic on X and $u(z) \leqslant C$ then $u(z) \equiv const$.*

It is called S-parabolic manifold if it contains a special exhaustion function $\rho(z)$ that satisfies the following conditions

a) $\rho(z) \in psh(X), \{\rho \leqslant c\} \subset\subset X \ \forall c \in \mathbb{R}$;

b) $(dd^c\rho)^n = 0$ outside some compact $K \subset\subset X$, i.e., function ρ is maximal function on $X \backslash K$.

X is called S^*-parabolic if there is a continuous special exhaustion function $\rho(z)$ on it.

It is clear that S^*-parabolic manifold is S-parabolic and, in turn, it is easy to prove that S-parabolic manifold is parabolic. It was noted [5, 6] that for $n = 1$ all these 3 concepts coincide (see [7]). However, for $n > 1$ the equivalence of these three definitions is still an open problem.

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The purpose of this paper is to study holomorphic and plurisubharmonic functions on an analytic surface. Concepts of parabolic surfaces $X \subset \mathbb{C}^N$, plurisubharmonic functions and Green function on them are introduced (Section 2). A series of properties of plurisubharmonic on X functions are proved. Some of these properties are non-trivial due to the presence of singular (critical) points of X. When proving properties in neighbourhoods of such points, the local principle of analytic covering is used. In Section 3, the concepts of polynomials are defined, the class of polynomials on parabolic surfaces is studied, and a number of examples of surfaces of this type are given. Theorem 3.1 states that complement $X = \mathbb{C}^N \backslash A$ of an arbitrary pure $(n-1)$-dimensional algebraic set $A = \{p(z) = 0\} \subset \mathbb{C}^N$ is *regular* S^*-parabolic surface.

2. Parabolic surfaces

In this section, parabolic surfaces, their classification and the Green's functions on them are we studied.

2.1. Plurisubharmonic functions on analytic surfaces.

Let us consider an analytic surface, i.e., an irreducible analytic set X, $\dim X = n$ embedded in a complex space $\mathbb{C}^N$, $X \subset \mathbb{C}^N$ such that for any ball $B(0,r) \subset \mathbb{C}^N$ the intersection $X \cap B(0,r)$ lies compactly on X, $X \cap B(0,r) \subset\subset X$. To define plurisubharmonic functions on X we denote the set of regular points of the set X by the $X^0 \subset X$. Then the set of critical (singular) points $X\backslash X^0$ is an analytic set of lower dimension $\dim X\backslash X^0 < n$. Set $X\backslash X^0$ does not split X, and set X^0 is a complex n dimensional submanifold in $\mathbb{C}^N$ (see [8,9]).

Definition 2 ([10]). *Function $u(z)$ defined in a domain $D \subset X$ is called plurisubharmonic (psh) in D if it is locally bounded from above in this domain and plurisubharmonic on the manifold $D \cap X^0$, $u(z) \in psh(D \cap X^0)$.*

The class of plurisubharmonic functions in D is denoted by $psh(D)$. In practice, at critical points $z \in X\backslash X^0$ function $u^*(z) = \varlimsup\limits_{\substack{w \to z \\ w \in X^0 \cap D}} u(w)$, $z \in D$ is usually considered, and in studies of plurisubharmonic functions $u^*(z)$ is studied. Function $u^*(z)$ is assumed to be upper semicontinuous in D, the set $\{z \in D : \ u^*(z) < C\}$ is open for all $C \in \mathbb{R}$ and $u^*(z) = u(z)$, $\forall\, z \in X^0 \cap D$.

Let us consider several properties of plurisubharmonic functions on X that are needed below.

1) *A linear combination of finite number of plurisubharmonic functions in $D \subset X$ with positive coefficients is a plurisubharmonic function, i.e., if $u_j^*(z) \in psh(D)$, $\alpha_j \geqslant 0$, $\quad j = 1, 2, \ldots, s$ then*

$$\alpha_1 u_1^*(z) + \ldots + \alpha_s u_s^*(z) \in psh(D).$$

2) *The uniform limit or the limit of a monotonically decreasing sequence $\{u_j^*(z)\}$ of plurisubharmonic functions is plurisubharmonic function, i.e., if $u_j^*(z) \in psh(D)$, $\quad j = 1, 2, \ldots$, $u_j^*(z) \rightrightarrows u^*(z)$ or if $u_j^*(z) \searrow u^*(z)$ then $u^*(z) \in psh(D)$.*

The following property is not trivial due to the presence of critical points on the surface X.

3)**Maximum principle.** *For $u^*(z) \in psh(D)$, $D \subset X$ the maximum principle holds, i.e., if at some interior point $z^0 \in D$ the value $u^*(z^0) = \sup\limits_D u^*(z)$ then $u^* \equiv const$.*

Now let us assume that u^* has a maximum at an interior point $z^0 \in D$ (without loss of generality one can assume $z^0 = 0$) and $u^*(0) \geqslant u^*(z)$ $\forall z \in D$. If $0 \in D$ is regular point then it is obvious that $u(z) \equiv const$ in $D \backslash S$, where $S = X \backslash X^0$, because for plurisubharmonic functions on a manifold the maximum principle is valid. Therefore, $u^*(z) \equiv const$ in D. If 0 is a critical point then there is a complex plane $0 \in \Pi \subset \mathbb{C}^N$ such that $\dim \Pi = N - n$, $X \cap \Pi$ is discrete. Hence, there exists a ball $B(''0, r) \subset \Pi$, $r > 0$ such that

$$X \cap B(''0, r) = \{0\}, \; X \cap \partial B(''0, r) = \emptyset. \tag{1}$$

Let us set $z = ('z, ''z)$, $'z = (z_1, \ldots, z_n)$, $''z = (z_{n+1}, \ldots, z_N)$. Let $\Pi = \{\, 'z = (z_1, \ldots, z_n) = 0 \,\}$. Since X is closed then according to (1), there exists a neighbourhood $'U \ni {'0} : X \cap \partial B('z, r) = \emptyset$, $\forall\, 'z \in {'U}$. Therefore, $\pi : X \cap [{'U} \times {''U}] \to {'U}$ is a k-sheeted analytic covering, $1 \leqslant k < \infty$.

Let $J \subset {'U}$ be the set of critical points of this covering. It means that

$$\pi \; : \; \{X \cap [\,{'U} \times {''U}]\} \backslash \pi^{-1}(J) \; \to \; {'U} \backslash J$$

is a regular k-sheeted covering

$$\pi^{-1}('z) \cap \{X \cap [\,{'U} \times {''U}]\} \backslash \pi^{-1}(J) \; = \{\alpha_1\,('z)\,, \ldots, \alpha_k\,('z)\,\} \;\; \forall\, 'z \in {'U} \backslash J. \tag{2}$$

Moreover, for each point $'z^0 \in {'U} \backslash J$ locally, in some neighbourhood of $W \ni {'z^0}$, the inverse-image $\pi^{-1}(W) \cap X \cap [{'U} \times {''U}]$ is split into k pieces of disjoint complex manifolds $M_1, M_2, \ldots, M_k$ (see, for example, [8,11]). Function $u^*(z) = u(z)$ is plurisubharmonic function on every piece of manifolds M_j, $j = 1, 2, \ldots, k$.

It follows from (2) that $w('z) = \sum\limits_{j=1}^{k} u^*(\alpha_j('z))$ is plurisubharmonic function in $'U \backslash J$ locally bounded in $'U$. Since $J \subset {'U}$ is an analytic set then $w('z)$ is plurisubharmonically extended to $'U$. (Recall that if $w('z) \in psh(D \backslash P)$ is locally bounded in D, P is closed pluripolar set then $w('z)$ is plurisubharmonically extended to D (see [12] and also [13,14]).

Thus, $w('z) \in psh('U)$ and by assumption it reaches its maximum at the point $0 \in U$. This is a contradiction. □

2.2. Holomorphic functions

It is convenient for us to define holomorphic functions on an analytic surface X in the sense of H. Cartan [9].

Definition 3. *Function $f(z)$ defined in a domain $D \subset X$ is called holomorphic in D, if:*
a) it is holomorphic on the manifold $D \cap X^0$;
b) it is locally bounded in D, i.e., for each point $z^0 \in D$ there exists a neighborhood $W \ni z^0$, $W \subset D$ such that $|f(z)| \leqslant const \quad, \forall z \in W \cap X^0$.

The class of holomorphic functions in D is denoted by $\mathcal{O}(D)$. Holomorphic functions in space X have many properties of holomorphic functions of several complex variables. In particular, a linear combination of holomorphic functions with constant coefficients is holomorphic function. In other words, if $f_1, \ldots, f_m \in \mathcal{O}(D)$ then $c_1 f_1 + \ldots + c_m f_m \in \mathcal{O}(D)$; the product of two holomorphic functions is also holomorphic function, i.e., if $f, g \in \mathcal{O}(D)$ then $f \cdot g \in \mathcal{O}(D)$. In addition, the theorem of uniqueness holds, i.e., if $f \in \mathcal{O}(D)$ and $f \equiv 0$ in some non-empty neighbourhood of $W \subset D$ then $f \equiv 0$ in domain $D \subset X$. Since holomorphic functions are defined only at regular points then $f \equiv 0$ in some neighbourhood $W \subset X$ means that $f \equiv 0 \; \forall z \in W \cap X^0$.

The following theorem of H. Cartan is very useful in the study of holomorphic functions.

Theorem 2.1 (H. Cartan [9]). *Let function $f(z)$ defined in a domain $D \subset X$ is continuous on $D \cap X^0$ and has the property that for each point $z^0 \in D$ there exist a neighbourhood $W \ni z^0$, $W \subset D$, and holomorphic in W functions $g_1, \ldots, g_m \in \mathcal{O}(W) : f^m(z) + g_1(z) f^{m-1}(z) + \ldots + g_m(z) = 0$ $\forall\, z \in W \cap X^0$. Then $f(z) \in \mathcal{O}(D)$.*

There is an intimate connection between holomorphic and plurisubharmonic functions.

Theorem 2.2. *If $f(z) \in \mathcal{O}(D)$ then function $u(z) = \ln |f(z)|$ is plurisubharmonic in domain D, $u(z) \in psh(D)$.*

2.3. S-parabolic analytic surfaces

Let $X \subset \mathbb{C}^N$ be an analytic surface embedded in $\mathbb{C}^N$, i.e., X is an irreducible analytic set in $\mathbb{C}^N$ for which the intersections $B(0,r) \cap X \subset\subset X$, $\forall\, r > 0$. The concept of parabolicity of surface X is introduced similarly to the parabolicity of manifolds.

Definition 4. *An analytic surface X is called parabolic if it does not contain a bounded plurisubharmonic function that is different from a constant.*

Analytic surface X is called S-parabolic if it has a special exhaustion function $\rho(z)$ satisfying the following conditions

a) $\rho(z) \in psh(X), \{\rho \leqslant c\} \subset\subset X \; \forall\, c \in \mathbb{R}$;

b) function ρ^* is a maximal function on $X \backslash K$ for some compact set $K \subset\subset X$. This is equivalent to $(dd^c \rho^*)^n = 0$ on $X^0 \backslash K$ (see [15]).

Analytic surface X is called S^*-parabolic if there exists a continuous special exhaustion function $\rho(z) \in C(X^0)$.

It is clear that S^*-parabolic analytic surface is S-parabolic. As we noted above, the converse assumption remains open even for a complex manifold.

The main result of Section 2 is the following theorem.

Theorem 2.3. *S-parabolic surface X is parabolic, i.e., on the S-parabolic surface X there is no bounded from above plurisubharmonic function $u^*(z)$ different from a constant.*

Proof. Let X be a S-parabolic analytic surface with special exhaustion function $\rho(z) \in psh(X)$, and ρ is a maximal plurisubharmonic function on $X \backslash K$, where $K \subset\subset X$ is some compact set. Suppose that there exists function $u(z) \in psh(X)$, $u(z) \leqslant M$ but $u(z) \not\equiv const$. Consider a ball $B_r = \{z \in X : \rho(z) < \ln r\} \subset\subset X$. Let us put $\rho_r = \max\limits_{\overline{B}_r} \rho(z)$, $u_r = \max\limits_{\overline{B}_r} u^*(z)$, $u_r \leqslant M$. Let us fix the numbers $r < r' < R < \infty$, $B_r \supset K$. Then for P-measure (see [15]) we have

$$\omega^*(z, \overline{B}_r, B_R) = \frac{\rho(z) - \rho_R}{\rho_R - \rho_r}. \tag{3}$$

Let us note that $u^*(z) \leqslant u_r$, $z \in \overline{B}_r$ and $u^*(z) \leqslant u_R$, $z \in \overline{B}_R$. Therefore, by the theorem on two constants [15] we have

$$u^*(z) \leqslant u_R \cdot (1 + \omega^*(z, \overline{B}_r, B_R)) - u_r \cdot \omega^*(z, \overline{B}_r, B_R).$$

Substituting (3) into the last inequality, we obtain for $z \in \overline{B}_r$

$$u_{r'} \leqslant \Big(1 + \frac{\rho_{r'} - \rho_R}{\rho_R - \rho_r}\Big) u_R - \frac{\rho_{r'} - \rho_R}{\rho_R - \rho_r} u_r.$$

Since function $u(z)$ is bounded on X then $u_R \leqslant M$, and when $R \to \infty$ we have $u_{r'} \leqslant u_r$. Hence, according to the maximum principle, $u^*(z) \equiv const$ in the ball B_r. Since $r < \infty$ is an arbitrary fixed number then $u^*(z) \equiv const$ on X. Theorem 2.3 is proved. □

2.4. Green's function on parabolic surfaces

In this subsection the Green's function on S-parabolic analytic surfaces is introduced.

Let (X,ρ) be a S-parabolic surface. Let us denote the class of plurisubharmonic functions $u \in psh(X)$ satisfying the condition

$$u(z) \leqslant c_u + \rho^+(z),\ z \in X,$$

by the $\mathfrak{A}_\rho(X)$. Here c_u is some constant that depends on function u and $\rho^+(z) = \max\{0,\ \rho(z)\}$. Class $\mathfrak{A}_\rho(X)$ is called the Lelong class of plurisubharmonic functions on X. For a fixed compact set $K \subset\subset X$, we define

$$V_\rho(z,K) = \sup\{u(z) :\ u \in \mathfrak{A}_\rho(X),\ u|_K \leqslant 0\}.$$

Then the regularization $V^*_\rho(z,K) = \overline{\lim_{w\to z}} V_\rho(w,K)$ is called ρ-*Green's function* of the compact $K \subset\subset X$.

Similarly, in the classical case there are

1. *Either* $V_\rho \in \mathfrak{A}_\rho(X)$ or $V_\rho \equiv +\infty$. $V_\rho(z,K) \equiv +\infty$ *if and only if* K *is pluripolar set on* X, *i.e., there exists a function* $u^* \in psh(X) :\ u^* \not\equiv -\infty,\ u^*(z) = -\infty\, \forall\, z \in K$.

2. Let $K \subset\subset X$ *be a non-pluripolar compact set. Then the Green's function* $V_\rho(z,K)$ *is maximal in* $X\backslash K$. *In particular,* $[dd^c V_\rho(z,K)]^n = 0$ *on the complex manifold* $X^0\backslash K$.

The proofs of these important properties of the $\rho-$ Green's function are identical to the proofs of the corresponding properties of the Green's function in space $\mathbb{C}^n$, and they are omitted.

Definition 5. *A compact set* $K \subset X$ *is called regular at a point* $z^0 \in X$ *if* $V^*_\rho(z^0,\ K) = 0$. *If all points of* $K \subset X$ *are regular then compact* $K \subset X$ *is called regular.*

Note that if compact set $K \subset X$ is regular then the open set $G_\varepsilon = \left\{z \in X : V^*_\rho(z,K) < \varepsilon\right\}$ contains K, $G_\varepsilon \supset K$.

2.5. Regular parabolic surfaces

2.5.1. Polynomials on parabolic analytic surfaces

Let $X \subset \mathbb{C}^N$ be a S-parabolic surface and $\rho(z)$ is a special exhaustion function.

Definition 6. *If function* $f \in \mathcal{O}(X)$ *satisfies the inequality*

$$\ln|f(z)| \leqslant d\rho^+(z) + c_f\ \forall\, z \in X, \tag{4}$$

where c_f *and* d *are positive real numbers (constant) then* f *is called the* ρ*-polynomial. The smallest value* d *that satisfies condition (4) is called the degree of polynomial* f.

Let us denote the set of all ρ-polynomials of degree less than or equal to d by $\mathcal{P}^d_\rho(X)$ and the union $\mathcal{P}_\rho(X) = \bigcup\limits_{d\geqslant 1} \mathcal{P}^d_\rho(X)$ by $\mathcal{P}_\rho(X)$. Then it is easy to prove (see [6,16]) that $\mathcal{P}^d_\rho(X)$ is a linear space of finite dimension $\dim \mathcal{P}^d_\rho(X) \leqslant C\,(d+1)^n$.

However, a parabolic manifold was constructed [6] where there are no non-trivial polynomials, i.e., any polynomial $P(z)$ on X is equal to a constant, $\mathcal{P}_\rho \simeq \mathbb{C}$.

Definition 7. *If space of all* ρ*-polynomials* $\mathcal{P}_\rho(X) = \bigcup\limits_{d\geqslant 1} \mathcal{P}^d_\rho(X)$ *is dense in space* $\mathcal{O}(X)$ *then* S*-parabolic surface* X *is called regular.*

2.6. Examples

Example 1. *Let $A \subset \mathbb{C}^N$ be irreducible, n dimensional, $\dim A = n$, $n < N$ algebraic set. According to the well-known criterion of W. Rudin [17] (see also [18]) and after corresponding linear transformation, algebraic set A can be included in a special cone*

$$A \subset \{w = ('w,'' w) = (w_1, \ldots, w_n, w_{n+1}, \ldots, w_N) : \ \|''w\| < C\,(1 + \|'w\|)\},$$

where C is constant.

Let us consider projection $\pi('w,'' w) = 'w : A \to \mathbb{C}^n$. If $('w^0, ''w^0)$ is a regular point of A, i.e., $('w^0, ''w^0) \in A^0$ then in some neighbourhood $U \ni ('w^0, ''w^0)$, $U \subset A^0$ projection $\pi : U \to \mathbb{C}^n$ is biholomorphic. Consequently, restriction $\rho|_A$ of the plurisubharmonic in $\mathbb{C}^N$ function $\rho(w) = \ln \|'w\|$ is plurisubharmonic function in a neighbourhood of $U \ni ('w^0, ''w^0)$. Since point $('w^0, ''w^0) \in A^0$ is arbitrary restriction of $\rho|_A$ is a plurisubharmonic function in A^0. In addition, it is locally bounded from above on A and, therefore, $\rho|_A \in psh(A)$.

It is clear that $\rho|_A$ is special exhaustion function on A and restriction on A of polynomials $p('w, ''w)$ from $\mathbb{C}^N$ are polynomials on A. This implies that set of polynomials $\mathcal{P}_\rho(A)$ is dense in $\mathcal{O}(A)$, i.e., affine-algebraic surface is regular parabolic surface.

Example 2. *Let $A = \{\Phi(z) = 0\} \subset \mathbb{C}^n$ be a pure $(n-1)$ dimensional analytic surface such that*

$$A \subset \{z = ('z, z_n) \in \mathbb{C}^n : |z_n| < \varphi('z)\},$$

where $'z = (z_1, \ldots, z_{n-1})$, $\varphi('z)$ is a locally bounded positive function. Then A is S^*-parabolic surface.

Let us consider projection $\pi('z, z_n) = 'z : A \to \mathbb{C}^{n-1}$. For each fixed point $'z^0 \in \mathbb{C}^{n-1}$ intersection $\{'z = 'z^0\} \cap A = \pi^{-1}\{'z^0\}$ consists of a finite number of points $\{'z = 'z^0\} \cap A = (\alpha_1('z^0), \ldots, \alpha_m('z^0))$ as a compact analytic set in plane $\mathbb{C}_{'z^0}$. Function $\Phi('z,\, z_n) \neq 0$ on the boundary of circle $\{|z_n| = \varphi('z)\}$. According to the argument principle, the number of zeros (taking into account multiplicities)

$$N('z) = \frac{1}{2\pi i} \int\limits_{|z_n| = \varphi('z^0)} \frac{\Phi'('z,\, z_n)}{\Phi('z,\, z_n)}\, dz_n, \quad 'z \in U,$$

as a continuous integer function is constant, $N('z) \equiv m$ and $\pi^{-1}('z) = (\alpha_1('z), \ldots, \alpha_m('z))$, $'z \in \mathbb{C}^{n-1}$. Moreover, function

$$F('z, z_n) = \prod_{k=1}^{m} (z_n - \alpha_k('z)) = z_n^m + f_{m-1}('z) z_n^{m-1} + \ldots + f_0('z)$$

is an entire function, where $f_k('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, 1, \ldots, m-1$.

If A is not an algebraic set then function $F('z, z_n)$ is not a polynomial, i.e., not all functions $f_k('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, 1, \ldots, m-1$ are polynomials. As in Example 1, contraction $\rho|_A$ of plurisubharmonic function $\rho(z) = \ln \|'z\|$ from $\mathbb{C}^N$ is special exhaustion function on A, i.e., surface A is parabolic. However, here restrictions of polynomials $P(z)$ in $\mathbb{C}^n$ on A are not, in general, $\rho|_A$ polynomials.

It was proved ([5], see also [19]) that $X = \mathbb{C}^n \backslash A$ complement of zeros of the Weierstrass polynomial $A = \{z_n^m + f_1('z) z_n^{m-1} + \ldots + f_m('z) = 0\}$, where $f_1('z), \ldots, f_m('z)$ are entire functions, is S^*-parabolic manifold with special exhaustion function

$$\rho(z) = \frac{1}{2}\ln\left(|'z|^2 + \left|F(z) + \frac{1}{F(z)}\right|^2\right),$$

where $F('z, z_n) = z_n^m + f_{m-1}('z)z_n^{m-1} + \ldots + f_0('z)$. However, $X = \mathbb{C}^n \backslash A$ is not always regular (see [19]). The main result of this section is

Theorem 2.4. *The complement $X = \mathbb{C}^n \backslash A$ of an arbitrary pure $(n-1)$ dimensional algebraic set $A = \{p(z) = 0\} \subset \mathbb{C}^n$ is regular S^*-parabolic manifold. If $p(0) \neq 0$ then function*

$$\rho(z) = -\frac{1}{\deg p}\ln|p(z)| + 2\ln\|z\| \tag{5}$$

is special exhaustion function on X.

The theorem is proved in several steps.

Step 1. Let us show that $\rho(z)$ from (5) is special exhaustion function. In fact, function $-\frac{1}{\deg p}\ln|p(z)|$ is pluriharmonic in X and function $2\ln\|z\|$ is maximal in $X\backslash\{0\}$. Therefore, function $\rho(z)$ is the maximal function in $X\backslash\{0\}$. In addition, since $p(0) \neq 0$ then $\{z \in X : \rho(z) < C\} \subset\subset X\ \forall C > 0$.

Step 2. Using the criterion of W. Rudin [17] and after the corresponding linear transformation of space $\mathbb{C}^n$, A is reduced into special form (see Example 1)

$$A \subset \{z = ('z, z_n) \in \mathbb{C}^n : |z_n| < C(1 + \|'z\|)\}, \quad C - const. \tag{6}$$

Then A has the form

$$A = \left\{p(z) = z_n^m + e_1('z)z_n^{m-1} + \ldots + e_m('z) = 0\right\},$$

where $m = \deg p > 1$, $e_1('z), \ldots, e_m('z)$ are polynomials and $p(0) \neq 0$.

Step 3. The expansion of holomorphic functions in $X = \mathbb{C}^n \backslash A$ into Jacobi–Hartogs series is used. First, let us consider some insights on the theory of Jacobi series ([20], see also [21]). Let $p(z) = z_n^m + e_1 z_n^{m-1} + \ldots + e_m$, $m > 1$ and $e_1, \ldots, e_m$ are constants. Let us denote the lemniscate ring $\{z \in \mathbb{C} : r < |p(z)| < R\}$ by $G_{r,R}$. If function $f(z)$ is holomorphic in some neighbourhood $\overline{G}_{r,R}$ then function of two variables

$$F(z, w) = \frac{1}{2\pi i}\int\limits_{\partial G_{r,R}} \frac{f(\xi)}{p(\xi) - w} \cdot \frac{p(\xi) - p(z)}{\xi - z} d\xi$$

is holomorphic in domain $G_{r,R} \times \{r < |w| < R\}$. According to the Cauchy integral formula, the equality $F(z, p(z)) \equiv f(z)$ $(z \in G_{r,R})$ takes place. The expansion of function $F(z, w)$ into Hartogs–Laurent series (see [11]) with respect to the variable w is

$$F(z, w) = \sum_{k=-\infty}^{\infty} c_k(z) w^k, \tag{7}$$

where

$$c_k(z) = \frac{1}{2\pi i}\int\limits_{|p(\xi)|=r_1} f(\xi)\frac{p(\xi) - p(z)}{p^{k+1}(\xi)(\xi - z)} d\xi, \qquad (r < r_1 < R,\quad z \in G_{r,R},\ k = 0, \pm 1, \pm 2, \ldots). \tag{8}$$

Series (7) converges uniformly inside domain $G_{r,R} \times \{r < |w| < R\}$. If we put $w = p(z)$ then we obtain the series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z) p^k(z), \quad z \in G_{r,R},$$

which is called the Jacobi–Hartogs series of function $f(z)$. It converges uniformly inside domain $G_{r,R}$. One can be see from (8) that coefficients $c_k(z)$ are polynomials of degree $\deg c_k(z) \leqslant m-1$.

It follows that if function $f(z)$ is holomorphic in $G_{0,\infty}$ then it is expanded into the series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z) p^k(z),$$

which converges uniformly inside $G_{0,\infty}$. Here $c_k(z)$ are polynomials of degree $\deg c_k(z) \leqslant m-1$,

$$c_k(z) = \frac{1}{2\pi i} \int\limits_{|p(\xi)|=r} f(\xi) \frac{p(\xi) - p(z)}{p^{k+1}(\xi)\,(\xi - z)} d\xi \quad (0 < r < \infty, \ \ z \in G_{0,\infty}),$$

and the Cauchy inequality holds:

$$|c_k(z)| \leqslant \frac{\max\{|f(\xi)| : \ |p(\xi)| = r\}}{2\pi r^{k+1}} \int\limits_{|p(\xi)|=r} \left| \frac{p(\xi) - p(z)}{\xi - z} \right| |d\xi|, \ \ k = 0, \pm 1, \pm 2, \dots \tag{9}$$

Step 4. Let us apply the Jacobi–Hartogs series to the holomorphic function $f\left('z, z_n\right) \in \mathcal{O}(X)$ outside the algebraic set $A = \left\{p(z) = z_n^m + e_1('z)\, z_n^{m-1} + \ldots + e_m('z) = 0\right\}$. We fix $'z \in \mathbb{C}^{n-1}$ and expand function $f\left('z, z_n\right)$ in the Jacobi–Hartogs–Laurent series:

$$f('z, z_n) = \sum_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n) \tag{10}$$

where coefficients

$$c_k('z, z_n) = \frac{1}{2\pi i} \int\limits_{|p('z,\xi_n)|=r} f('z, \xi_n) \cdot \frac{p('z, \xi_n) - p('z, z_n)}{p^{k+1}('z, \xi_n)(\xi_n - z_n)} d\xi_n$$

are polynomials in variable z_n with holomorphic $\mathbb{C}^{n-1}$ coefficients

$$c_k('z, z_n) = a_{k,m-1}('z)\, z_n^{m-1} + \ldots + a_{k,0}('z), \ a_{k,j}('z) \in \mathcal{O}\left(\mathbb{C}^{n-1}\right), \ j = 0, 1, \ldots, m-1.$$

Series (10) converges uniformly inside domain

$$X = \{('z, z_n) \in \mathbb{C}^n : \quad 0 < |p('z, z_n)| < \infty\}.$$

Step 5. Let us show that rational functions of the form $\dfrac{q(z)}{p^k(z)}$, where $q(z)$ is a polynomial in $\mathbb{C}^n$, $k \geqslant 0$ are integer functions, and only they are ρ-polynomials in X, where $\rho(z) = = -\dfrac{1}{\deg p} \ln|p(z)| + 2\ln\|z\|$. In fact, since

$$\ln\left|\frac{q(z)}{p^k(z)}\right| = -k\ln|p(z)| + \ln|q(z)| \leqslant \max\{k, \deg q\}\, \rho^+(z) + const$$

then function $\dfrac{q(z)}{p^k(z)}$ is $\rho(z)$-polynomial in $X = \mathbb{C}^n \backslash A$.

On the contrary, if $P(z)$ is a $\rho(z)$-polynomial in $X = \mathbb{C}^n \backslash A$, then according to (4)

$$\ln|P(z)| \leqslant d\rho^+(z) + c \ \ \forall\, z \in X, \ \ d, c - const.$$

Let us expand $P(z)$ into the Jacobi–Hartogs–Laurent series (10)

$$f('z, z_n) = \sum_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n),$$

with coefficients

$$c_k('z, z_n) = \frac{1}{2\pi i} \int\limits_{|p('z,\xi_n)|=r} P('z, \xi_n) \cdot \frac{p('z,\xi_n) - p('z, z_n)}{p^{k+1}('z,\xi_n)(\xi_n - z_n)} d\xi_n.$$

According to (9), we have the following estimate

$$|c_k('z, z_n)| \leqslant \frac{\max\left\{|P('z,\xi_n)| : \ |p('z,\xi_n)| = r\right\}}{2\pi r^{k+1}} \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p('z,\xi_n) - p('z,z_n)}{\xi_n - z_n} \right| |d\xi_n| \leqslant$$
$$\leqslant \frac{\max\left\{\exp\left[c + d\rho^+('z,\xi_n)\right] : \ |p('z,\xi_n)| = r\right\}}{2\pi r^{k+1}} \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p('z,\xi_n) - p('z,z_n)}{\xi_n - z_n} \right| |d\xi_n| .$$

Substituting $\rho(z) = -\dfrac{1}{\deg p} \ln|p(z)| + 2\ln\|z\|$, we obtain

$$|c_k('z, z_n)| \leqslant \frac{\max\left\{\exp\left[c + d\left(-\frac{1}{m}\ln|p('z,\xi_n)| + 2\ln\|('z,\xi_n)\|\right)^+\right] : \ |p('z,\xi_n)| = r\right\}}{2\pi r^{k+1}} \times$$
$$\times \int\limits_{|p('z,\xi_n)|=r} \left| \frac{p('z,\xi_n) - p('z,z_n)}{\xi_n - z_n} \right| |d\xi_n| . \tag{11}$$

However,

$$\begin{aligned} &\max\left\{\exp\left[c + d\left(-\frac{1}{\deg p}\ln|p(z)| + 2\ln\|z\|\right)^+\right] : \ |p('z,\xi_n)| = r\right\} \leqslant \\ &\leqslant \exp\left[c + d\left(-\frac{\ln r}{m} + \ln\|('z,\xi_n)\|^2_{|p('z,\xi_n)|=r}\right)^+\right] \leqslant \\ &\leqslant \exp\left[c + d\left(-\frac{\ln r}{m} + \ln\left(r^2 + C_1\left(1 + \|'z\|^2\right)\right)\right)^+\right] \leqslant C_2 \begin{cases} \left(r^2 + \|'z\|^2\right)^d & \text{if } r \to \infty \\ \|'z\|^{2d} \cdot r^{-d/m} & \text{if } r \to 0 \end{cases} \end{aligned} \tag{12}$$

Here the estimate

$$\|('z,\xi_n)\|^2_{|p('z,\xi_n)|=r} \leqslant \left[\|'z\|^2 + |\xi_n|^2\right]_{|p('z,\xi_n)|=r} \leqslant \|'z\|^2 + r^2 + C^2\left(1 + \|'z\|\right)^2,$$

is used which is easy to obtain by applying relation (6). The integral in (11) is estimated as (see [19] Lemma 4.1)

$$\int\limits_{|p('z,\xi_n)|=r} \left| \frac{p('z,\xi_n) - p('z,z_n)}{\xi_n - z_n} \right| |d\xi_n| \leqslant C_3 r, \ \ |p('z,z_n)| \leqslant r, \ C_3 - const. \tag{13}$$

Thus, substituting (12), (13) into (11), we obtain the final estimate

$$|c_k\left('z, z_n\right)| \leqslant \frac{C_4}{r^k} \begin{cases} \left(r^2 + \|'z\|^2\right)^d & \text{if } r \to \infty \\ \|'z\|^{2d} \cdot r^{-d/m} & \text{if } r \to 0 \end{cases}, \quad k = 0, \pm 1, \pm 2, \ldots, \ C_4 - const. \tag{14}$$

For indices $k \geqslant 0$ the upper inequality (14)is used:
$|c_k\left('z, z_n\right)| \leqslant \dfrac{C_4}{r^k}\left(r^2 + \|'z\|^2\right)^d \to 0$, for $r \to \infty$ and $k > 2d$. For indices $k < 0$ the lower inequality (14) is taken:
$|c_k\left('z, z_n\right)| \leqslant \dfrac{C_3}{r^k}\|'z\|^{2d} \cdot r^{-d/m} \to 0$, with $r \to 0$ and $k < -\dfrac{d}{m}$. Consequently, $c_k\left('z, z_n\right) \equiv 0$ for all $|k| > 2d$ and

$$f('z, z_n) = \sum_{k=-2d}^{+2d} c_k('z, z_n) \cdot p^k('z, z_n).$$

However, according to (14), each function $c_k('z, z_n)$, $|k| \leqslant 2d$, is a polynomial, i.e.,

$$f('z, z_n) = \sum_{k=-2d}^{+2d} c_k('z, z_n) \cdot p^k('z, z_n) = \frac{q('z, z_n)}{p^{2d}('z, z_n)}.$$

Step 6. It remains to show that space of polynomials $\mathcal{P}_\rho(X)$ is dense in space $\mathcal{O}(X)$, i.e., an arbitrary holomorphic function $f(z)$ is uniformly approximated by ρ-polynomials inside $X = \mathbb{C}^n \backslash A$. This follows from the fact that, as we noted above (step 4), the Jacobi–Hartogs–Laurent series $f('z, z_n) = \sum\limits_{k=-\infty}^{\infty} c_k('z, z_n) \cdot p^k('z, z_n)$ of an arbitrary function $f('z, z_n) \in \mathcal{O}(X)$ converges uniformly inside $X = \mathbb{C}^n \backslash A$. Here coefficients are

$$c_k('z, z_n) = a_{k,m-1}('z) z_n^{m-1} + \ldots + a_{k,0}('z), \ a_{k,j}('z) \in \mathcal{O}(\mathbb{C}^{n-1}), \ j = 0, 1, \ldots, m-1.$$

Consequently, the partial sums of $S_M('z, z_n) = \sum\limits_{k=-M}^{M} c_k('z, z_n) \cdot p^k('z, z_n)$ converge uniformly inside $X = \mathbb{C}^n \backslash A$. Approximating coefficients $a_{k,j}('z) \in \mathcal{O}(\mathbb{C}^{n-1})$, $k = 0, \pm 1, \ldots, \pm M$, $j = 1, 2, \ldots, m-1$ by polynomials, we thereby obtain approximation of function $f('z, z_n) \in \mathcal{O}(X)$ by polynomials, i.e., $\overline{\mathcal{P}}_\rho(X) = \mathcal{O}(X)$. Theorem is proved. $\square$

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Функция Грина на параболической аналитической поверхности

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Аннотация. В данной работе рассматривается класс плюрисубгармонических функций на комплексной параболической поверхности, вводятся понятия функции Грина и плюриполярных множеств, изучается ряд их потенциальных свойств.

Ключевые слова: параболическое многообразие, параболические поверхности, регулярные параболические поверхности, функции Грина, плюриполярные множества.

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On Maximal Operators Associated with a Family of Singular Surfaces

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Abstract. Maximal operator associated with singular surfaces is considered in this paper. The boundedness of this operator in the space of summable functions is proved when singular surfaces are given by parametric equations. Boundedness index of the maximal operator is also found for these spaces.

Keywords: maximal operator, averaging operator, fractional power series, singular surface, boundedness indicator.

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1. Introduction and preliminaries

The purpose of the paper is to study the boundedness of maximal operators defined by

$$\mathcal{M}f(y) := \sup_{t>0} \mid \mathcal{A}_t f(y) \mid, \tag{1}$$

where

$$\mathcal{A}_t f(y) := \int_S f(y - tx)\psi(x) dS(x) \tag{2}$$

is so called averaging operator, $S \subset \mathbb{R}^{n+1}$ is a hyper-surface, $\psi \geqslant 0$ is a fixed smooth function with compact support, i.e., $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $f \in C_0^\infty(\mathbb{R}^{n+1})$.

Maximal operator (1) is bounded in $L^p := L^p(\mathbb{R}^{n+1})$ if there exists a number $C > 0$ such that for any function $f \in C_0^\infty(\mathbb{R}^{n+1})$ the L^p inequality $\|\mathcal{M}f\|_{L^p} \leqslant C \|f\|_{L^p}$ holds.

For a hyper-surface S and for a fixed function $0 \leqslant \psi \in C_0^\infty(\mathbb{R}^{n+1})$ a critical exponent of maximal operator (1) is defined by

$$p(S) := \inf\{p : \mathit{operator}\,(1)\,\mathit{is\,bounded\,in}\,L^p\}.$$

Firstly, it was showed that when S is the unit $(n-1)$-dimensional sphere centred at the origin then maximal operator (1) is bounded in $L^p(\mathbb{R}^n)$ for $p > \dfrac{n}{n-1}, n \geqslant 3$ and it is not bounded in $L^p(\mathbb{R}^n)$ whenever $p \leqslant \dfrac{n}{n-1}$ [1]. The two dimensional case of this result was proved by J. Bourgain [2].

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It was proved that maximal operator (1) is bounded in $L^p(\mathbb{R}^{n+1})$ for $n \geqslant 2$ and $p > (n+1)/n$ when hyper-surface has everywhere non-vanishing Gaussian curvature [3]. Moreover, it was showed that if hyper-surface has at least $k (k \geqslant 2)$ non-vanishing principal curvatures then the maximal operator is bounded in $L^p(\mathbb{R}^{n+1})$ $(n \geqslant 2)$ for all $p > (k+1)/k$. A similar result for more difficult case $k = 1$ was obtained by C. D. Sogge [4].

Also, maximal operators (1) were considered in [5–11]. Maximal operators associated with smooth hyper-surfaces in $\mathbb{R}^{n+1}$ were studied and critical exponent of these operators in $L^p(\mathbb{R}^{n+1})$ was defined [12]. The boundedness of the maximal operators related to singular surfaces in 3-dimensional Euclidean space was investigated [13] and [14].

2. Problem statement

Let us consider a family of singular surfaces in $\mathbb{R}^3$ defined by the following parametric equations

$$\begin{gathered} x_1(u_1,u_2) = r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2), \quad x_2(u_1,u_2) = r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2), \\ x_3(u_1,u_2) = r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2), \end{gathered} \tag{3}$$

where r_1, r_2, r_3 are any real numbers, $a_1, a_2, b_1, b_2, c_1, c_2$ are non-negative rational numbers, $u_1 \geqslant 0$, $u_2 \geqslant 0$ and $\{g_k(u_1,u_2)\}_{k=1}^3$ are fractional power series. For the definition of the fractional power series see [13] and [15].

Let us introduce the following designations

$$B_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad B_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Remark 1. *If at least one of the numbers B_1, B_2, B_3 is nonzero then the points of surface* (3) *that lie in a sufficiently small neighborhood of the singular point (r_1, r_2, r_3) outside the coordinate planes of a coordinate system which has its origin at the point (r_1, r_2, r_3) are non-singular. Points of surface* (3) *that lie on the coordinate planes of this coordinate system are singular points* (see definition 2 in [13]).

Let us define the averaging operator in (2) associated with surfaces (3) in the form

$$\begin{aligned} \mathcal{A}_t^{\varphi} f(y) = \int\limits_{\mathbb{R}_+^2} f\Big(&y_1 - t\big(r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)\big), y_2 - t\big(r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)\big), \\ &y_3 - t\big(r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2)\big)\Big)\psi_1(u_1,u_2)u_1^{d_1}u_2^{d_2}\varphi(u_1,u_2)du_1du_2, \end{aligned} \tag{4}$$

where $\varphi(u_1,u_2)$ is fractional power series such that $\varphi(0,0) \neq 0$,

$$\psi_1(u_1,u_2) = \psi\big(r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2),\ r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2),\ r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2)\big)$$

is non-negative fractional power series with a sufficiently small support, d_1, d_2 are real numbers and $f \in C_0^\infty(\mathbb{R}^3)$. The purpose is to prove L^p inequality for the maximal operator defined by

$$\mathcal{M}^{\varphi} f(y) := \sup_{t>0} |\mathcal{A}_t^{\varphi} f(y)|, \quad y \in \mathbb{R}^3.$$

Here this maximal operator is investigated in a small neighbourhood of singular point (r_1, r_2, r_3) of surfaces (3) in the case when $p > 2$.

3. The boundedness of the maximal operator $\mathcal{M}^{\varphi} f$

Let us denote the critical exponent of maximal operator $\mathcal{M}^{\varphi} f$ by $p'(S)$ and

$$p'(S) = \max\left\{\frac{a_1}{d_1+1}, \frac{a_2}{d_2+1}, \frac{b_1}{d_1+1}, \frac{b_2}{d_2+1}, \frac{c_1}{d_1+1}, \frac{c_2}{d_2+1}\right\}.$$

The extension of Theorem 1 in [13] and the main result of this paper is

Theorem 3.1. *Let $\varphi(u_1,u_2)$, $\{g_k(u_1,u_2)\}_{k=1}^{3}$ be fractional power series which defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ and it satisfy the following conditions: $\varphi(0,0)\neq 0$, $g_k(0,0)\neq 0$. Suppose $d_1>-1$, $d_2>-1$ and at least one of the following conditions is hold:*

1. $r_3\neq 0, B_1\neq 0$ and either $B_2B_3\neq 0$, or $B_2(B_2+B_1)\neq 0$, or $B_3(B_3-B_1)\neq 0$;

2. $r_2\neq 0, B_3\neq 0$ and either $B_2B_1\neq 0$, or $B_2(B_2-B_3)\neq 0$, or $B_1(B_1-B_3)\neq 0$;

3. $r_1\neq 0, B_2\neq 0$ and either $B_1B_3\neq 0$, or $B_3(B_3-B_2)\neq 0$, or $B_1(B_1+B_2)\neq 0$.

Then there exists a small neighbourhood U of the singular point (r_1,r_2,r_3) such that for any function $\psi_1\in C_0^{\infty}(U)$ the maximal operator $\mathcal{M}^{\varphi} f$ is bounded in $L^p(\mathbb{R}^3)$ for $p>\max\{p'(S),2\}$. Moreover, if $\psi_1(0,0)=\psi(r_1,r_2,r_3)>0$ and $p'(S)>2$, then the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ whenever $2<p\leqslant p'(S)$.

Proof. Suppose that condition 1 is satisfied and at least one of the numbers r_1, r_2 is not equal to zero. Let us consider the boundedness of the maximal operator $\mathcal{M}^{\varphi} f$ at non-singular points of surface (3) (see Remark 1).

Let us consider the partition of the unity $\sum\limits_{k=0}^{\infty}\chi_k(s)=1$ on the interval $0<s\leqslant 1$, where $\chi_k(s):=\chi(2^k s)$, $\chi\in C_0^{\infty}(\mathbb{R})$ supported on the interval $[0.5;2]$ and $\chi_{j_1,j_2}(u_1,u_2)=\chi_{j_1}(u_1)\chi_{j_2}(u_2)$, $j_1,j_2\in N$. Then averaging operator $A_t^{\varphi} f$ is decomposed as follows

$$\mathcal{A}_t^{\varphi,j_1,j_2} f(y)=\int\limits_{\mathbb{R}^2_+} f\Big(y_1-t\big(r_1+u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)\big), y_2-t\big(r_2+u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)\big),$$
$$y_3-t\big(r_3+u_1^{c_1}u_2^{c_2}g_3(u_1,u_2)\big)\Big)\psi_1(u_1,u_2)\chi_{j_1,j_2}(u_1,u_2)u_1^{d_1}u_2^{d_2}\varphi(u_1,u_2)du_1du_2.$$

Next, by applying the change of variables $u_1=2^{-j_1}v_1$, $u_2=2^{-j_2}v_2$, one can obtain

$$A_t^{\varphi,j_1,j_2} f(y)=2^{-(j_1+j_2)-(j_1d_1+j_2d_2)}\int\limits_{\mathbb{R}^2_+} f\Big(y_1-t\big(r_1+2^{-(j_1a_1+j_2a_2)}v_1^{a_1}v_2^{a_2}\times$$
$$\times g_1(2^{-j_1}v_1,2^{-j_2}v_2)\big), y_2-t\big(r_2+2^{-(j_1b_1+j_2b_2)}v_1^{b_1}v_2^{b_2}g_2(2^{-j_1}v_1,2^{-j_2}v_2)\big),$$
$$y_3-t\big(r_3+2^{-(j_1c_1+j_2c_2)}v_1^{c_1}v_2^{c_2}g_3(2^{-j_1}v_1,2^{-j_2}v_2)\big)\Big)\psi_1(2^{-j_1}v_1,2^{-j_2}v_2)\chi(v_1)\chi(v_2)\times$$
$$\times v_1^{d_1}v_2^{d_2}\varphi(2^{-j_1}v_1,2^{-j_2}v_2)dv_1dv_2,$$

where $0.5\leqslant v_1\leqslant 2$, $0.5\leqslant v_2\leqslant 2$, $j_1,j_2\geqslant j_0$, j_0 is a large number such that implies from the smallness of the support of ψ_1.

Let us change the variables as follows

$$\begin{cases} w_1=v_1^{a_1}v_2^{a_2}g_1(2^{-j_1}v_1,2^{-j_2}v_2)\\ w_2=v_1^{b_1}v_2^{b_2}g_2(2^{-j_1}v_1,2^{-j_2}v_2),\end{cases} \tag{5}$$

and assume that $g_1(0,0) = g_2(0,0) = 1$. Then in the first quadrant $\mathbb{R}^2_+$ the system

$$\begin{cases} w_1 = v_1^{a_1} v_2^{a_2} \\ w_2 = v_1^{b_1} v_2^{b_2} \end{cases},$$

yields

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \end{cases}. \tag{6}$$

In particular, relations (6) are valid in the set $\{(w_1, w_2) \in \mathbb{R}^2_+ : 2^{-(a_1+a_2)} \leqslant w_1 \leqslant 2^{a_1+a_2},\ 2^{-(b_1+b_2)} \leqslant w_2 \leqslant 2^{b_1+b_2}\}$.

Consequently, the change of variables

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1 \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2 \end{cases}, \tag{7}$$

is introduced, where $\hat{g}_1, \hat{g}_2$ are new variables and it is supposed that $\hat{g}_1 \sim 1$, $\hat{g}_2 \sim 1$. As a result system (5) implies

$$\begin{aligned} &(\hat{g}_1)^{a_1}(\hat{g}_2)^{a_2} g_1\Big(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1,\ 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2\Big) = 1, \\ &(\hat{g}_1)^{b_1}(\hat{g}_2)^{b_2} g_2\Big(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1,\ 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2\Big) = 1. \end{aligned} \tag{8}$$

According to the implicit function theorem, system (8) has a unique smooth solutions with respect to $\hat{g}_1, \hat{g}_2$ in a sufficiently small neighbourhood of the point $(0, 0, 1, 1)$

$$\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1}\tilde{h}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2}\tilde{h}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2),$$

$$\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1}\tilde{\rho}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2}\tilde{\rho}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2).$$

Here $\tilde{h}_1, \tilde{h}_2, \tilde{\rho}_1, \tilde{\rho}_2$ are smooth functions. It is assumed that $\tilde{g}_1(0, 0, 1, 1) = 1$, $\tilde{g}_2(0, 0, 1, 1) = 1$.

Then taking into account (7), one can obtain

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2) \end{cases}. \tag{9}$$

Applying relations (5) and (9) to the last integral, we obtain

$$\mathcal{A}_t^{\varphi, j_1, j_2} f(y) = 2^{-(j_1+j_2)-(j_1 d_1 + j_2 d_2)} \int\limits_{\mathbb{R}^2_+} f\Big(y_1 - t\big(r_1 + 2^{-(j_1 a_1 + j_2 a_2)} w_1\big),$$

$$y_2 - t\big(r_2 + 2^{-(j_1 b_1 + j_2 b_2)} w_2\big),\ y_3 - t\big(r_3 + 2^{-(j_1 c_1 + j_2 c_2)} \alpha(w_1, w_2)\big)\Big) \beta(w_1, w_2) dw_1 dw_2,$$

where $\alpha(w_1, w_2) = w_1^{\frac{-B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1, w_2)$,

$$g(w_1, w_2) = \big(\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2)\big)^{c_1} \big(\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2)\big)^{c_2} \times$$

$$\times g_3\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\beta(w_1,w_2)=\tilde{\psi_1}(w_1,w_2)\tilde{\chi}_1(w_1,w_2)\tilde{\chi}_2(w_1,w_2)(\varphi_1(w_1,w_2))^{d_1}(\varphi_2(w_1,w_2))^{d_2}\tilde{\varphi}(w_1,w_2)J(w_1,w_2),$$

$$\tilde{\psi_1}(w_1,w_2)=\psi_1\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\tilde{\chi}_1(w_1,w_2)=\chi\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\tilde{\chi}_2(w_1,w_2)=\chi\Big(2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\varphi_1(w_1,w_2)=w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ \ \varphi_2(w_1,w_2)=w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2),$$

$$\tilde{\varphi}(w_1,w_2)=\varphi\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big)$$

are fractional power series, $J(w_1,w_2)$ is the Jacobian of the change of variables (9).

The dilation operators

$$T_1^{j_1,j_2}f(y):=2^{\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}f\Big(2^{j_1a_1+j_2a_2}y_1,\ 2^{j_1b_1+j_2b_2}y_2,\ 2^{j_1c_1+j_2c_2}y_3\Big)$$

are isometric in $L^p(\mathbb{R}^3)$ and they transform the averaging operators $\mathcal{A}_t^{\varphi,j_1,j_2}f$ into new ones

$$\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f(y)=2^{-(j_1+j_2)-(j_1d_1+j_2d_2)+\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}\times$$

$$\times\int\limits_{\mathbb{R}^2_+}f\Big(2^{j_1a_1+j_2a_2}\big(y_1-tr_1-t\cdot 2^{-(j_1a_1+j_2a_2)}w_1\big),2^{j_1b_1+j_2b_2}\big(y_2-tr_2-t\cdot 2^{-(j_1b_1+j_2b_2)}w_2\big),$$

$$2^{j_1c_1+j_2c_2}\big(y_3-tr_3-t\cdot 2^{-(j_1c_1+j_2c_2)}\alpha(w_1,w_2)\big)\Big)\beta(w_1,w_2)dw_1dw_2.$$

Also, the dilation operators

$$T_2^{-j_1,-j_2}f(y):=2^{-\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}f\Big(2^{-j_1a_1-j_2a_2}y_1,\ 2^{-j_1b_1-j_2b_2}y_2,\ 2^{-j_1c_1-j_2c_2}y_3\Big)$$

are isometric in space $L^p(\mathbb{R}^3)$ and they turn operators $\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f$ into new operators

$$T_2^{-j_1,-j_2}\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f(y)=2^{-(j_1+j_2)-(j_1d_1+j_2d_2)}\int\limits_{\mathbb{R}^2_+}\times$$

$$\times f\Big(y_1-t\big(s_1+w_1\big),\ y_2-t\big(s_2+w_2\big),y_3-t\big(s_3+\alpha(w_1,w_2)\big)\Big)\beta(w_1,w_2)dw_1dw_2,$$

where $s_1=2^{j_1a_1+j_2a_2}r_1, s_2=2^{j_1b_1+j_2b_2}r_2,\ s_3=2^{j_1c_1+j_2c_2}r_3$.

Suppose that $\max\{|s_1|,|s_2|,|s_3|\}=|s_3|$ and define the following rotation operator

$$R^\theta f(y):=f(e_{11}x_1+e_{12}x_2+e_{13}x_3,\ e_{21}x_1+e_{22}x_2+e_{23}x_3,\ e_{31}x_1+e_{32}x_2+e_{33}x_3)$$

which is isometric in space $L^p(\mathbb{R}^3)$. Rotation orthogonal matrix $(e_{ij})_{i,j=1}^3$ is

$$\begin{pmatrix}\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2\cos\theta_3 & \sin\theta_1\sin\theta_3\\ \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_3\\ \sin\theta_2\sin\theta_3 & -\sin\theta_3\cos\theta_2 & \cos\theta_3\end{pmatrix},$$

where $\theta_1, \theta_2, \theta_3$ are Euler angles, θ_3 is the angle between vectors $(0,0,d)$ and (s_1, s_2, s_3), $d = \sqrt{s_1^2+s_2^2+s_3^2}$ (see [16], pp. 288–289).

The rotation operator $R^{\theta}f$ and its inverse $R^{-\theta}f$ turn operators $T_2^{-j_1,-j_2}\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f$ into the following new operators

$$R^{-\theta}T_2^{-j_1,-j_2}\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}R^{\theta}f(y) = 2^{-(j_1+j_2)-(j_1d_1+j_2d_2)}\times$$
$$\times\int\limits_{\mathbb{R}_+^2} f\Big(y_1 - t\alpha_1(w_1,w_2),\, y_2 - t\alpha_2(w_1,w_2), y_3 - t\big(d+\alpha_3(w_1,w_2)\big)\Big)\beta(w_1,w_2)dw_1dw_2, \tag{10}$$

where

$$\alpha_1(w_1,w_2) = e_{11}w_1 + e_{12}w_2 + e_{13}\alpha(w_1,w_2),$$
$$\alpha_2(w_1,w_2) = e_{21}w_1 + e_{22}w_2 + e_{23}\alpha(w_1,w_2),$$
$$\alpha_3(w_1,w_2) = e_{31}w_1 + e_{32}w_2 + e_{33}\alpha(w_1,w_2).$$

It is well-known that the second fundamental form of the surface given by parametric equations

$$\bar{r}(w_1,w_2) = \bar{r}\big(\alpha_1(w_1,w_2),\, \alpha_2(w_1,w_2),\, \alpha_3(w_1,w_2)\big) \tag{11}$$

has the following form

$$L = L_{11}dw_1^2 + 2L_{12}dw_1dw_2 + L_{22}dw_2^2,$$

where

$$L_{11} = (\bar{r}_{11},\bar{n}), \quad L_{12} = (\bar{r}_{12},\bar{n}), \quad L_{22} = (\bar{r}_{22},\bar{n}), \tag{12}$$
$$\bar{r}_{11} = \frac{\partial^2\bar{r}}{\partial w_1^2}, \quad \bar{r}_{12} = \frac{\partial^2\bar{r}}{\partial w_1\partial w_2}, \quad \bar{r}_{22} = \frac{\partial^2\bar{r}}{\partial w_2^2},$$

$\bar{n} = \bar{N}\cdot|\bar{N}|^{-1}$ is the unit normal vector. A normal vector $\bar{N}$ in any point of surface (11) defined by

$$barN = \begin{vmatrix} i & j & k \\ \dfrac{\partial\alpha_1}{\partial w_1} & \dfrac{\partial\alpha_2}{\partial w_1} & \dfrac{\partial\alpha_3}{\partial w_1} \\ \dfrac{\partial\alpha_1}{\partial w_2} & \dfrac{\partial\alpha_2}{\partial w_2} & \dfrac{\partial\alpha_3}{\partial w_2} \end{vmatrix}.$$

Coefficients L_{11}, L_{22}, L_{12} in (12) are

$$L_{11} = \frac{\partial^2\alpha}{\partial w_1^2} = Cw_1^{-\frac{B_2}{B_1}-2}w_2^{\frac{B_3}{B_1}}\Big(B_2(B_2+B_1)g(w_1,w_2) - B_2B_1w_1\frac{\partial g(w_1,w_2)}{\partial w_1} + B_1^2w_1^2\frac{\partial^2 g(w_1,w_2)}{\partial w_1^2}\Big),$$
$$L_{22} = \frac{\partial^2\alpha}{\partial w_2^2} = Cw_1^{-\frac{B_2}{B_1}}w_2^{\frac{B_3}{B_1}-2}\Big(B_3(B_3-B_1)g(w_1,w_2) + B_3B_1w_2\frac{\partial g(w_1,w_2)}{\partial w_2} + B_1^2w_2^2\frac{\partial^2 g(w_1,w_2)}{\partial w_2^2}\Big),$$
$$L_{12} = \frac{\partial^2\alpha}{\partial w_1\partial w_2} = -Cw_1^{-\frac{B_2}{B_1}-1}w_2^{\frac{B_3}{B_1}-1}\Big(B_2B_3g(w_1,w_2) - B_3B_1w_1\frac{\partial g(w_1,w_2)}{\partial w_1} +$$
$$+ B_2B_1w_2\frac{\partial g(w_1,w_2)}{\partial w_2} - B_1^2w_1w_2\frac{\partial^2 g(w_1,w_2)}{\partial w_1 w_2}\Big),$$

where $C = \dfrac{|\bar{N}|^{-1}}{B_1^2}$.

It follows from condition 1 of the Theorem that at least one of the numbers $B_2(B_2+B_1)$, $B_3(B_3-B_1)$, B_2B_3 is not equal to zero. Therefore, at least one of the coefficients L_{11}, L_{12}, L_{22} is not equal to zero for sufficiently large j_0.

Hence, surface (11) satisfies the assumptions of Proposition 4.5 in [9]. Applying this proposition to integral (10) for $p>2$, we obtain

$$\| \sup_{t>0} |R^{-\theta} T_2^{-j_1,-j_2} \mathcal{A}_t^{\varphi,j_1,j_2} T_1^{j_1,j_2} R^{\theta} f| \|_{L^p} \leqslant D_p \left(\frac{d}{|e_{33}|}\right)^{\frac{1}{p}} 2^{-\frac{p(j_1 d_1+j_2 d_2)+p(j_1+j_2)}{p}} \| f \|_{L^p},$$

where $\max\{|s_1|,|s_2|,|s_3|\}=|s_3|$. Taking into account this inequality and isometry of operators $T_1^{j_1,j_2} f$, $T_2^{-j_1,-j_2} f$, $R^{\theta} f$, $R^{-\theta} f$ and considering condition $\max\{|s_1|,|s_2|,|s_3|\}=|s_3|$, we obtain

$$\| \sup_{t>0} |\mathcal{A}_t^{\varphi,j_1,j_2} f| \|_{L^p} \leqslant D_p 2^{\frac{j_1 c_1+j_2 c_2-p(j_1 d_1+j_2 d_2)-p(j_1+j_2)}{p}} \| f \|_{L^p} .$$

Consequently, we have

$$\sum_{j_1,j_2\geqslant j_0} \| \mathcal{M}^{\varphi,j_1,j_2} f \|_{L^p} \leqslant D_p \sum_{j_1,j_2\geqslant j_0} 2^{\frac{j_1 c_1+j_2 c_2-p(j_1 d_1+j_2 d_2)-p(j_1+j_2)}{p}} \| f \|_{L^p} .$$

The series on the right side of the last inequality converges for all p satisfying the condition $p>\max\left\{\frac{c_1}{d_1+1},\frac{c_2}{d_2+1}\right\}$. Therefore, for such p the following inequalities

$$\| \mathcal{M}^{\varphi} f \|_{L^p} \leqslant \sum_{j_1,j_2\geqslant j_0} \| \mathcal{M}^{\varphi,j_1,j_2} f \|_{L^p} \leqslant C_p \| f \|_{L^p}$$

hold true, where C_p is some positive number.

Analogously, one can show that if $\max\{|s_1|,|s_2|,|s_3|\}=|s_1|$ or $\max\{|s_1|,|s_2|,|s_3|\}=|s_2|$ then the maximal operator $\mathcal{M}^{\varphi} f$ is bounded in $L^p(\mathbb{R}^3)$ for $p>\max\left\{\frac{a_1}{d_1+1},\frac{a_2}{d_2+1}\right\}$ or for $p>\max\left\{\frac{b_1}{d_1+1},\frac{b_2}{d_2+1}\right\}$ respectively.

Thus, the proof of the positive result of Theorem is completed.

Let us prove now the negative result. For this reason suppose that $\max\left\{\frac{c_1}{d_1+1},\frac{c_2}{d_2+1}\right\}>2$. Then following [1] consider the function

$$f(x_1,x_2,x_3)=\frac{\eta_1(x_1,\, x_2)\eta_2(x_3)}{|x_3|^{\frac{1}{p}}\, |\ln|x_3||^{\frac{1}{p}}},$$

where η_1,η_2 are smooth functions satisfying the following condition

$$\eta_1(x_1,x_2)\eta_2(x_3)=\begin{cases} 1, & |x|\leqslant \frac{\kappa}{2} \\ 0, & |x|\geqslant \kappa. \end{cases}$$

Here $\kappa>0$ is some sufficiently small number. Taking into account relations (2) and (3) the averaging operator corresponding to function $f(x_1,x_2,x_3)$ is represented as follows

$$\mathcal{A}_t^{\varphi} f(y)=\int\limits_{\mathbb{R}_+^2} \frac{\eta_1\big(y_1-tx_1(u_1,u_2),y_2-tx_2(u_1,u_2)\big)\eta_2(y_3-tx_3(u_1,u_2))}{|y_3-tx_3(u_1,u_2)|^{\frac{1}{p}}\,\big|\ln|y_3-tx_3(u_1,u_2)|\big|^{\frac{1}{p}}}\times$$

$$\times\psi_1(u_1,u_2)u_1^{d_1}u_2^{d_2}\varphi(u_1,u_2)du_1du_2.$$

Let us assume that $\psi_1(0,0) > 0,\ t = \dfrac{y_3}{r_3} > 0$. Since κ is sufficiently small number consider (y_1, y_2) that lies in a small neighbourhood of the point $\left(\dfrac{r_1 y_3}{r_3}, \dfrac{r_2 y_3}{r_3}\right)$. Then one can obtain

$$\sup_{t>0} |\mathcal{A}_t^{\varphi} f(y)| \geqslant C \frac{1}{|\frac{y_3}{r_3}|^{\frac{1}{p}}} \int\limits_{|u|\leqslant \frac{\kappa}{2}} \frac{|u_1|^{d_1 - \frac{c_1}{p}} |u_2|^{d_2 - \frac{c_2}{p}}}{|\ln|\frac{y_3}{r_3} u_1^{c_1} u_2^{c_2} g_3(u_1, u_2)||^{\frac{1}{p}}} du_1 du_2,$$

where C is some positive number. The last integral diverges for all p satisfying $2 < p \leqslant \max\left\{\dfrac{c_1}{d_1+1}, \dfrac{c_2}{d_2+1}\right\}$. Hence, the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ for these p.

Analogously, one can show that if $\max\{|s_1|, |s_2|, |s_3|\} = |s_1|$ or $\max\{|s_1|, |s_2|, |s_3|\} = |s_2|$ then the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ whenever $2 < p \leqslant \max\left\{\dfrac{a_1}{d_1+1}, \dfrac{a_2}{d_2+1}\right\}$ or $2 < p \leqslant \max\left\{\dfrac{b_1}{d_1+1}, \dfrac{b_2}{d_2+1}\right\}$, respectively.

Thus, making similar arguments under conditions 2 or 3, the proof of Theorem 3.1 is completed. □

Consider now a number of corollaries in connection with Theorem 3.1.

Corollary 1. *Let $\varphi(u_1, u_2)$, $\{g_k(u_1, u_2)\}_{k=1}^3$ be fractional power series defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ such that $\varphi(0,0) \neq 0$, $g_k(0,0) \neq 0$ and $d_1 > -1$, $d_2 > -1$. Then the following assertions hold true*

1. If $r_1 = 0, r_2 = 0, r_3 \neq 0, B_1 \neq 0$ and either $B_2 B_3 \neq 0$ or $B_2(B_2 + B_1) \neq 0$ or $B_3(B_3 - B_1) \neq 0$ then $p'(S) = \max\left\{\dfrac{c_1}{d_1+1}, \dfrac{c_2}{d_2+1}\right\}$.

2. If $r_1 = 0, r_3 = 0, r_2 \neq 0, B_3 \neq 0$ and either $B_2 B_1 \neq 0$ or $B_2(B_2 - B_3) \neq 0$ or $B_1(B_1 - B_3) \neq 0$ then $p'(S) = \max\left\{\dfrac{b_1}{d_1+1}, \dfrac{b_2}{d_2+1}\right\}$.

3. If $r_2 = 0, r_3 = 0, r_1 \neq 0, B_2 \neq 0$ and either $B_1 B_3 \neq 0$ or $B_3(B_3 - B_2) \neq 0$ or $B_1(B_1 + B_2) \neq 0$ then $p'(S) = \max\left\{\dfrac{a_1}{d_1+1}, \dfrac{a_2}{d_2+1}\right\}$.

Corollary 2. *Let us assume that $\varphi(u_1, u_2)$, $\{g_k(u_1, u_2)\}_{k=1}^3$ are real analytic functions defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ and they satisfy the following conditions: $\varphi(0,0) \neq 0$, $g_k(0,0) \neq 0$. Then under the assumptions of Theorem 3.1 its assertions are true.*

Corollary 3. *If conditions 1–3 of Theorem 3.1 are replaced with the relations*

$$r_3 \neq 0,\ B_1 \neq 0,\ A_1^{-1}\bar{c} \neq (1,0),\ A_1^{-1}\bar{c} \neq (0,1),\ A_1^{-1}\bar{c} \neq (0,0);$$

$$r_1 \neq 0,\ B_2 \neq 0,\ A_2^{-1}\bar{a} \neq (1,0),\ A_2^{-1}\bar{a} \neq (0,1),\ A_2^{-1}\bar{a} \neq (0,0);$$

$$r_2 \neq 0,\ B_3 \neq 0,\ A_3^{-1}\bar{b} \neq (1,0),\ A_3^{-1}\bar{b} \neq (0,1),\ A_3^{-1}\bar{b} \neq (0,0),$$

respectively, and other conditions are satisfied then assertions of Theorem hold true. Here A_1, A_2, A_3 are matrices B_1, B_2, B_3, respectively, and $\bar{a} = (a_1, a_2)$, $\bar{b} = (b_1, b_2)$, $\bar{c} = (c_1, c_2)$.

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О максимальных операторах, ассоциированных с семейством сингулярных поверхностей

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Аннотация. В этой статье рассматривается максимальный оператор, ассоциированный с сингулярными поверхностями. Доказываем ограниченность этого оператора в пространстве суммируемых функций, когда сингулярные поверхности задаются параметрическими уравнениями. А также найден показатель ограниченности максимального оператора для таких пространств.

Ключевые слова: максимальный оператор, оператор усреднения, дробно-степенной ряд, сингулярная поверхность, показатель ограниченности.

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A Note on the Diophantine Equation $(4^q - 1)^u + \left(2^{q+1}\right)^v = w^2$

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Abstract. Let a, b and c be positive integers such that $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = 1$, a even. Terai's conjecture claims that the Diophantine equation $x^2 + b^y = c^z$ has only the positive integer solution $(x, y, z) = (a, 2, 2)$. In this short note, we prove that the equation of the title, has only the positive integer solution $(u, v, w) = (2, 2, 4^q + 1)$, where q is a positive integer.

Keywords: Terai's conjecture, Pythagorean triple.

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1. Introduction and preliminaries

In 1956, Sierpinski [2] studied the equation

$$3^u + 4^v = 5^w$$

and proved that it only possesses $(u, v, w) = (2, 2, 2)$ as a solution in integers. In turn, Jésmanowicz [1] showed that the only positive solution in integers of any of the following equations

$$5^u + 12^v = 13^w, \quad 7^u + 24^v = 25^w, \quad 9^u + 40^v = 41^w, \quad 11^u + 60^v = 61^w$$

is $(u, v, w) = (2, 2, 2)$, and posed the following Conjecture 1.1 (see [3]).

Recall that when positive integers a, b, c satisfy $a^2 + b^2 = c^2$ we say that (a, b, c) is a Pythagorean triple, and if in addition $\gcd(a, b, c) = 1$ it is said a primitive Pythagorean triple.

Historically, Euclid of Alexandria (323–300 BC) was the first mathematician who proved that (a, b, c) is a primitive Pythagorean triple with a odd, if and only if, there exists a pair of numbers $(\alpha, \beta) \in \mathbb{N}^{*2}$ with $\alpha > \beta$, α and β are coprime and of different parity, such that

$$a = \alpha^2 - \beta^2, \quad b = 2\alpha\beta \quad \text{and} \quad c = \alpha^2 + \beta^2.$$

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Conjecture 1.1. *If* (a, b, c) *is Pythagorean triple, then the equation*

$$a^u + b^v = c^w$$

has the only solution $(u, v, w) = (2, 2, 2)$.

In 2013, Z. Xinwen and Z. Wenpeng [6] showed that, for any positive integers n and m the exponential Diophantine equation

$$\left((2^{2m} - 1)n\right)^x + \left(2^{m+1}n\right)^y = ((2^{2m} + 1)n)^z$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

Recently, Hai Yang and Ruiqin Fu [7] by combining Baker's method with an elementary approach, have proven that if $\alpha\beta \equiv 2 \pmod 4$ and $\alpha > 17.8\beta$, then the Conjecture 1.1 is true, this is for $(a, b, c) = (2\alpha\beta, \alpha^2 - \beta^2, \alpha^2 + \beta^2)$.

Thirty years before, Terai had conjectured [4]

Conjecture 1.2. *Let* α, β *be positive integers such that* $\alpha > \beta$, $\gcd(\alpha, \beta) = 1$ *and* $\alpha \not\equiv \beta \pmod 2$*, then the equation*

$$x^2 + (\alpha^2 - \beta^2)^m = (\alpha^2 + \beta^2)^n$$

has the only positive solution in integers $(x, m, n) = (2\alpha\beta, 2, 2)$.

In 2020, M. Le and G. Soydan [5] studied Conjecture 1.2 in the case $\alpha = 2^r s$ and $\beta = 1$, where r, s are positive integers satisfying $2 \nmid s, r \geqslant 2$ and $s < 2^{r-1}$.

First Terai conjecture is "Let a, b, c be relatively prime positive integers such that $a^p + b^q = c^r$ for fixed integers $p, q, r \geqslant 2$. Terai conjectured that The equation $a^x + b^y = c^z$ in positive integers has only the solution $(x, y, z) = (p, q, r)$ except for some specific cases".

There are many results and studies related to this conjecture we can cite among them: Nobuhiro Terai [12, 13] and Takafumi Miyazaki [8–11].

In this short note we prove

Theorem 1.3. *Let* q *be a positive integer. Then the Diophantine equation*

$$(4^q - 1)^u + \left(2^{q+1}\right)^v = w^2$$

has only the positive integer solution $(u, v, w) = (2, 2, 4^q + 1)$.

2. Proof of the main result

Proof. Suppose that there are positive integers u, v and w such that

$$(4^q - 1)^u + \left(2^{q+1}\right)^v = w^2 \tag{1}$$

then w is odd and

$$w^2 \equiv 1 \pmod{4}.$$

Reducing equation (1) modulo 4, we get

$$(4^q - 1)^u \equiv 1 \pmod{4},$$

or equivalently

$$(-1)^u \equiv 1 \pmod{4}.$$

This implies $u = 2t$ for some positive integer t.
Thus,

$$2^{(q+1)v} = \left(2^{q+1}\right)^v = w^2 - \left(\left(4^q - 1\right)^t\right)^2 = \left(w + \left(4^q - 1\right)^t\right)\left(w - \left(4^q - 1\right)^t\right)$$

Hence,

$$w + \left(4^q - 1\right)^t = 2^s$$

and

$$w - \left(4^q - 1\right)^t = 2^r,$$

with $s > r$ and $s + r = (q+1)v$. Solving for w and $\left(4^q - 1\right)^t$, we get

$$w = 2^{r-1}\left(2^{s-r} + 1\right) \quad \text{and} \quad \left(4^q - 1\right)^t = 2^{r-1}\left(2^{s-r} - 1\right).$$

Since the left side of both previous equalities is odd, r must be equal to 1. Let $x = s - r$. Then the equation

$$\left(4^q - 1\right)^t = 2^{r-1}\left(2^{s-r} - 1\right)$$

becomes

$$\left(4^q - 1\right)^t = 2^x - 1.$$

The reduction modulo 3 gives

$$0 \equiv (-1)^x - 1 \pmod{3},$$

and so x is even, say $x = 2k$ for some positive integer k. Thus,

$$\left(4^q - 1\right)^t = \left(2^k\right)^2 - 1$$

by the Mihailescu's Theorem $t = 0$ or $t = 1$. Consequently, $t = 1$, and so $x = 2q$. This gives us the unique solution $(u, v, w) = (2, 2, 4^q + 1)$. $\square$

If we maintain the same conditions as before we believe in the validity of the following:

Conjecture 2.1. *If $a^2 + b^2 = c^2$ with $(a, b, c) = 1$, then the Diophantine equation*

$$a^u + b^v = w^2.$$

has only the positive integer solutions $(u, v, w) = (2, 2, c)$.

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Заметка о диофантовом уравнении $(4^q-1)^u+\left(2^{q+1}\right)^v=w^2$

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Аннотация. Пусть a, b и c — натуральные числа такие, что $a^2+b^2=c^2$ с $\gcd(a,b,c)=1$, a четным. Гипотеза Тераи утверждает, что диофантово уравнение $x^2+b^y=c^z$ имеет только натуральное решение $(x,y,z)=(a,2,2)$. В этой короткой заметке мы доказываем, что уравнение заголовка имеет только положительное целочисленное решение $(u,v,w)=(2,2,4^q+1)$, где q положительное целое число.

Ключевые слова: гипотеза Тераи, тройка Пифагора.