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On Generation of the Groups $GL_n(\mathbb{Z})$ and $PGL_n(\mathbb{Z})$ by Three Involutions, Two of which Commute

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Abstract. It is proved that the general linear group $GL_n(\mathbb{Z})$ (its projective image $PGL_n(\mathbb{Z})$ respectively) over the ring of integers $\mathbb{Z}$ is generated by three involutions, two of which commute, if and only if $n \geqslant 5$ (if $n = 2$ and $n \geqslant 5$ respectively).

Keywords: general linear group, ring of integers, generating triples of involutions.

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Introduction

We call groups, generated by three involutions, two of which commute, $(2\times2,2)$-*generated*.The class of such groups is closed with respect to homomorphic images if, by definition, we consider the identity group as such and we do not exclude the coincidence of two or all three involutions. M. C. Tamburini and P. Zucca [5] have proved that some matrix groups of big enough degree n, depending on the parameter d, over the d-generated commutative are $(2 \times 2, 2)$-generated. Particularly they established $(2 \times 2, 2)$-generation of the special linear group $SL_n(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$ when $n \geqslant 14$. Ya. N. Nuzhin [2] has proved that the projective special linear group $PSL_n(\mathbb{Z})$ over the ring of integers is $(2 \times 2, 2)$-generated if and only if $n \geqslant 5$. Applying methods of the paper [2], we obtain similar criteria for the general linear group $GL_n(\mathbb{Z})$ and its projective image $PGL_n(\mathbb{Z})$.

Theorem 1. *The general linear group $GL_n(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$ is generated by three involutions, two of which commute if and only if $n \geqslant 5$.*

Theorem 2. *The projective general linear group $PGL_n(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$ is generated by three involutions, two of which commute if and only if $n = 2$ and $n \geqslant 5$.*

1. Notations and preliminary results

Further, R is an arbitrary commutative ring with the identity 1, $SL_n(R)$ is a subgroup of matrices with determinant 1 of the general linear group $GL_n(R)$ over the ring R.

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Elementary transvections

$$t_{ij}(k) = E_n + ke_{ij},\ \ i,j = 1,2,\dots,n,\ i \neq j,\ k \in R,$$

will be called simply transvections, where E_n is an identity matrix of degree n, and e_{ij} is a $(n \times n)$-matrix with 1 on the position (i,j) and zeros elsewhere. Also let

$$t_{ij}(R) = \langle t_{ij}(k) \mid k \in R \rangle,\ \ i,j = 1,2,\dots,n,\ i \neq j.$$

For any non-empty subset M of some group, by $\langle M \rangle$ we denote the subgroup generated by the set M. The next lemma is well known (see, for example, [6, p. 107]).

Lemma 1. *The group $SL_n(R)$ over the Euclidean ring R, in particular over any field, is generated by subgroups $t_{ij}(R)$, $i,j = 1,\dots,n$.*

The ring of integers $\mathbb{Z}$ is Euclidean and $t_{rs}(\mathbb{Z}) = \langle t_{rs}(1) \rangle$, so the corollary of Lemma 1 is

Lemma 2. *The group $SL_n(\mathbb{Z})$ is generated by transvections $t_{ij}(1)$, $i \neq j$, $i, j = 1, 2, \dots, n$.*

Since the index of the subgroup $SL_n(\mathbb{Z})$ in the group $GL_n(\mathbb{Z})$ is equal to 2, Lemma 2 implies

Lemma 3. *The group $GL_n(\mathbb{Z})$ is generated by transvections $t_{ij}(1)$, $i \neq j$, $i, j = 1, 2, \dots, n$, and any matrix with determinant -1.*

Set

$$\mu = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The matrix μ has group order n and the group $\langle \mu \rangle$ acts by conjugations regularly on the set of transvections

$$T = \{t_{1n}(1), t_{i+1i}(1), i = 1,2,\dots,n-1\},$$

and on the transposed set

$$T' = \{t_{n1}(1), t_{ii+1}(1), i = 1,2,\dots,n-1\}.$$

By commuting transvections from the set T or from T', all the transvections $t_{ij}(1)$ can be obtained. Therefore, each of the sets T and T' generates the group $SL_n(\mathbb{Z})$. Moreover, by virtue of Lemmas 2 and 3, the following lemma is valid

Lemma 4. *The group $SL_n(\mathbb{Z})$ ($GL_n(\mathbb{Z})$ respectively) is generated by one of the transvections*

$$t_{1n}(1),\ t_{i+1i}(1),\ t_{n1}(1),\ t_{ii+1}(1),\ \ i = 1,2,\dots,n-1,$$

and by the matrix $\varepsilon\mu$ for any $(1,-1)$-diagonal matrix ε under condition that $\varepsilon\mu \in SL_n(\mathbb{Z})$ ($det(\varepsilon\mu) = -1$ respectively).

Let I be an ideal of the ring R. Then the natural ring homomorphism $\rho_I:\ R \to R/I$ defines a surjective homomorphism

$$\psi_I:\ M_n(R) \to M_n(R/I)$$

of the ring of $n \times n$-matrices $M_n(R)$ with the usual operations of addition and multiplication, where for any matrix $(a_{ij}) \in M_n(R)$ by definition

$$\psi_I:\ (a_{ij}) \to (\rho_I(a_{ij})).$$

On the other hand, the homomorphism ρ_I induces a group homomorphism

$$\varphi_I:\ GL_n(R) \to GL_n(R/I),$$

$$\varphi_I:\ SL_n(R) \to SL_n(R/I),$$

where also by definition

$$\varphi_I:\ (a_{ij}) \to (\rho_I(a_{ij})).$$

D. A. Suprunenko calls φ_I a *Minkowski homomorphism* [7, p. 95]. However, the homomorphism φ_I is no longer required to be surjective like the homomorphism ψ_I (see [1, Example 1]).

A linear group of type X_n over a finite field of q elements will be denoted by $X_n(q)$

Lemma 5. *The group $PSL_n(2)$ is a homomorphic image of the groups $GL_n(\mathbb{Z})$ and $PGL_n(\mathbb{Z})$.*

Proof. Evidently, $GL_n(2) = PGL_n(2) = SL_n(2) = PSL_n(2)$. Since both groups $GL_n(2)$ and $SL_n(\mathbb{Z})$ are generated by their transvections, and $GL_n(\mathbb{Z}) = \langle E_n - 2e_{nn}\rangle SL_n(\mathbb{Z})$ and the quotient ring $\mathbb{Z}/I$ by the ideal I generated by the element 2 is isomorphic to a field of two elements, then the homomorphisms $\varphi_I : GL_n(\mathbb{Z}) \to GL_n(2)$ and $\varphi_I : PGL_n(\mathbb{Z}) \to PGL_n(2)$ are surjective.

The lemma is proved. □

For brevity, the group generated by three involutions, two of which commute, will be called $(2\times 2, 2)$-*generated*, and, by definition, we consider the identity group as such and do not exclude the coincidence of two or all three involutions. With this definition, the following lemma is valid

Lemma 6. *The class of $(2 \times 2, 2)$-generated groups is closed under homomorphic images.*

We use the following notations: $a^b = bab^{-1}$, $[a, b] = aba^{-1}b^{-1}$.

2. Proof of Theorem 1

The case of $\mathbf{n = 2}$. The fact that the group $GL_2(\mathbb{Z})$ is not generated by three involutions, two of which commute, was established in [3, Sentence 2.3].

Cases $\mathbf{n = 3, 4}$. For $n = 3, 4$ the group $PSL_n(2)$ is not $(2 \times 2, 2)$-generated [4]. Therefore, by virtue of Lemmas 5 and 6, the group $GL_n(\mathbb{Z})$ will also be such. Note that for $n = 2$ this argument fails, since the group $PSL_2(2)$ is isomorphic to a dihedral group of order 6, which is $(2 \times 2, 2)$-generated by definition.

Case $\mathbf{n = 5}$. Let us show that the group $GL_5(\mathbb{Z})$ is generated by the following three involutions

$$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the first two of which commute. Suppose $M = \langle \alpha, \beta, \gamma \rangle$. Let

$$\eta = \beta\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then

$$\alpha^\eta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$[\alpha, \alpha^\eta] = t_{31}(-1)t_{41}(1),$$

$$[\alpha, \alpha^\eta]^\eta = t_{42}(-1)t_{52}(1),$$

$$[\alpha, [\alpha, \alpha^\eta]^\eta] = t_{42}(1)t_{51}(-1)t_{52}(-2),$$

$$[\alpha, [\alpha, \alpha^\eta]^\eta]^{\eta^{-1}} = t_{31}(1)t_{41}(-2)t_{45}(-1),$$

$$\left[[\alpha, \alpha^\eta]^\eta, [\alpha, [\alpha, \alpha^\eta]^\eta]^{\eta^{-1}} \right] = t_{42}(1),$$

$$(t_{42}(1))^{\eta^3} = t_{25}(1),$$

$$[t_{42}(1), t_{25}(1)] = t_{45}(1).$$

Thus, M contains the transvection $t_{45}(1)$ and the monomial matrix $\eta = -\mu$ with determinant -1. By Lemma 4 $M = GL_5(\mathbb{Z})$. What was required to show.

Case n = 6. Let us show that the group $GL_6(\mathbb{Z})$ is generated by the following three involutions

$$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The involutions β and γ commute. Let us introduce the following notation for some diagonal and monomial matrices by setting

$$d_i = E_n - 2e_{ii},$$
$$d_{ij} = E_n - 2e_{ii} - 2e_{jj},$$
$$n_{ij}^+ = t_{ij}(1)t_{ji}(-1)t_{ij}(1),$$
$$n_{ij}^- = t_{ij}(-1)t_{ji}(1)t_{ij}(-1),$$
$$n_{ij} = d_i n_{ij}^+.$$

Obvious relations are valid for the specified elements, which we will use below.

$$(d_i)^2 = (d_{ij})^2 = (n_{ij})^2 = n_{ij}^+ n_{ij}^- = 1,$$
$$(n_{ij}^+)^2 = (n_{ij}^-)^2 = d_{ij},$$
$$n_{ij}^+ = n_{ji}^-,$$
$$n_{ij} = n_{ji}.$$

Note also that n_{ij} is the permutation matrix corresponding to the transposition (ij). In these notations

$$\alpha = d_{16}t_{21}(1)n_{45},$$
$$\beta = n_{12}n_{34}n_{56},$$
$$\gamma = n_{34}n_{25}n_{16}.$$

Matrix calculations show that

$$\alpha^\beta = d_{25}t_{12}(1)n_{36},$$
$$\alpha^\gamma = d_{16}t_{56}(1)n_{23},$$
$$\upsilon = (\alpha^\gamma\alpha)^2 = t_{21}(1)t_{31}(1)t_{46}(-1)t_{56}(-1),$$
$$\eta = (\alpha\alpha^\beta)^2 = t_{21}(-1)n_{21}^- d_{34}d_{56},$$
$$(\alpha^\beta)^\eta = d_{15}t_{21}(1)n_{36},$$
$$(\alpha(\alpha^\beta)^\eta)^2 = d_{34}d_{56},$$
$$(d_{34}d_{56})^\gamma = d_{12}d_{34},$$
$$\alpha^{d_{12}d_{34}} = d_{16}d_{45}t_{21}(1)n_{45},$$
$$\alpha\alpha^{d_{12}d_{34}} = d_{45},$$
$$(d_{45})^\gamma = d_{23},$$
$$(d_{12}d_{34})d_{45}d_{23} = d_{15},$$
$$d_{15}(\alpha^\beta)^\eta = t_{21}(1)n_{36},$$
$$(t_{21}(1)n_{36})^\beta = t_{12}(1)n_{45},$$
$$(d_{15})^\gamma = d_{26},$$
$$d_{26}\alpha = d_{12}t_{21}(1)n_{45},$$
$$(d_{12}x_{r_1}(1)n_{45}\eta)^2 = d_{12},$$

$$d_{12}d_{26} = d_{16},$$
$$d_{16}\alpha = t_{21}(1)n_{45},$$
$$(t_{12}(1)n_{45})^{\gamma} = t_{65}(1)n_{23},$$
$$(t_{21}(1)n_{45}t_{65}(1)n_{23})^2 = t_{21}(1)t_{31}(1)t_{64}(1)t_{65}(1),$$
$$\upsilon^{\beta} = t_{12}(1)t_{35}(-1)t_{42}(1)t_{65}(-1),$$
$$(\upsilon^{\beta})^{\eta} = t_{21}(-1)t_{35}(-1)t_{41}(-1)t_{65}(-1),$$
$$[(\upsilon^{\beta})^{\eta}, (t_{21}(1)n_{45}t_{65}(1)n_{23})^2] = t_{61}(1),$$
$$(t_{61}(1))^{\gamma} = t_{16}(1),$$
$$t_{16}(1)t_{61}(-1)t_{16}(1) = n_{61}^{+},$$
$$\beta\gamma = n_{15}n_{26},$$
$$[\alpha, t_{16}(1)] = t_{26}(-1),$$
$$(t_{26}(-1))^{\beta\gamma} = t_{62}(-1),$$
$$t_{26}(1)t_{62}(-1)t_{26}(1) = n_{26}^{+},$$
$$(n_{61}^{+})^{n_{26}^{+}} = n_{21}^{+},$$
$$(n_{21}^{+})^{\gamma} = n_{65}^{-},$$
$$(t_{16}(1))^{n_{26}^{+}} = t_{12}(1),$$
$$d_{16}^{\beta} = d_{25},$$
$$t_{12}(-1)d_{25}\alpha^{\beta} = n_{36},$$
$$n_{36}^{\beta} = n_{45},$$
$$n_{45}^{\gamma} = n_{23},$$
$$(n_{36})^{n_{45}n_{65}^{+}} = n_{34}.$$

Thus, we have obtained monomial elements $n_{21}^{+}, n_{23}, n_{34}, n_{45}, n_{65}^{-}$, which generate a subgroup N containing a representative of each coset of the whole monomial subgroup of $GL_6(\mathbb{Z})$ by its diagonal subgroup. Such a subgroup N acts transitively by conjugations on the subgroups $t_{rs}(\mathbb{Z}) = \langle t_{rs}(1)\rangle$. We have already obtained several transvections $t_{rs}(1)$, and there are also matrices with determinant -1. Thus, by Lemma 3, the involutions α, β, γ generate $GL_6(\mathbb{Z})$.

The case of n $\geqslant$ 7. In the paper [2] for $n \geqslant 7$ in the proof of the generation of the groups $PSL_n(\mathbb{Z})$ for $n = 4k+2$ and $SL_n(\mathbb{Z})$ for $n \neq 4k+2$ by three involutions α, β, γ, the first two of which commute, all calculations, namely, the commutation of two transvections and their conjugation by monomial matrices are carried out up to sign. Therefore, by changing only one of the generating monomial involutions so that its determinant is equal to -1, one can obtain the $(2\times 2, 2)$-generatedness of the group $GL_n(\mathbb{Z})$. The following changes are suitable for our purposes. We replace:

1) β on $\beta' = \sum\limits_{i=1}^{n}(-1)e_{i,n-i+1}$ for $n = 4k+2$ (in this case in [2] the preimage of the involution β in the group $SL_n(\mathbb{Z})$ has order 4);

2) γ on γ', where γ' differs from γ only in the sign of the element at position (n, n) for $n \neq 4k+2$.

The theorem is proved.

Note. There is a typo in [2] on page 70. For $n = 2(2k+1)+1$ (k=7, 11, ...) instead of $\eta_2 = E_n$ there should be $\eta_2 = -E_n$.

3. Proof of Theorem 2

The case of $\mathbf{n=2}$. The fact that the group $PGL_2(\mathbb{Z})$ is generated by three involutions, two of which commute, was established in [3, Proposition 2.1].

Cases $\mathbf{n=3,4}$. For $n=3,4$ the group $PSL_n(2)$ is not $(2\times 2,2)$-generated [4]. Therefore, by virtue of Lemmas 5 and 6, the group $PGL_n(\mathbb{Z})$ will also be such.

Case $\mathbf{n\geqslant 5}$. The group $PGL_n(\mathbb{Z})$ is a homomorphic image of the group $GL_n(\mathbb{Z})$. Therefore, by virtue of Lemma 6, it follows from Theorem 1 that in this case the group $PGL_n(\mathbb{Z})$ is $(2\times 2,2)$-generated.

The theorem is proved.

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О порождаемости групп $\mathbf{GL_n}(\mathbb{Z})$ и $\mathbf{PGL_n}(\mathbb{Z})$ тремя инволюциями, две из которых перестановочны

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Аннотация. Доказано, что общая линейная группа $GL_n(\mathbb{Z})$ (соответственно, ее проективный образ $PGL_n(\mathbb{Z})$) над кольцом целых чисел $\mathbb{Z}$ тогда и только тогда порождается тремя инволюциями, две из которых перестановочны, когда $n\geqslant 5$ (соответственно, когда $n=2$ и $n\geqslant 5$).

Ключевые слова: общая линейная группа, кольцо целых чисел, порождающие тройки инволюций.

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Analog of the Weierstrass Theorem and the Blaschke Product for $A(z)$-analytic Functions

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Abstract. We consider $A(z)$-analytic functions in the case when $A(z)$ is an antiholomorpic function. For $A(z)$-analytic functions analogs of the Weierstrass theorem and of the Blaschke theorem are proved.

Keywords: $A(z)$-analytic function, Cauchy's integral theorem, Weierstrass theorem, Jensen's theorem, Blaschke theorem.

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1. Introduction and preliminaries

The paper is devoted to the solutions of the Beltrami equation

$$\bar{D}_A f(z) := \frac{\partial f(z)}{\partial \bar{z}} - A(z) \frac{\partial f(z)}{\partial z} = 0 \tag{1}$$

which is directly related to the theory of quasi-conformal mappings (see [1,13]). The function $A(z)$ in general is assumed to be measurable with the condition $|A(z)| \leqslant C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. Solutions of equation (1) are often called $A(z)$-analytic functions. The most interesting case is $\partial A = 0$, i.e. $A(z)$ is an anti-analytic function in D and such that $|A(z)| \leqslant C < 1\ \forall z \in D$. Then according to (1) the class $f \in O_A(D)$ of $A(z)$-analytic functions in D is characterized by the fact that $\bar{D}_A f = 0$. Since any anti-analytic function is smooth, it follows that $O_A(D) \subset C^\infty(D)$ (see [13]).

Here we study the analogs of the well-known Weierstrass and Blaschke theorems for $A(z)$-analytic functions in convex domains, when $A(z)$ is an anti-analytic function. The requirement for the convexity of the domain is due to the fact that for non-convex domains the required kernel of the integral formula, which is involved in the proof of the main results, may not exist. For analytic functions, the Weierstrass and Blaschke factorizations are well studied (see [7, 8]).

Let us present some facts from the theory of $A(z)$- analytic functions that we will need below. Consider the integral

$$\psi(z,\xi) = z - \xi + \overline{\int_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau} \in O_A(D),$$

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where $\gamma(\xi, z)$ is a smooth curve connecting points ξ, $z \in D$. If the domain D simply connected, then the integral

$$I(z) = \int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau$$

does not depend on the integration path; it coincides with the primitive, $I'(z) = \bar{A}(z)$. The function $\psi(z, \xi)$ for convex domains has a single zero at the point $z = \xi$. In particular, the set $L(\xi, r) = \left\{ z \in D : \left| \psi(z, \xi) \right| = \left| z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau} \right| < r \right\}$ is an open connected set in D. For sufficiently small $r > 0$ it belongs compactly to D and contains the point ξ. This set is called the $A(z)$-lemniscate centered at ξ and denoted as $L(\xi, r)$. Put

$$K(z, \xi) = \frac{1}{2\pi i} \cdot \frac{1}{z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau}}. \tag{2}$$

Theorem 1.1 (analog of Cauchy's formula, see [4, 9])**.** *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be its subdomain with a piecewise smooth boundary ∂G. Then for any function $f(z) \in O_A(G) \bigcap C(\bar{G})$ we have*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\xi)}{z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau}} \left(d\xi + A(\xi)\, d\bar{\xi} \right), \quad z \in G. \tag{3}$$

2. Generalized Weierstrass theorem for $A(z)$-analytic functions.

The main result of the section is following theorem.

Theorem 2.1. *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ its compact subdomain. Then, whatever sequence of points $a_n \in G$ that has no limit points in G, there exists an $A(z)$-analytic in G function f that has zeros at all points of a_n and only at these points.*

Proof. Note that if the set $\{a_n\} = \{a_1, a_2, \ldots, a_n\}$ is finite, then the product $\prod\limits_{n=1}^{m} \psi(z, a_n)$ can be taken as the function $f(z)$. However, when the set $\{a_n\}$ is countable this product may diverge. In this case, the function $f(z)$ is constructed in the form of an infinite product, also with the help of $\psi(z, \xi)$, which for convex domains has a single zero $z = \xi$. But the $\psi(z, a_n)$ is multiplied by some additional function, that do not vanish, so that the considered infinite product converges uniformly.

For each point a_n, we find a point $b_n \in \partial G$, which is closest to the point a_n. Then the value of $r_n = \psi(b_n, a_n) \to 0$ at $n \to \infty$. Since

$$\psi(z, b_n) - \psi(a_n, b_n) = z - b_n + \overline{\int\limits_{\gamma(b_n,z)} \bar{A}(\tau)\, d\tau} - (a_n - b_n) - \overline{\int\limits_{\gamma(b_n,a_n)} \bar{A}(\tau)\, d\tau} =$$

$$= z - a_n + \overline{\int\limits_{\gamma(a_n,z)} \bar{A}(\tau)\, d\tau} = \psi(z, a_n),$$

we get

$$\frac{\psi(z, a_n)}{\psi(z, b_n)} = \frac{\psi(z, b_n) - \psi(a_n, b_n)}{\psi(z, b_n)} = 1 - \frac{\psi(a_n, b_n)}{\psi(z, b_n)}.$$

We fix $n \in \mathbb{N}$ and consider the decomposition

$$\ln \frac{\psi(z,a_n)}{\psi(z,b_n)} = \ln\left(1 - \frac{\psi(a_n,b_n)}{\psi(z,b_n)}\right) = -\sum_{k=1}^{\infty} \frac{\psi^k(a_n,b_n)}{k\psi^k(z,b_n)}. \tag{4}$$

The series converges uniformly on the compact set $\{z \in G : \ |\psi(z,b_n)| \geqslant 2r_n\}$. Therefore, we can choose a natural number p_n so that

$$\left|\ln \frac{\psi(z,a_n)}{\psi(z,b_n)} + \sum_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}\right| < \frac{1}{2^n}\,, \ \ |\psi(z,b_n)| \geqslant 2r_n, \ \ (n = 1,2,\dots). \tag{5}$$

With this choice of p_n, the infinite product

$$f(z) = \prod_{n=1}^{\infty} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}} \tag{6}$$

converges uniformly inside the domain $G\setminus\{a_n\}$.

Indeed, for any compact set $K \subset\subset G$, there is N such that $a_n \notin K, \ |\psi(z,b_n)| \geqslant 2r_n$ for all $n \geqslant N$ and all $z \in K$. Then the series of $A(z)$-analytic functions

$$\sum_{n=N}^{\infty}\left(\ln \frac{\psi(z,a_n)}{\psi(z,b_n)} + \sum_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}\right)$$

and, therefore, the infinite product $\prod\limits_{n=N}^{\infty} \frac{\psi(z,a_n)}{\psi(z,b_n)} e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}}$ due to (5) converges on K uniformly. Therefore, the product

$$f(z) = \prod_{n=1}^{\infty} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}} = \prod_{n=1}^{N-1} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}} \times \prod_{n=N}^{\infty} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}}$$

is an $A(z)$-analytic function in G that vanishes only at points $a_n \in G$. □

Corollary 1. *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ an arbitrary simply connected compact subdomain. Then, any function $f(z) \in O_A(G)$ admits a factorization*

$$f(z) = e^{g(z)} \prod_{n} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}}, \tag{7}$$

where $\{a_n\}$ is a set (finite or countable) of zeros of the function $f(z) \in O_A(G)$, p_n, b_n the values defined in the proof of Theorem 2, and $g(z)$ is some $A(z)$-analytic function in G. Note that if $\{a_n\}$ is finite, then representation (7) is very simple,

$$f(z) = e^{g(z)} \prod_{n} \psi(z,a_n)\,.$$

Proof. The corollary is easily obtained if we take into account that the ratio

$$f(z) / \prod_{n} \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}}$$

is an $A(z)$-analytic and non-vanishing function in G. Since $G \subset\subset D$ is simply connected, the logarithm

$$g(z) = \ln \left\{ f(z) / \prod_n \frac{\psi(z, a_n)}{\psi(z, b_n)} e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}} \right\} \in O_A(G)$$

and

$$f(z) = e^{g(z)} \prod_n \frac{\psi(z, a_n)}{\psi(z, b_n)} e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}}.$$

□

3. The Blaschke product for $A(z)$-analytic functions.

In this section, we study the zero densities of an $A(z)$-analytic function $f(z) \in O_A(L)$, bounded in lemniscate $L = L(a, R) = \{|\psi(a, z)| < R\}$ in a convex domain $D \subset \mathbb{C}$. Let us start with the formulation of the following Jensen formula

Theorem 3.1 (Jensen's formula)**.** *Let $f \in O_A(L(a, R))$. Denote by $n(t)$ the number of zeros, taking into account the multiplicities of the function $f(z)$ in $\bar{L}(a, t)$, $t < R$. Assume that $f(a) \neq 0$, i.e. $n(0) = 0$. Then, the following formula holds*

$$\int_0^r \frac{n(t)\,dt}{t} = \frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + A(z)\,d\bar{z}| - \ln|f(a)|. \tag{8}$$

Proof. Suppose that $a_1, a_2, a_3, \ldots$ are the zeros of the function f in $L(a, R)$, in the non-decreasing order of $r_n = |\psi(a, a_n)|$, and each $a_1, a_2, a_3, \ldots$ zero in the sequence occurs as many times as its multiplicity. First we show that under the condition $r_n < r_{n+1}$ for $r \in (r_n, r_{n+1})$ we have

$$\frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + A(z)\,d\bar{z}| = \ln \frac{r^n |f(a)|}{r_1 r_2 r_3 \ldots r_n} = \ln|f(a)| + n \ln r - \ln r_1 r_2 \ldots r_n. \tag{9}$$

To do this, consider the finite product

$$B(z) = \prod_{k=1}^{n} r \cdot \frac{|\psi(a_k, a)|}{\psi(a_k, a)} \frac{\psi(a_k, a) - \psi(z, a)}{r^2 - \overline{\psi}(a_k, a)\psi(z, a)}.$$

It represents an $A(z)$-analytic function in the lemniscate $L(a, r_{n+1})$ that vanishes only at the points $a_1, a_2, \ldots, a_n$. Therefore, the following representation is true

$$f(z) = e^{g(z)} B(z) = e^{g(z)} \prod_{k=1}^{n} r \cdot \frac{|\psi(a_k, a)|}{\psi(a_k, a)} \frac{\psi(a_k, a) - \psi(z, a)}{r^2 - \overline{\psi}(a_k, a)\psi(z, a)}, \quad g(z) \in O(L(a, r_{n+1})).$$

From here

$$\ln|f(z)| = \operatorname{Re} g(z) + \sum_{k=1}^{n} \ln \left| r \frac{\psi(a_k, a) - \psi(z, a)}{r^2 - \overline{\psi}(a_k, a)\psi(z, a)} \right|, \quad \ln|f(a)| = \operatorname{Re} g(a) + \sum_{k=1}^{n} \ln \frac{r_k}{r}.$$

Since $\operatorname{Re} g(z)$ is $A(z)$-analytic function, we have (see [6])

$$\frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \operatorname{Re} g(z)\,|dz + A(z)\,d\bar{z}| = \operatorname{Re} g(a).$$

Since $\left|r\dfrac{\psi(a_k,a)-\psi(z,a)}{r^2-\overline{\psi}(a_k,a)\psi(z,a)}\right|=1$ for $|\psi\,(z,a)|=r$, we get

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln\left|r\frac{\psi(a_k,a)-\psi(z,a)}{r^2-\overline{\psi}(a_k,a)\psi(z,a)}\right||dz+A\,(z)\,d\bar{z}|=0.$$

Therefore,

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln|f\,(z)|\,|dz+A\,(z)\,d\bar{z}|=\operatorname{Re}g\,(a)=\ln|f\,(a)|+n\ln r-\ln r_1r_2\ldots r_n,$$

which proves the validity of formula (9).

It is clear that

$$\ln|f\,(a)|+n\ln r-\ln r_1r_2\ldots r_n=\ln|f\,(a)|+n\ln r-\sum_{k=1}^{n}\ln r_k=\ln|f\,(a)|+$$
$$+\sum_{k=1}^{n-1}k\,(\ln r_{k+1}-\ln r_k)+n\,(\ln r-\ln r_n)=\ln|f\,(a)|+\sum_{k=1}^{n-1}k\int\limits_{r_k}^{r_{k+1}}\frac{dt}{t}+n\int\limits_{r_n}^{r}\frac{dt}{t}=\ln|f\,(a)|+$$
$$+\sum_{k=1}^{n-1}\int\limits_{r_k}^{r_{k+1}}\frac{n\,(t)\,dt}{t}+\int\limits_{r_n}^{r}\frac{n\,(t)}{t}dt=\int\limits_{0}^{r_n}\frac{n\,(t)}{t}dt+\int\limits_{r_n}^{r}\frac{n\,(t)}{t}dt+\ln|f\,(a)|=\int\limits_{0}^{r}\frac{n\,(t)\,dt}{t}+\ln|f\,(a)|\,.$$

It follows that formula (9) can be written as

$$\int\limits_{0}^{r}\frac{n\,(t)\,dt}{t}=\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln|f\,(z)|\,|dz+A\,(z)\,d\,\bar{z}|-\ln|f\,(a)|\,. \tag{10}$$

Note that we proved formula (10) under the condition $r_n<r<r_{n+1}$. If we show the continuous increase of both parts of this formula with the continuous increase of r from $r_{n+1}-0$ to $r_{n+1}+0$, then this will prove the validity of formula (10) for an arbitrary $r<R$. For the left side of (10) this is obvious. For the right side, let $r_n<r_{n+1}=r_{n+2}=\ldots.=r_{n+m}<r_{n+m+1},\ \ m\geqslant 1$. Then in some ring $L\,(a,r'')\setminus\bar{L}\,(a,r')\,,\ r_n<r'<r_{n+1}<r''<r_{n+m+1}$, (see [7])

$$f\,(z)=g\,(z)\prod_{k=1}^{m}\left[\psi\,(a_{n+k},a)-\psi\,(z,a)\right]=g\,(z)\prod_{k=1}^{m}\psi\,(a_{n+k},a)\left[1-\frac{\psi\,(z,a)}{\psi\,(a_{n+k},a)}\right]$$

for all $z\in L\,(a,r'')\setminus\bar{L}\,(a,r')\,$. Therefore,

$$\ln|f\,(z)|=\ln|g\,(z)|+\sum_{k=1}^{m}\ln\left[|\psi\,(a_{n+k},a)|+\left|1-\frac{\psi\,(z,a)}{\psi\,(a_{n+k},a)}\right|\right]=\ln|g\,(z)|+$$
$$+\sum_{k=1}^{m}\ln r_{n+k}+\sum_{k=1}^{m}\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|=\ln|g\,(z)|+m\ln r_{n+1}+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|,\ \ 0\leqslant t\leqslant 2\pi.$$

From here,

$$\ln|f\,(z)|=\ln|g\,(z)|+m\ln r_{n+1}+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|=\eta\,(z)+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|,$$

where

$$\eta(z) = \ln|g(z)| + m \ln r_{n+1}$$

is continuous in a neighborhood of $r' < r < r''$. Now it is sufficient to prove that the integral

$$I(r) = \int_0^{2\pi} \ln\left|1 - \frac{r}{r_n} e^{it}\right| dt, \quad I(r_n) = 0,$$

is continuous at the point $r = r_{n+1}$. For

$$\frac{r}{r_{n+1}} \geqslant \left|1 - \frac{r}{r_{n+1}} e^{it}\right|^2 = 1 - 2\frac{r}{r_{n+1}} \cos t + \frac{r^2}{r_{n+1}^2} = \sin^2 t + \left(\cos t - \frac{r}{r_{n+1}}\right)^2 \geqslant \sin^2 t.$$

Hence, for fixed $\varepsilon > 0$, $\delta \in (0, \pi)$ we have

$$I(r) - I(r_{n+1}) = I(r) = \int_0^{2\pi} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt =$$
$$= \int_{-\delta}^{\delta} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt + \int_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt.$$

$$\left|\int_{-\delta}^{\delta} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt\right| < \int_{-\delta}^{\delta} (\ln 3 + |\ln|\sin t||)\, dt < \int_{-\delta}^{\delta} (\ln 3 + |\ln|t||)\, dt <$$
$$< (2 + \ln 9)\,\delta + 2\delta \ln\frac{1}{\delta} < (4 + \ln 9)\,\delta \ln\frac{1}{\delta}.$$

We fix δ so small that the right side is smaller than $\frac{\varepsilon}{2}$. The integral

$$\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt$$

is continuous at the point $r = r_n$. Therefore, for $r \to r_{n+1}$ we have

$$\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt \to \int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r_n}{r_{n+1}} e^{it}\right| dt = 0$$

and we get that for sufficiently close r to r_{n+1} the integral

$$\left|\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt\right| < \frac{\varepsilon}{2}.$$

Hence, $|I(r) - I(r_{n+1})| < \varepsilon$ i.e. $I(r) \to I(r_{n+1})$ for $r \to r_{n+1}$ and the integral

$$\int_0^{2\pi} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt$$

is continuous at the point $r = r_{n+1}$. $\square$

4. Properties of the Blaschke product for $A(z)$-analytic functions

If $0 < |\psi(a_n, a)| < R$, $n = 1, 2, 3, \ldots$, and an infinite product

$$\prod_{n=1}^{\infty} R \cdot \frac{|\psi(a_n,a)|}{\psi(a_n,a)} \frac{\psi(a_n,a)-\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)} \tag{11}$$

converges uniformly inside $\{|\psi(z,a)| < R\} \setminus \{a_n\}$, then it represents some $A(z)$-analytic in the lemniscate $L(a,R)$ function $B(z)$. It is called the Blaschke product. One can admit a finite number of zeros in the lemniscate $L(a,R)$. In this case, the number of factors in (11) will be finite.

Now we study the convergence of the Blaschke product (11).We have

$$R\frac{|\psi(a_n,a)|}{\psi(a_n,a)} \frac{\psi(a_n,a)-\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)} = R\left[|\psi(a_n,a)| \frac{1-\frac{\psi(z,a)}{\psi(a_n,a)}}{R^2-\overline{\psi}(a_n,a)\psi(z,a)}\right] =$$

$$= R\frac{1}{R^2}\left[|\psi(a_n,a)| + \frac{\left(\overline{\psi}(a_n,a)-\frac{R^2}{\psi(a_n,a)}\right)|\psi(a_n,a)|\,\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)}\right] =$$

$$= \frac{1}{R}\left[|\psi(a_n,a)| + \frac{|\psi(a_n,a)|^2-R^2}{R^2-\overline{\psi}(a_n,a)\psi(z,a)} \frac{|\psi(a_n,a)|\,\psi(z,a)}{\psi(a_n,a)}\right].$$

Here

$$R\frac{|\psi(a_n,a)|}{\psi(a_n,a)} \frac{\psi(a_n,a)-\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)} =$$
$$= \frac{1}{R}\left\{R + (|\psi(a_n,a)|-R)\left\{1+\frac{(|\psi(a_n,a)|+R)\,|\psi(a_n,a)|}{\psi(a_n,a)\left[R^2-\overline{\psi}(a_n,a)\psi(z,a)\right]}\psi(z,a)\right\}\right\}.$$

Therefore, the considered infinite product converges uniformly inside $\{|\psi(z,a)| < R\} \setminus \{a_n\}$ if and only if

$$\sum_{n=1}^{\infty} (R - |\psi(a_n,a)|) < \infty\,.$$

Note that

$$\left|R\frac{\psi(a_n,a)-\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)}\right|^2 = \frac{|\psi(a_n,z)|^2}{|\psi(a_n,z)|^2+|R-\psi(a_n,a)|^2+|R-\psi(a_n,a)|^2} \leqslant 1 \ \forall z \in L(a,R)\,.$$

Under the condition

$$\sum_{n=1}^{\infty} (R - |\psi(a_n,a)|) < \infty\,,$$

the $A(z)$-analytic Blaschke product $B(z)$ in $L(a,R)$ does not exceed 1 in absolute value, i.e., $|B(z)| \leqslant 1$.

Let $\sum\limits_{n=1}^{\infty} (R - |\psi(a_n,a)|) < \infty$, so that

$$\prod_{n=1}^{\infty} R\frac{|\psi(a_n,a)|}{\psi(a_n,a)} \frac{\psi(a_n,a)-\psi(z,a)}{R^2-\overline{\psi}(a_n,a)\psi(z,a)}$$

converges in $L(a,R)$ and represents the Blaschke product $B(z)$, which is $A(z)$-analytic in $L(a,R)$, $|B(z)|<1$.

The following assertion implies that at almost all points of the boundary $\partial L(a,R)$ the Blaschke product has radial limits

Lemma 4.1. *If a function $f \in O_A(L(a,R))$ and is bounded in $L(a,R)$, $|f| \leqslant M$, then it has the radial limit $\lim\limits_{z\to\xi\in\partial L(a,R)} f(z)$ almost everywhere on $\partial L(a,R)$.*

Proof. We expand the function $f(z)$ into a series: $f(z)=\sum\limits_{n=0}^{\infty} c_n\psi^n(z,a), z\in L(a,R)$ (see [9]). First we show that $\sum\limits_{n=1}^{\infty}|c_n|^2R^{2n}<\infty$. Setting $\psi(z,a)=re^{it}$, we have

$$|f(z)|^2 = f(z)\overline{f(z)} = \sum_{n=0}^{\infty} c_n r^n e^{int} \sum_{n=0}^{\infty} \overline{c_n} r^n e^{-int} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} c_j\overline{c_{n-j}}e^{it(2j-n)}\right) r^n, \quad r<R.$$

The series

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} c_j\overline{c_{n-j}}e^{it(2j-n)t}\right) r^n$$

converges uniformly in $[0,2\pi]$ and integrating it, we get

$$\int\limits_{|\psi(z,a)|=r} |f(z)|^2\,|dz+A(z)\,d\bar z| = \sum_{n=0}^{\infty}|c_n|^2 r^{2n},$$

That is why

$$\sum_{j=0}^{\infty}|c_n|^2 r^{2n} \leqslant M^2.$$

Since this inequality is true for all $r<R$, we have

$$\sum_{n=0}^{\infty}|c_n|^2 R^{2n} \leqslant M^2.$$

According to the Riesz–Fischer theorem, it follows from the condition $\sum\limits_{n=1}^{\infty}|R^n c_n|^2<\infty$ that $\sum\limits_{n=-\infty}^{\infty} c_nR^ne^{itn}=\varphi(t)\in L_2[0;2\pi]$ is a Fourier series. So that $\int\limits_{[0;2\pi]}\left|\sum\limits_{n=1}^{\infty}c_nR^ne^{itn}-\varphi(t)\right|^2 dt=0$. This means that the series is Cesaro summable and converges to $\varphi(t)$ for almost all $t\in[0;2\pi]$. But then it is Abel summable (see [8,12]), i.e.

$$\lim_{z\to\xi\in\partial L(a,R)} f(z) = \lim_{r\to R-0}\sum_{n=0}^{\infty} c_n r^n e^{int}$$

for almost all $t\in[0,2\pi]$. □

The Lemma just proven states that for almost all $\xi\in\partial L(a,R)$ the limit function

$$\lim_{z\to\xi\in\partial L(a,R)} B(z) = B^*(\xi)$$

exists.

Theorem 4.2. $|B^*(z)| \underset{\partial L(a,R)}{\overset{a.e}{=}} 1$ *holds almost everywhere on* $L(a,R)$.

Proof. Without loss of generality, we can assume that all points $a_n \neq a$ (otherwise we would consider the function $B^*(z) = \dfrac{B(z)}{\psi^N(z,a)}$, where N is the order of zero of the function $B(z)$ at the point a). Then $\ln|B(a)| = \sum\limits_{n=1}^{\infty} \ln \dfrac{|\psi(a_n,a)|}{R}$ and the fact that $\sum\limits_{n=1}^{\infty}(R - |\psi(a_n,a)|) < \infty$ implies

$$\sum_{n=1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R} > -\infty.$$

Take $r \in (0;R)$ not equal to any of the values $|\psi(a_n,a)|$. Then, according to the analogue of the Jensen formula

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)=r|} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = \ln|B(a)| - \sum_{|\psi(a_n,a)|<r} \ln\frac{|\psi(a_n,a)|}{r}.$$

Substituting

$$\ln|B(a)| = \sum_{n=1}^{\infty} \ln\frac{|\psi(a_n,a)|}{R},$$

we get

$$\sum_{n=1}^{\infty} \ln\frac{|\psi(a_n,a)|}{R} = \sum_{|\psi(a_n,a)|<r} \ln\frac{|\psi(a_n,a)|}{r} + \frac{1}{2\pi r}\int\limits_{|\psi(z,a)=r|} \ln|B(z)|\,|dz + A(z)\,d\bar{z}|,$$

or

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = \sum_{n=1}^{\infty} \ln\frac{|\psi(a_n,a)|}{R} - \sum_{|\psi(a_n,a)|<r} \ln\frac{|\psi(a_n,a)|}{r}.$$

We fix some number n_0 such that

$$\sum_{n=n_0+1}^{\infty} \ln\frac{|\psi(a_n,a)|}{R} < \varepsilon$$

and take $r < R$ so large that for $n \in \{1,2,\ldots,n_0\}$ all points of z_n lie in $L(a,r)$. Then from the previous relation we get

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant \sum_{n=1}^{n_0} \ln\frac{|\psi(a_n,a)|}{R} - \sum_{n=1}^{n_0} \ln\frac{|\psi(a_n,a)|}{r} - \varepsilon.$$

From here it follows that

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant -2\varepsilon,$$

if we take $r < R$ close enough to R. Due to the arbitrariness of the number $\varepsilon > 0$, we obtain

$$\varliminf_{r\to R-0} \frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant 0. \tag{12}$$

But from the conditions $\lim\limits_{z\to\xi\in\partial L(a,R)} B(z) = B^*(\xi)$ almost everywhere and $\ln|B(z)| \leqslant 0$, $z \in L(a,r)$ according to (12) we get $\frac{1}{2\pi R}\int\limits_{|\psi(z,a)|=R} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = 0$. This means that $|B^*(z)| \underset{\partial L(a,R)}{\overset{a.e}{=}} 1$. $\square$

Theorem 4.3 (An analogue of Blaschke's theorem)**.** *Let the function $f(z) \in \mathrm{O}_A(L(a,R))$ and $a_1, a_2, a_3, \ldots$ be the zeros of the function f in $L(a,R)$, $r_n = |\psi(a,a_n)|$. If*

$$M = \sup_{0<r<R} \frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + Ad\bar{z}| < \infty$$

then

$$\sum_n (R - |\psi(a_n,a)|) < \infty$$

and the Blaschke product

$$B(z) = \prod_n R \cdot \frac{|\psi(a,a_n)|}{\psi(a,a_n)} \frac{\psi(a,a_n) - \psi(z,a)}{R^2 - \overline{\psi}(a,a_n)\psi(z,a)}$$

is $A(z)$-analytic in $\{|\psi(z,a)| < R\}$, $f(z) = B(z)\cdot G(z)$, where the function $G(z)$ is $A(z)$-analytic and has no zeros at $\{|\psi(z,a)| < R\}$.

Proof. Without loss of generality, we can assume that $f(a) \neq 0$. Then by the Jensen formula

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + Ad\bar{z}| = \ln\frac{r^n f(a)}{r_1 r_2 \ldots r_n},\ r < R,$$

it follows, that

$$\sum_{|\psi(a_n,a)|<r} \ln\left|\frac{r}{\psi(a_n,a)}\right| \leqslant -\ln|f(a)|\,.$$

Letting r tend to R, we get that

$$\sum_n \ln\frac{R}{|\psi(a_n,a)|} < \infty.$$

Note that the convergence of this series is equivalent to the convergence of the series

$$\sum_n (R - |\psi(a_n,a)|) < \infty.$$

The existence of the Blaschke product $B(z)$ now follows according to Theorem 3.

Finally, if we define a function $G(z)$ in $\{|\psi(z,a)| < R\}$ by the formula $G(z) = \frac{f(z)}{B(z)} \in O_A(L(a,R))$, then $G(z) \neq 0$ and $f(z) = B(z)\cdot G(z)$. $\square$

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Обобщенная теорема Вейерштрасса и произведение Бляшке для $A(z)$-аналитических функций

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Аннотация. Мы рассматриваем $A(z)$-аналитические функции в случае, когда $A(z)$ является антиголоморфной функцией. В статье для $A(z)$-аналитических функций доказаны аналог теоремы Вейерштрасса и аналог теоремы Бляшке.

Ключевые слова: $A(z)$-аналитическая функция, интегральная теорема Коши, теорема Вейерштрасса, теорема Йенсена, теорема Бляшки.

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Local Asymptotic Normality of Statistical Experiments in an Inhomogeneous Competing Risks Model under Random Censoring on the Right

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Abstract. The local asymptotic normality property of the likelihood ratio statistic in the competing risk model that corresponds to inhomogeneous and randomly right-censored observations is proved in the paper.

Keywords: local asymptotic normality, likelihood ratio statistic, competing risk model, random censoring, asymptotic representation.

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1. Introduction and preliminaries

There are many works on the study of asymptotic properties of the likelihood ratio statistics (LRS) for full samples. It was shown that local asymptotic normality (LAN) allows one to develop an asymptotic theory of maximum likelihood estimates and Bayesian estimates, as well as the contiguity of families of probability distributions [1–4]. The study of similar properties in the case of incomplete — censored observations is of considerable interest. Effective estimates for the unknown parameter were obtained from censored observations when the distribution of the censoring random variable also depends on the unknown parameter [5]. The properties of local asymptotic normality for LRS were established in some models of censoring observations in the presence of competing risks [6–8]. The properties of local asymptotic normality of the likelihood ratio statistic in the competing risks model under random censoring on the right are studied in this paper.

Let us consider a inhomogeneous competing risks model (CRM). Let $\{X_m, m \geqslant 1\}$ be a sequence of random variables (r.v.) defined on a probability space (Ω, A, P) with distribution functions (d.f.) $H(x;\theta)$, $\theta \in \Theta \subseteq R^1$ with values in a measurable space $(X_m,\, B_m)$. The joint properties of pairs $\left(X_m, A_m^{(i)}\right)$, $i=\overline{1,k}$; are of interest, where $A_m^{(1)},\dots,A_m^{(k)}$ — pairwise disjoint events $P\left(\bigcup\limits_{i=1}^{k} A_m^{(i)}\right)=1$. Let $\delta_m^{(i)} = I\left(A_m^{(i)}\right)$ be an indicator of the event $A_m^{(i)}$, $i=\overline{1,k}$; $m \geqslant 1$. Suppose that set $\left(X_m,\, A_m^{(1)},\dots,A_m^{(k)}\right)$ is randomly censored from the right by an r.v. Y with

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continuous d.f. K. Observation is an available set $\left(Z_m;\ B_m^{(0)}, B_m^{(1)}, \ldots, B_m^{(k)}\right)$, where $Z_m = \min(X_m, Y)$, events $B_m^{(0)} = \{\omega : Y(\omega) \leqslant X_m(\omega)\}$ and $B_m^{(i)} = A_m^{(i)} \cap \{\omega : X_m(\omega) \leqslant Y\}$, $i = \overline{1,k}$; $m \geqslant 1$. Let $\left\{X_m, Y_m;\ B_m^{(0)}, B_m^{(1)}, \ldots, B_m^{(k)}\right\}_{m=1}^{\infty}$ be a sequence of independent copies of population $\left\{X_m, Y;\ B_m^{(0)}, B_m^{(1)}, \ldots, B_m^{(k)}\right\}_{m=1}^{\infty}$ and there is a sample $\widetilde{Z}^{(n)} = \left(\widetilde{Z}_1, \ldots, \widetilde{Z}_n\right)$ in the n-step of the experiment, where $\widehat{Z}_m = \left\{Z_m;\ \Delta_m^{(0)}, \Delta_m^{(1)}, \ldots, \Delta_m^{(k)}\right\}$, $Z_m = \min(X_m, Y_m)$, $\Delta_m^{(i)} = I\left(B_m^{(i)}\right)$, $i = 0, 1, \ldots, k$. Let us note that considering sample $\widehat{Z}^{(n)}$ the pairs $\left(X_m, A_m^{(i)}\right)$ are observable only in the case of $\Delta_m^{(i)} = 1$, $i = \overline{1,k}$; $m = \overline{1,n}$. It is easy to see that r.v. have d.f. where d.f. is interfering.

Let us introduce sub-distributions

$$M_m^{(i)}(x;\theta) = P_\theta\left(Z_m < x,\ M_m^{(i)}\right), \quad i = 0, 1, \ldots, k,$$

where

$$M_m^{(0)}(x;\theta) = P_\theta(Y_m \leqslant x \wedge X_m) = M_\theta\left[I(Y_j \leqslant x,\ X_m > Y_m)\right] =$$
$$= M_\theta\left\{M_\theta\left[I(X_m > Y_m / Y_m)\right] \cdot I(Y_m < x)\right\} = M_\theta\left[I(Y_m < x)(1 - H_m(Y_m;\theta))\right] =$$
$$= \int_{-\infty}^{x} (1 - H_m(u;\theta))dK(u),$$

and for $i = 1, \ldots, k$

$$M^{(i)}(x;\theta) = P_\theta\left(X_n < x \wedge Y_m;\ A_m^{(i)}\right) = M_\theta\left[I\left(X_m < x;\ A_m^{(i)},\ Y_m > X_m\right)\right] =$$
$$= M_\theta\left\{M_\theta\left[I(Y_m > X_m / X_m)\right] \cdot I\left(Y_m < x;\ A_m^{(i)}\right)\right\} = M_\theta\left[I\left(X_m < x;\ A_m^{(i)}\right)(1 - K(X_m))\right] =$$
$$= \int_{-\infty}^{x} (1 - K(u))dH_m(u;i).$$

Then it is easy to see that integral intensity functions $\Lambda_m^{(i)}$ can be represented as

$$\Lambda_m^{(i)}(x;\theta) = \int_{-\infty}^{x} \frac{dM_m^{(i)}(u;\theta)}{1 - N_m(u;\theta)}, \quad m = \overline{1,n}, \ i = \overline{1,k}.$$

Let $\left(Y^{(n)}, U^{(n)}, \widetilde{Q}_\theta^{(n)}\right)$ be a sequence of statistical experiments generated by observations $\widetilde{Z}^{(n)}$. Moreover, if the set of possible values of the r.v. Z is denoted by $\widetilde{Z}$ then we have

$$Y^{(n)} = \left\{\widetilde{Z} \otimes \{0,1\}^{(k+1)}\right\}^{(n)} = \left\{\overbrace{\widetilde{Z} \otimes \{0,1\}^{(k+1)} \otimes \cdots \otimes \widetilde{Z} \otimes \{0,1\}^{(k+1)}}^{n}\right\},$$

where $\{0,1\}^{(k+1)} = \underbrace{\{0,1\} \otimes \cdots \otimes \{0,1\}}_{k+1}$, $U^{(n)}$ is σ-algebra of Borel sets in $Y^{(n)}$, $Q_\theta^{(n)}$ distribution on $\left(Y^{(n)}, U^{(n)}\right)$ is the n-fold product of "one-dimensional" distributions

$$\widetilde{Q}_{\theta m}\left(x, y^{(0)}, y^{(1)}, \ldots, y^{(k)}\right) = P_\theta\left(Z_m < x,\ \Delta_m^{(0)} = y^{(0)}, \Delta_m^{(1)} = y^{(1)}, \ldots, \Delta_m^{(k)} = y^{(k)}\right),$$

$x \in \overline{R}^1$, $y^{(i)} \in \{0,1\}$, $1 = \overline{1,k}$; $m = \overline{1;n}$, Θ is open set in R^1.

Let $h_m^{(i)}(x;\theta) = f_m^{(i)}(x;\theta) \prod\limits_{j \neq i} \left(1 - F_m^{(j)}(x;\theta)\right)$, $i = \overline{1,k}$; $m = \overline{1,n}$. Let us introduce the likelihood ratio statistics (LRS)

$$\frac{d\widetilde{Q}_{\theta_2}^{(n)}\left(\widetilde{Z}^{(n)}\right)}{d\widetilde{Q}_{\theta_1}^{(n)}\left(\widetilde{Z}^{(n)}\right)} = \prod_{m=1}^{n} \left\{ \prod_{i=1}^{k} \left[\frac{h_m^{(i)}(Z_m;\theta_2)}{h_m^{(i)}(Z_m;\theta_1)} \right] \right\}^{y_m^{(i)}} \cdot \left\{ \frac{1 - H_m(Z_m;\theta_2)}{1 - H_m(Z_m;\theta_1)} \right\}^{y_m^{(0)}},$$

and its logarithm

$$L_n(u) = \log \left\{ \frac{d\widetilde{Q}_{\theta_2}^{(n)}\left(\widetilde{Z}^{(n)}\right)}{d\widetilde{Q}_{\theta_1}^{(n)}\left(\widetilde{Z}^{(n)}\right)} \right\} = \sum_{m=1}^{n} \sum_{i=1}^{k} \int_{-\infty}^{\infty} \log \left[\frac{h_m^{(i)}(x;\theta_2)}{h_m^{(i)}(x;\theta_1)} \right] dI\left(Z_m < x,\, \Delta_m^{(i)} = 1\right) +$$

$$+ \sum_{m=1}^{n} \int_{-\infty}^{\infty} \log \cdot \left[\frac{1 - H_m(x;\theta_2)}{1 - H_m(x;\theta_1)} \right] dI\left(Z_m < x,\, \Delta_m^{(0)} = 1\right).$$

2. LAN of a family of probability measures

Let us now formulate the regularity conditions. If these conditions are fulfilled then one can establish the local asymptotic normality (LAN) of the family of distributions $\left\{\widetilde{Q}_\theta^{(n)},\, \theta \in \Theta\right\}$. For simplicity, consider the case of homogeneous distributions θ.

(C1) Supports $N_{h_m^{(i)}} = \left\{x : h_m^{(i)}(x;\theta) > 0\right\}$, $i = \overline{1,k}$; $m = \overline{1,n}$, are independent of parameter θ and $\bigcap\limits_{m=1}^{n} \bigcap\limits_{n=1}^{k} N_{h_m^{(i)}}$ is not empty.

(C2) For any two points $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, $h_m^{(i)}(x;\theta_1) \neq h_m^{(i)}(x;\theta_2)$.

(C3) There exist derivatives $\left\{ \dfrac{\partial^l h_m^{(i)}(x;\theta)}{\partial\theta^l},\ l{=}1,2;\ i{=}\overline{1,k};\ m{=}\overline{1,n} \right\}$, and they are finite for all x, while

$$\int_{-\infty}^{\infty} \left| \frac{\partial^l h_m^{(i)}(x;\theta)}{\partial\theta^l} \right| \nu_m(dx),\quad l = 1,2;\quad i = \overline{1,k};\quad m = \overline{1,n}.$$

(C4) Fisher information $J(\theta) = \sum\limits_{m=1}^{n} J_m(\theta)$ is finite and positive, where

$$J_m(\theta) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta} \right)^2 h_m^{(i)}(x;\theta)\, \nu_m(dx) +$$

$$+ \int_{-\infty}^{\infty} \left(\frac{\partial \log\left(1 - H_m(x;\theta)\right)}{\partial\theta} \right)^2 \left(1 - H_m(x;\theta)\right) dx.$$

Let us note that according to (C3)

$$\left| \frac{\partial^2 \left(1 - H_m(x;\theta)\right)}{\partial\theta^l} \right| \leqslant \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left| \frac{\partial^l h_m^{(i)}(x;\theta)}{\partial\theta^l} \right| \nu_m(dx) < \infty,\quad l = 1,2.$$

The lemma on the equality to zero of the mean of the contribution of the sample is valid.

Lemma 2.1. *Let regularity conditions (C1)–(C3) are fulfilled. Then*

$$\sum_{i=1}^{k} M_\theta \left[\Delta_m^{(i)} \frac{\partial h_m^{(i)}(Z_m;\theta)}{\partial\theta}\right] + M_\theta \left[\Delta_m^{(0)} \frac{\partial \log(1-H_m(Z_m;\theta))}{\partial\theta}\right] = 0. \tag{1}$$

Proof. At $l=1,2$ for all $\theta \in \Theta$

$$\sum_{l=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\partial^l h_m^{(i)}(x;\theta)}{\partial\theta^l}\right)^2 \nu_m(dx) + \int_{-\infty}^{\infty} \frac{\partial^l (1-H_m(x;\theta))}{\partial\theta^l} \nu_m(dx) = 0. \tag{2}$$

This equality is a differentiated version of the identity

$$M_m^{(0)}(+\infty;\theta) + \sum_{i=1}^{k} M_m^{(i)}(+\infty;\theta) = H_m(+\infty;\theta) = 1.$$

Now (1) is a consequence of (2).

Let us introduce $\psi^2(n;\theta) = \sum\limits_{m=1}^{n} I_m(\theta)$, $\varphi(n) = \varphi(n,t) = \psi^{-1}(n,t)$, and formulate a theorem on the LAN of a family of probability measures $\left\{\widetilde{Q}_\theta^{(n)},\, \theta \in \Theta\right\}$.

Theorem 1. *Let regularity conditions (C1)–(C3) are fulfilled for any $T > 0$*

$$\lim_{n\to\infty} \sup_{|u|<T} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} \left(\frac{\partial}{\partial\theta}\sqrt{h_m^{(i)}\left(x;\ t+\frac{u}{\psi(n;t)}\right)} - \frac{\partial}{\partial t}\sqrt{h_m^{(i)}(x;t)}\right)^2 dx = 0, \tag{3}$$

$$\lim_{n\to\infty} \sup_{|u|<T} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} \int \left(\frac{\partial}{\partial\theta}\sqrt{1-H_m\left(x;\ t+\frac{u}{\psi(n;t)}\right)} - \frac{\partial}{\partial t}\sqrt{1-H_m(x;t)}\right)^2 dx = 0 \tag{4}$$

and the Lindberg condition holds

$$\lim_{n\to\infty} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n}\sum_{i=1}^{k} M_\theta \left\{\left|\frac{\partial}{\partial t}\log h_m^{(i)}(X_m;t)\right| \cdot I\left(\left|\frac{\partial \log h_m(X_m;t)}{\partial t}\right| > n\psi(n;t)\right)\right\} = 0, \tag{5}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} M_\theta \Bigg\{ & \left|\frac{\partial}{\partial t}\log(1-H_m(X_m;t))\right| \times \\ & \times I\left(\left|\frac{\partial \log(1-H_m(X_m;t))}{\partial t}\right| > n\psi(n;t)\right)\Bigg\} = 0. \end{aligned} \tag{6}$$

Then the family of probability measures

$$\widetilde{Q}_\theta^{(n)}(A) = \int \underset{A}{\cdots} \int \prod_{m=1}^{n} \left\{\prod_{i=1}^{k}\left[h_m^{(i)}(Z_m;\theta)\right]\right\}^{y_m^{(i)}} \cdot \left[(1-H_m(Z_m;\theta))\right]^{y_m^{(0)}} \cdot \nu_m(dZ_m)$$

satisfies the LAN property at the point $\theta = t$.

Let us introduce

$$\Delta_{n,t} = \varphi(n,t) \sum_{m=1}^{n} \left[\sum_{i=1}^{k} \Delta_m^{(i)} \cdot \frac{\partial \log h_m^{(i)}(Z_m;t)}{\partial t} + \Delta_m^{(0)} \cdot \frac{\partial \log(1-H_m(Z_m;\theta))}{\partial t}\right] =$$

$$= \varphi(n;t)\sum_{m=1}^{n}\sum_{i=1}^{k}\Delta_m^{(i)}\cdot\frac{\partial\log h_m^{(i)}(Z_m;t)}{\partial t}+\varphi(n;t)\sum_{m=1}^{n}\Delta_m^0\frac{\partial\log(1-H_m(Z_m;\theta))}{\partial t}=$$

$$=\Delta_{n,t}^{(1)}+\Delta_{n,t}^{(2)},$$

also $\xi_{n,m}^{(i)}=\left[\frac{h_m^{(i)}(Z_m;t+\varphi(n)u)}{h_m^{(i)}(Z_m;t)}\right]^{1/2}-1$ and $\eta_{n,m}=\left[\frac{1-H_m(Z_m;t+\varphi(n)u)}{1-H(Z_m;t)}\right]^{1/2}-1$. Further, the following assertion is also necessary.

Lemma 2.2. *Suppose that conditions of Theorem 1 are hold then for any $u\in R^1$ we have*

$$\overline{\lim_{n\to\infty}}\sum_{m=1}^{n}M_t\left[\xi_{n,m}^{(i)}\right]^2\leqslant\frac{u^2}{4},\tag{7}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left[\eta_{n,m}^2\right]\leqslant\frac{u^2}{4},\tag{8}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left|\xi_{n,m}^{(i)}-\frac{1}{2}\varphi(n)u\cdot\frac{\partial\log h_m^{(i)}(Z_m;t)}{\partial t}\right|^2=0,\tag{9}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left|\eta_{n,m}-\frac{1}{2}\varphi(n)u\cdot\frac{\partial\log(1-H_m(Z_m;t))}{\partial t}\right|^2=0.\tag{10}$$

Proof of Lemma 2.2. We have

$$\begin{aligned}\sum_{m=1}^{n}M_t\left[\xi_{n,m}^{(i)}\right]^2&=\sum_{m=1}^{n}\int\limits_{\left\{x:\,h_m^{(i)}(x;\theta)\neq0\right\}}\left(\sqrt{h_m^{(i)}(x;\,t+\varphi(n)u)}-\sqrt{h_m^{(i)}(x;\,t)}\right)^2\cdot\nu_m(dx)\leqslant\\&\leqslant\sum_{m=1}^{n}\int\left(\int\limits_0^{\varphi(n)u}\frac{\frac{\partial}{\partial t}h_m^{(i)}(x;\,t+v)\,dv}{2\sqrt{h_m^{(i)}(x;\,t+\varphi(n)u)}}\right)^2\nu_m(dx)\leqslant\frac{u\varphi(n)}{4}\int\limits_0^{\varphi(n)u}\sum_{m=1}^{n}I_{m1}^{(i)}(t+v)\,dv,\end{aligned}\tag{11}$$

where

$$I_{m1}^{(i)}(t)=\int_{-\infty}^{\infty}\left(\frac{\partial\log h_m^{(i)}(x;t)}{\partial t}\right)^2h_m^{(i)}(x;t)\,\nu_m(dx).$$

Also

$$\begin{aligned}\sum_{m=1}^{n}M_t\left[\eta_{n,m}\right]^2&=\sum_{m=1}^{n}\int\limits_{\{x:\,H_m(x;t)=1\}}\left(\sqrt{1-H_m(x;t+\varphi(n)u)}-\sqrt{1-H_m(x;t)}\right)^2\nu_m(dx)\leqslant\\&\leqslant\sum_{m=1}^{n}\int\int\limits_0^{\varphi(n)u}\left(\frac{\frac{\partial}{\partial t}(1-H_m(x;\,t+v)\,dv)}{2\sqrt{(1-H_m(x;t+v))}}\right)^2\nu_m(dx)\leqslant\frac{u\varphi(n)}{4}\int\limits_0^{\varphi(n)u}\sum_{m=1}^{n}I_{m2}^{(i)}(t+v)\,dv,\end{aligned}\tag{12}$$

where

$$I_{m2}^{(i)}(t)=\int_{-\infty}^{\infty}\left(\frac{\partial\log(1-H_m(x;t))}{\partial t}\right)^2(1-H_m(x;t))\,\nu_m(dx).\tag{13}$$

Next, using the inequality

$$|ab|<\alpha\cdot\frac{a^2}{2}+\frac{1}{2\alpha}b^2,$$

where $a_m = \dfrac{\frac{\partial}{\partial\theta} h_m^{(i)}(x;\theta)}{\sqrt{h_m^{(i)}(x;\theta)}}$ and $b_m = \dfrac{\frac{\partial}{\partial t} h_m^{(i)}(x;t)}{\sqrt{h_m^{(i)}(x;t)}}$ one can find

$$\left|\frac{1}{\psi^2(n;t)}\sum_{m=1}^{n} I_{m1}^{(i)}(\theta) - 1\right| \leqslant \frac{1}{\psi^2(n;t)}\left|\int (a_m - b_m)(a_m + b_m)\,\nu_m(dx)\right| \leqslant$$

$$\leqslant \frac{1}{\psi^2(n;t)}\left[\alpha\sum_{m=1}^{n}\int (a_m - b_m)^2 \cdot \nu_m(dx) + \frac{1}{\alpha}\sum_{m=1}^{n}\left(\int a_m^2 \nu_m(dx) + \int b_m^2 \cdot \nu_m(dx)\right)\right].$$

Assuming $\alpha = 2$ in this inequality and taking into account (3) and the equality

$$\int a_m^2 \nu_m(dx) = I_{m1}^{(i)}(\theta),$$

we make sure that fraction $\dfrac{1}{\psi^2(n;t)}\sum_{m=1}^{n} I_{m1}^{(i)}(\theta)$ is bounded under the condition $|\theta - t| < \varphi(n)|u|$. Using this inequality for large enough α, we verify that

$$\lim_{n\to\infty}\sup_{|\theta - t| < \varphi(n)|u|}\left|\frac{1}{\psi^2(n;t)}\sum_{m=1}^{n} I_{m1}^{(i)}(\theta) - 1\right|. \tag{14}$$

Now (7) follows from (11) and (13).

Similarly, setting in inequality (13) $a_m = \dfrac{\frac{\partial}{\partial\theta}(1 - H_m(x;\theta))}{\sqrt{(1 - H_m(x;\theta))}}$ and $b_m = \dfrac{\frac{\partial}{\partial t}(1 - H_m(x;t))}{\sqrt{(1 - H_m(x;t))}}$ and repeating all the inequalities, one can obtain

$$\lim_{n\to\infty}\sup_{|\theta - t| < \varphi(n)|u|}\left|\frac{1}{\psi^2(n;t)}\sum_{m=2}^{n} I_{m1}^{(i)}(\theta) - 1\right| = 0 \tag{15}$$

which implies (8). It remains to prove (9) and (10). By virtue of (3) we have

$$\sum_{m=1}^{n} M_t\left(\xi_{n,m}^{(i)} - \frac{1}{2}\varphi(n)\,u\,\frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2 \leqslant$$

$$\leqslant \frac{1}{4}\sum_{m=1}^{n}\int\left[\int_0^{\varphi(n)u} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t+v)}{\sqrt{h_m^{(i)}(Z_m;t+v)}} - \frac{\frac{\partial}{\partial t} h_m^{(i)}(x_m;t)}{\sqrt{h_m^{(i)}(x_m;t)}}dv\right]^2 \nu_m(dx) \leqslant$$

$$\leqslant \frac{\varphi(n)\,u}{4}\int_0^{\varphi(n)u} dv \sum_{m=1}^{n}\int\left(\frac{\frac{\partial}{\partial t} h_m^{(i)}(x;t+v)}{\sqrt{h_m^{(i)}(x;t)}} - \frac{\frac{\partial}{\partial t} h_m^{(i)}(x;t)}{\sqrt{h_m^{(i)}(x;t)}}\right)^2 \nu_m(dx) \to 0, \quad n \to \infty.$$

Similarly, due to (4) we have

$$\sum_{m=1}^{n} M_t\left(\eta_{n,m} - \frac{1}{2}\varphi(n)\,u \cdot \frac{\frac{\partial}{\partial t}(1 - H_m(Z_m;t))}{1 - H_m(Z_m;t)}\right)^2 \leqslant$$

$$\leqslant \frac{1}{4}\sum_{m=1}^{n}\int\left[\int_0^{\varphi(n)u} \frac{\frac{\partial}{\partial t}(1 - H_m(x;t+v))}{\sqrt{(1 - H_m(x;t+v))}} - \frac{\frac{\partial}{\partial t}(1 - H_m(x;t))}{\sqrt{(1 - H_m(x;t))}}dv\right]^2 \nu_m(dx) \leqslant$$

$$\leqslant \frac{\varphi(n)\,u}{4} \int\limits_0^{\varphi(n)u} dv \sum_{m=1}^{n} \int \left(\frac{\frac{\partial}{\partial t}\left(1 - H_m(x;t+v)\right)}{\sqrt{\left(1 - H_m(x;t+v)\right)}} - \frac{\frac{\partial}{\partial t}\left(1 - H_m(x;t)\right)}{\sqrt{\left(1 - H_m(x;t)\right)}} \right)^2 \nu_m(dx) \to 0,\ n \to \infty$$

which proves Lemma 2.2. □

Proof of Theorem 1. Under conditions

$$\max_{m=\overline{1,n}} \left|\xi_{n,m}^{(i)}\right| < \varepsilon \quad \text{and} \quad \max_{m=\overline{1,n}} \left|\eta_{n,m}^{(i)}\right| < \varepsilon$$

we obtain

$$\sum_{m=1}^{n} \log\left[\frac{h_m^{(i)}(Z_m;t+\varphi(n)\,u)}{h_m(Z_m;t)}\right] = 2\log\left(1+\xi_{n,m}^{(i)}\right) = 2\sum_{m=1}^{n}\xi_{n,m}^{(i)} - \sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2 + \sum_{m=1}^{n}\gamma_{n,m}^{(i)}\left|\xi_{n,m}^{(i)}\right|^3,$$

and

$$\begin{aligned}\sum_{m=1}^{n} \log\left[\frac{1 - H_m(Z_m;t+\varphi(n)\,u)}{1 - H_m(Z_m;t)}\right] &= 2\log\left(1+\eta_{n,m}\right) = \\ &= 2\sum_{m=1}^{n}\eta_{n,m} - \sum_{m=1}^{n}\left[\eta_{n,m}\right]^2 + \sum_{m=1}^{n}\beta_{n,m}\cdot\left|\eta_{n,m}\right|^3,\end{aligned} \tag{16}$$

where $\left|\gamma_{n,m}^{(i)}\right| < 1$ and $|\beta_{n,m}| < 1$, $m = \overline{1,n}$; $i = \overline{1,k}$ with probability 1. Let us prove the following relations for terms of expansions (16)

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\xi_{n,m}^{(i)}\right| > \varepsilon\right\} = 0, \tag{17}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\eta_{n,m}\right| > \varepsilon\right\} = 0, \tag{18}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2 - \frac{u^2}{4}\right| > \varepsilon\right\} = 0, \tag{19}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\eta_{n,m}^2 - \frac{u^2}{4}\right| > \varepsilon\right\} = 0, \tag{20}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\left|2\sum_{m=1}^{n}\xi_{n,m}^{(i)} - \varphi(n)\,u\cdot\sum_{m=1}^{n}\frac{\frac{\partial}{\partial t}\log h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} + \frac{u^2}{4}\right| > \varepsilon\right\} = 0, \tag{21}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\left|2\sum_{m=1}^{n}\eta_{n,m} - \varphi(n)\,u\cdot\sum_{m=1}^{n}\frac{\frac{\partial}{\partial t}\log\left(1 - H_m(Z_m;t)\right)}{\left(1 - H_m(Z_m;t)\right)} + \frac{u^2}{4}\right| > \varepsilon\right\} = 0 \tag{22}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\sum_{m=1}^{n}\left|\xi_{n,m}^{(i)}\right|^3 > \varepsilon\right\} = 0 \tag{23}$$

$$\lim_{n\to\infty} \widetilde{Q}_t^{(n)}\left\{\sum_{m=1}^{n}\left|\eta_{n,m}\right|^3 > \varepsilon\right\} = 0. \tag{24}$$

Using above relations, one needs to establish (17), (19), (21) and (23). The rest relations are proved quite similarly. Consider the inequality

$$\widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\xi_{n,m}^{(i)}\right|>\varepsilon\right\}\leqslant\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\xi_{n,m}^{(i)}\right|>\varepsilon\right\}\leqslant$$

$$\leqslant\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\xi_{n,m}^{(i)}-\frac{\varphi(n)\,u}{2}\frac{\partial h_m^{(i)}(Z_m;t)}{\frac{\partial t}{h_m(Z_m;t)}}\right|>\varepsilon/2\right\}+\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m(Z_m;t)}\right|>\frac{\varepsilon}{4\varphi(n)\,|u|}\right\},$$

where the Chebyshev inequality is used for the first component, and (9) is used for the second one. Now to prove (19) consider the following inequalities

$$\widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2-\frac{1}{4}\varphi^2(n)\,u^2\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2\right|>\varepsilon\right\}\leqslant$$

$$\leqslant\frac{1}{\varepsilon}\sum_{m=1}^{n}M_t\left|\left[\xi_{n,m}^{(i)}\right]^2-\frac{1}{4}\varphi^2(n)\,u^2\cdot\left(\frac{\frac{\partial}{\partial t}h_m(Z_m;t)}{h_m(Z_m;t)}\right)^2\right|\leqslant$$

$$\leqslant\frac{\alpha}{2\varepsilon}\sum_{m=1}^{n}M_t\left|\xi_{n,m}-\frac{1}{2}\varphi(n)\,u\cdot\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m(Z_m;t)}\right|^2+\frac{1}{2\alpha\varepsilon}\left(1+\sum_{m=1}^{n}M_t\xi_{n,m}^{(i)}\right).$$

In this case, using the law of large numbers for sums

$$\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2$$

with the corresponding normalization, we have

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|\varphi^2(n)\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2-1\right|>\varepsilon\right\}=0.$$

Equality (19) is proved. Equality (23) is a consequence of (17) and (19). It remains to establish (21). It follows from (19) that $\sum\limits_{m=1}^{n}\left(\xi_{n,m}^{(i)}\right)^2$ converges in probability to $\frac{u^2}{4}$. Using (7), we obtain the equality

$$\lim_{n\to\infty}M_t\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2=\frac{u^2}{4}.$$

Using this equality and (11), the following relation is obtained

$$\lim_{n\to\infty}\sum_{m=1}^{n}\int\limits_{\left\{x:\,h_m^{(i)}(x;t)=0\right\}}h_m^{(i)}(x;t+\varphi(n)\,u)\,\nu_m(dx)=0.$$

Considering these two equalities and passing to the mathematical expectations in the identity

$$\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2=\sum_{m=1}^{n}\left(\frac{h_m^{(i)}(Z_m;t+\varphi(n)\,u)}{h_m(Z_m;t)}-1\right)-2\sum_{m=1}^{n}\xi_{n,m}^{(i)},$$

we obtain

$$\lim_{n\to\infty} M_t \sum_{m=1}^{n} \xi_{n,m}^{(i)} = -\frac{u^2}{8}.$$

Next, for $n \geqslant n_0$ we find

$$\widetilde{Q}_t^{(n)} \left\{ \left| 2\sum_{m=1}^{n} \xi_{n,m}^{(i)} - \varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} + \frac{u^2}{4} \right| > \varepsilon \right\} \leqslant$$

$$\leqslant \widetilde{Q}_t^{(n)} \left\{ \left| 2\sum_{m=1}^{n} \left(\xi_{n,m}^{(i)} - M_t \xi_{n,m}^{(i)} \right) - \varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right| > \frac{\varepsilon}{2} \right\} \leqslant$$

$$\leqslant \frac{16}{\varepsilon^2} M_t \left[\sum_{m=1}^{n} \left(\xi_{n,m}^{(i)} - M_t \xi_{n,m}^{(i)} \right) - \frac{1}{2}\varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right]^2 \leqslant$$

$$\leqslant \frac{16}{\varepsilon^2} \sum_{m=1}^{n} M_t \left(\xi_{n,m}^{(i)} - \frac{1}{2}\varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right)^2$$

Now (21) follows from the last relation and (9). To prove the remaining relations (18), (20), (22) and (24) one should proceed in a similar way. Theorem is completely proved. □

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Локальная асимптотическая нормальность статистических экспериментов в неоднородной модели конкурирующих рисков при случайном цензурировании справа

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Аннотация. Статья посвящена доказательству свойства локальной асимптотической нормальности статистики отношения правдоподобия в модели конкурирующих рисков, отвечающих неоднородным и случайно цензурированных справа наблюдениям.

Ключевые слова: локальная асимптотическая нормальность, статистика отношения правдоподобия, модель конкурирующих рисков, случайное цензурирование, асимптотическое представление.

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Creeping Three-dimensional Convective Motion in a Layer with Velocity Field of a Special Type

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Abstract. Problem of three-dimensional motion of a heat-conducting fluid in a channel with solid parallel walls is considered. Given temperature distribution is maintained on solid walls. The liquid temperature depends quadratically on the horizontal coordinates, and the velocity field has a special form. The resulting initial-boundary value problem for the Oberbeck–Boussinesq model is inverse and reduced to a system of five integro-differential equations. For small Reynolds numbers (creeping motion), the resulting system becomes linear. A stationary solution has been found for this system, and a priori estimates have been obtained. On the basis of these estimates, sufficient conditions for exponential convergence of a smooth non-stationary solution to a stationary solution have been established. The solution of the inverse problem has been found in the form of quadratures for the Laplace images under weaker conditions for the temperature regime on the walls of the layer. Behaviour of the velocity field for a specific liquid medium have been presented. The results were obtained with the use of numerical inversion of the Laplace transform.

Keywords: Oberbec–Boussinesq model, three-dimensional motion, inverse problem, a priori estimates, stability, Laplace transform.

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Problem statement and derivation of basic equations

Two-dimensional flows of the Himentz type [1] are known as flows near the critical point and they are characterized by the presence of zones with higher pressure and temperature than in the surrounding region. Such flows can be observed both in macro-scales (for example, the use of hydraulic fracturing technologies in the oil industry) and in micro-scales (for example, liquid biochips in medicine). The study of characteristics of such flows is necessary to assess the technological parameters, as well as to predict the dynamics and evolution of the liquid layer. Exact solutions of the defining equations are the most effective way to study processes in a liquid, as well as to obtain estimated characteristics. At present, solutions of problems describing Himentz-type flows in various geometries are presented: axisymmetric [2] and three-dimensional [3, 4] analogues of the Himentz solution, including flows in cylindrical geometry [5, 6]. A brief overview of the exact solutions that are close to the Himentz solution is given in [7].

Three-dimensional motion of a viscous incompressible heat-conducting fluid with special velocity field is studied in this paper. The velocity field is of the Himentz type: the horizontal

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components of the velocity field are linear in the corresponding coordinates, temperatures are set on solid walls.

The system of Oberbeck–Boussinesq equations of three-dimensional motion has the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\cdot\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\Delta\mathbf{u} + \mathbf{g}(1-\beta T), \quad \operatorname{div}\mathbf{u} = 0, \tag{1}$$

$$T_t + \mathbf{u}\cdot\nabla T = \chi\Delta T, \tag{2}$$

where $\mathbf{u}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$ is the velocity vector, u, v, w are components of the velocity vector in the Cartesian coordinate system; $\mathbf{g} = (0,0,-g)$; t is time; $T(x,y,z,t)$ is temperature; positive constants ρ, ν, χ, β, g are density, kinematic viscosity, thermal conductivity coefficient, coefficient of thermal expansion and acceleration of gravity, respectively. The solution of problem (1), (2) is taken in the following form

$$\begin{gathered} u(x,y,z,t) = (f(z,t)+h(z,t))x, \quad v(x,y,z,t) = (f(z,t)-h(z,t))y, \\ w(x,y,z,t) = -2\int_0^z f(\xi,t)d\xi, \quad p(x,y,z,t) = \bar{p}(x,y,z,t) - \rho g z, \\ T(x,y,z,t) = a(z,t)x^2 + b(z,t)xy + c(z,t)y^2 + \theta(z,t). \end{gathered} \tag{3}$$

Relations (3) are interpreted as fluid motion between two flat parallel fixed plates $z = 0$ and $z = l$ (see Fig. 1). Then adhesion conditions are set on fixed plates: $u(x,y,0,t) = v(x,y,0,t) = w(x,y,0,t) = 0$, $u(x,y,l,t) = v(x,y,l,t) = w(x,y,l,t) = 0$. Temperature is given in the form $T(x,y,0,t) = a_1(t)x^2 + b_1(t)xy + c_1(t)y^2$, $T(x,y,l,t) = a_2(t)x^2 + b_2(t)xy + c_2(t)y^2$. Considering (3), using conditions of adhesion and setting the temperature, boundary conditions for functions $a(z,t)$, $b(z,t)$, $c(z,t)$ $\theta(z,t)$ $f(z,t)$ $h(z,t)$ are derived

$$\begin{gathered} f(0,t) = f(l,t) = h(0,t) = h(l,t) = 0, \quad \int_0^l f(\xi,t)d\xi = 0, \\ a(0,t) = a_1(t), \quad b(0,t) = b_1(t), \quad c(0,t) = c_1(t), \quad \theta(0,t) = 0, \\ a(l,t) = a_2(t), \quad b(l,t) = b_2(t), \quad c(l,t) = c_2(t), \quad \theta(l,t) = 0, \end{gathered} \tag{4}$$

where functions $a_j(t)$, $c_j(t)$, $j = 1,2$ are set at some interval $[0,t_0]$. In addition, initial conditions are set

$$\begin{gathered} a(z,0) = a_0(z), \quad c(z,0) = c_0(z), \quad b(z,0) = b_0(z), \quad \theta(z,0) = 0, \\ f(z,0) = f_0(z), \quad h(z,0) = h_0(z). \end{gathered} \tag{5}$$

Remark 1. *Since* $\operatorname{rot}\mathbf{u} = ((h_z - f_z)y, (h_z + f_z)x, 0) \neq 0$, *then the motion is vortex.*

Remark 2. *Suppose, without the loss of generality, that* $a_j(t) \neq 0$, $j = 1,2$ *and* $b(z,t) = 0$. *Then when* $a_j(t) < 0$, $c_j(t) < 0$ *functions* $T_j(x,y,t)$ *have a maximum at the point* $x = 0$, $y = 0$, *and when* $a_j(t) > 0$, $c_j(t) > 0$ *functions* $T_j(x,y,t)$ *have a minimum. If* $a_j(t)$ *and* $c_j(t)$ *have the same signs then* $T_j(x,y,t)$ *is an elliptical paraboloid. If* $a_j(t)$ *and* $c_j(t)$ *have different signs then* $T_j(x,y,t)$ *is a hyperbolic paraboloid. In other words, the above solution describes the convection of a liquid near the points of temperature extremes on solid walls. There may be other cases, for example, the temperature has a maximum on the lower wall and a minimum on the upper wall or vice versa.*

The first step is to derive a system of equations for f, h, a, b, c, θ.

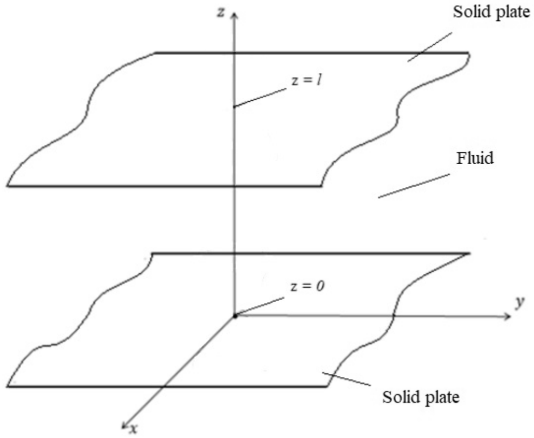

Fig. 1. Flow area diagram

Taking into account equations (2) and (3), the following relations are obtained

$$
\begin{gathered}
a_t + 2a(f+h) - 2a_z \int_0^z f(\xi,t)d\xi = \chi a_{zz},\ b(z,t) = 0,\\
c_t + 2c(f-h) - 2c_z \int_0^z f(\xi,t)d\xi = \chi c_{zz},\\
\theta_t - 2\theta_z \int_0^z f(\xi,t)d\xi = 2\chi(a+c) + \chi\theta_{zz}.
\end{gathered}
\tag{6}
$$

The mass conservation equation is satisfied identically, and momentum equation (1) is equivalent to the following equation

$$
\begin{gathered}
f_t + f^2 + h^2 - 2f_z \int_0^z f(\xi,t)d\xi = \nu f_{zz} - \beta g \int_0^z [a(\xi,t) + c(\xi,t)]d\xi + n_1(t),\\
h_t + 2fh - 2h_z \int_0^z f(\xi,t)d\xi = \nu h_{zz} - \beta g \int_0^z [a(\xi,t) - c(\xi,t)]d\xi + n_2(t),
\end{gathered}
\tag{7}
$$

where $n_1(t), n_2(t)$ are arbitrary functions of time that represent incremental pressure gradients. The modified pressure $\bar{p}(x,y,z,t)$ is found in the form of quadratures

$$
\begin{gathered}
\frac{1}{\rho}\bar{p}(x,y,z,t) = x^2(g\beta \int_0^z a(\xi,t)d\xi - \frac{1}{2}(n_1(t)+n_2(t))) +\\
+ y^2(g\beta \int_0^z c(\xi,t)d\xi - \frac{1}{2}(n_1(t)-n_2(t))) - 2\nu f(z,t) - gz +\\
+ g\beta \int_0^z \theta(\xi,t)d\xi + 2\int_0^z (z-\xi) f_t(\xi,t)d\xi - 2(\int_0^z f(\xi,t)d\xi)^2 + \alpha_0(t),
\end{gathered}
$$

where $\alpha_0(t)$ is an arbitrary function of time.

Thus, the Oberbeck–Boussinesq system is reduced to five non-linear integro-differential equations.

The following notations are introduced

$$\xi = \frac{z}{l};\quad \tau = \frac{\chi}{l^2}t;\quad a^* = \max(\mid a_j(t) \mid, \mid c_j(t) \mid),\ \ j=1,2,\quad u^* = \beta a^* l\chi;$$
$$a(z,t) = a^*A(\xi,\tau);\quad c(z,t) = a^*C(\xi,\tau);\quad \theta(z,t) = a^*\Theta(\xi,\tau);\quad f(z,t) = \frac{\chi}{l^2}ReF(\xi,\tau); \tag{8}$$
$$h(z,t) = \frac{\chi}{l^2}ReH(\xi,\tau);\quad n_j(t) = \frac{\chi^2}{l^4}N_j(\tau),\ j=1,2.$$

Here u^* is the characteristic rate of thermal expansion of the fluid, since a^*l^2 is the characteristic temperature of the walls, $\epsilon = \beta a^* l^2$ is the Boussinesq parameter [11], $Re = u^*l/\nu$ is the Reynolds number, $Re = \epsilon P$, where $P = \nu/\chi$ is the Prandtl number.

After substituting (8) into system (6), (7), the initial boundary value problem in dimensionless form is obtained

$$\begin{gathered}
A_\tau + 2ReA(F+H) - 2ReA_\xi \int_0^\xi F(\xi,\tau)d\xi = A_{\xi\xi},\\
C_\tau + 2ReC(F-H) - 2ReA_\xi \int_0^\xi F(\xi,\tau)d\xi = C_{\xi\xi},\\
\Theta_\tau - 2Re\Theta_\xi \int_0^\xi F(\xi,\tau)d\xi = 2(A+C) + \Theta_{\xi\xi},\\
F_\tau + ReF^2 + ReH^2 - 2ReF_\xi \int_0^\xi F(\xi,\tau)d\xi = PF_{\xi\xi} - \eta P\int_0^\xi [A(\xi,\tau)+C(\xi,\tau)]d\xi + N_1(\tau),\\
H_\tau + 2ReFH - 2ReH_\xi \int_0^\xi F(\xi,\tau)d\xi = PH_{\xi\xi} - \eta P\int_0^\xi [A(\xi,\tau)-C(\xi,\tau)]d\xi + N_2(\tau).
\end{gathered} \tag{9}$$

Parameter $\eta = gl^3(\nu\chi)^{-1}$ plays an important role in the theory of micro convection [11].

In system (9) $\tau \in [0, \tau_0 = \chi t_0 l^{-2}]$, $\xi \in [0,1]$. To fully define unknowns A, C, Θ, F, H, N_1, N_2 it is necessary to consider initial and boundary conditions

$$\begin{gathered}
A(\xi,0) = A_0(\xi),\quad C(\xi,0) = C_0(\xi),\quad \Theta(\xi,0) = 0,\\
F(\xi,0) = F_0(\xi),\quad H(\xi,0) = H_0(\xi).
\end{gathered} \tag{10}$$

$$\begin{gathered}
A(0,\tau) = A_1(\tau),\quad C(0,\tau) = C_1(\tau),\quad \Theta(0,\tau) = F(0,\tau) = H(0,\tau) = 0,\\
A(1,\tau) = A_2(\tau),\quad C(1,\tau) = C_2(\tau),\quad \Theta(1,\tau) = F(1,\tau) = H(1,\tau) = 0.
\end{gathered} \tag{11}$$

$$\int_0^1 F(\xi,\tau)d\xi = 0,\quad \int_0^1 H(\xi,\tau)d\xi = 0. \tag{12}$$

Let us note that problem (9)–(12) is the inverse problem, since functions $N_j(t)$ are unknown.

Remark 3. *Conditions (12) actually mean that motion is considered in some cell bounded by x and y.*

Conditions for matching the input data are satisfied for a smooth solution

$$A_0(0) = A_1(0),\quad C_0(0) = C_1(0),\quad A_0(1) = A_2(0),\quad C_0(1) = C_2(0), \tag{13}$$

$$\int_0^1 F_0(\xi)d\xi = 0,\quad \int_0^1 H_0(\xi)d\xi = 0. \tag{14}$$

Remark 4. *Taking into account (8), it is assumed that $a_j(t) = a^*A_j(\tau)$, $c_j(t) = a^*C_j(\tau)$.*

For most liquid media, the Boussinesq number is $\epsilon \ll 1$. Therefore, one can look for a solution of the inverse initial-boundary value problem in the form of a series with respect to the Reynolds number Re. The main terms of the decomposition satisfy the linear system of equations (the designations of the desired functions are left the same)

$$\begin{gathered} A_\tau = A_{\xi\xi}, \quad C_\tau = C_{\xi\xi}, \quad \Theta_\tau = 2(A+C) + \Theta_{\xi\xi}, \\ F_\tau = PF_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) + C(\xi,\tau)]d\xi + N_1(\tau), \\ H_\tau = PH_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) - C(\xi,\tau)]d\xi + N_2(\tau). \end{gathered} \tag{15}$$

The initial and boundary conditions remain unchanged (see (4), (5)). The problem describes the so-called "crawling" movements and it is the subject of study of this work.

Stationary creeping motion

In this case, all functions do not depend on the dimensionless time τ and initial data (5) is not taken into account. Let us assume that $A^s(\xi)$, $C^s(\xi)$, $\Theta^s(\xi)$, $F^s(\xi)$, $H^s(\xi)$, $N_1^s(\xi)$, N_2^s is the required solution, A_j^s, C_j^s are the given constants. Without the loss of of generality, it is assumed that $A_1^s \neq 0$. Simple mathematical treatment shows that there are relations

$$\begin{gathered} A^s(\xi) = A_1^s(1+\alpha_1\xi), \quad C^s(\xi) = A_1^s(\alpha_2+\alpha_3\xi), \\ \alpha_1 = \frac{A_2^s - A_1^s}{A_1^s}, \quad \alpha_2 = \frac{C_1^s}{A_1^s}, \quad \alpha_3 = \frac{C_2^s - C_1^s}{A_1^s}; \\ \Theta^s(\xi) = A_1^s\left[(1+\alpha_2)(\xi-\xi^2) + \frac{\alpha_1+\alpha_2}{3}(\xi-\xi^3)\right]; \\ F^s(\xi) = \frac{\eta A_1^s P}{12}\left[(1+\alpha_2)(2\xi^3-3\xi^2+\xi) + \frac{\alpha_1+\alpha_3}{10}(5\xi^4-9\xi^2+4\xi)\right], \\ H^s(\xi) = \frac{\eta A_1^s P}{12}\left[(1-\alpha_2)(2\xi^3-3\xi^2+\xi) + \frac{\alpha_1-\alpha_3}{10}(5\xi^4-9\xi^2+4\xi)\right]; \\ N_1^s = \frac{1}{2}\eta A_1^s P^2\left[1+\alpha_2+\frac{3}{10}(\alpha_1+\alpha_3)\right], \\ N_2^s = \frac{1}{2}\eta A_1^s P^2\left[1-\alpha_2+\frac{3}{10}(\alpha_1-\alpha_3)\right]. \end{gathered} \tag{16}$$

When $A_1^s = C_1^s$ there is radial heating of the fluid on the wall. If $A_1^s, C_1^s < 0$ then heating is maximal at the point $x=0, y=0$. If $A_1^s, C_1^s > 0$ then heating is minimal. If $A_j^s = -C_j^s$ then heating of the fluid on the wall has the form of a hyperbola.

The characteristic vertical velocity profile $W^s(\xi) = w^s(\xi)/W^0$ is shown in Fig. 2 ($W^0 = -\eta A_1^s \chi$)

Physical constants were taken for water at a temperature of 20 ℃: $P \sim 7$, $Re \sim 25.5 \cdot 10^{-}4$, values A_j^s, C_j^s, $j = 1, 2$ are shown in Fig. 2.

The solid line shows the case of radial heating of the fluid on the walls with a minimum of its value at the point $x = 0$, $y = 0$ while the fluid in the layer moves upwards.

The dashed line shows the vertical velocity profile when distribution of the fluid temperature has the form of a hyperbola on the lower wall and weak elliptical heating on the upper wall.

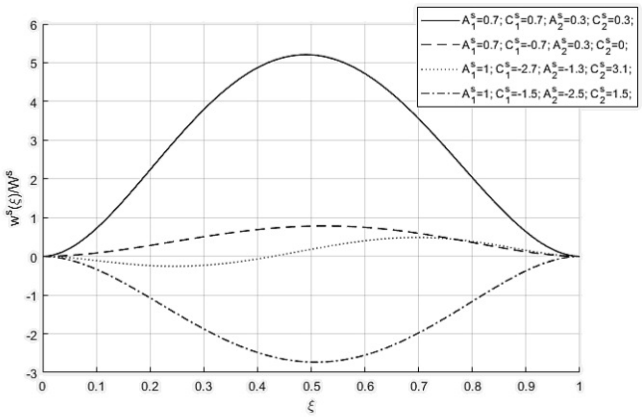

Fig. 2. Vertical velocity W^s as a function of dimensionless coordinate ξ.

In other cases, heating on both walls has the form of a hyperbola. The dotted line corresponds to such a temperature distribution that fluid in the lower part of the layer moves down, and in the upper part it moves up.

A priori estimates

The purpose of this paragraph is to establish sufficient conditions for the input data $A_j(t)$, $C_j(t)$, under which the solution of non-stationary problem converges to stationary solution (16) when dimensionless time increases. Functions $A(\xi,t)$, $C(\xi,t)$, $\Theta(\xi,t)$ are solutions of the first initial boundary value problem. They can be found in the form of trigonometric Fourier series. Using methods proposed in [12], it is possible to obtain a priori estimates of solutions. However, here it is easier to use results presented in [13] (pp. 201, 209). In fact, if $A_j(\tau)$, $C_j(\tau)$ are continuous for any $\tau \geqslant 0$ and

$$\lim_{\tau\to\infty} A_j(\tau) = A_j^s, \quad \lim_{\tau\to\infty} C_j(\tau) = C_j^s, \tag{17}$$

then

$$\lim_{\tau\to\infty} A_j(\xi,\tau) = A^s(\xi), \quad \lim_{\tau\to\infty} C_j(\xi,\tau) = C^s(\xi),$$

uniformly for any $\xi \in [0,1]$, where $A^s(\xi)$, $C^s(\xi)$ is stationary solution (16). If

$$|\,A_j(\tau) - A_j^s\,| \leqslant d(1+\tau)^{-\mu}, \quad |\,C_j(\tau) - C_j^s\,| \leqslant d(1+\tau)^{-\mu}, \tag{18}$$

with positive coefficients d, μ then

$$|\,A^s(\xi,\tau) - A^s(\xi)\,| \leqslant d_1(1+\tau)^{-\mu}, \quad |\,C^s(\xi,\tau) - C^s(\xi)\,| \leqslant d_1(1+\tau)^{-\mu}, \tag{19}$$

$d_1 > 0$ is a constant, $\xi \in [0,1]$. Considering inequalities

$$|\,A_j(\tau) - A_j^s\,| \leqslant d_2 e^{-\mu\tau}, \quad |\,C_j(\tau) - C_j^s\,| \leqslant d_2 e^{-\mu\tau}, \tag{20}$$

estimates

$$|\,A^s(\xi,\tau) - A^s(\xi)\,| \leqslant d_3 e^{-\mu_1\tau}, \quad |\,C^s(\xi,\tau) - C^s(\xi)\,| \leqslant d_3 e^{-\mu_1\tau}, \tag{21}$$

are obtained with constants $d_3 > 0$, $0 < \mu_1 \leqslant \mu$.

These estimates can be interpreted as the stability conditions of stationary solution $A^s(\xi)$, $C^s(\xi)$ under conditions (17), (18), (20).

Initial-boundary inverse problems for functions $F(\xi,\tau)$, $N_1(\tau)$ and $H(\xi,\tau)$, $N_2(\tau)$ are non classical ($A(\xi,\tau)$, $C(\xi,\tau)$ are known). Therefore, a priori estimates of their solutions have to be obtained.

Multiplying the last equation of system (15) by $H(\xi,\tau)$ and integrating with respect to ξ from zero to one, the following identity is obtained

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 H^2(\xi,\tau)d\xi + P\int_0^1 H_\xi^2(\xi,\tau)d\xi = -\eta P\int_0^1 H(\xi,\tau)\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\xi d\epsilon. \tag{22}$$

Here, boundary conditions (11) and redefinition condition (12) are taken into account. Since Steklov's inequality takes place

$$\int_0^1 H^2(\xi,\tau)d\xi \leqslant \frac{1}{\pi^2}\int_0^1 H_\xi^2(\xi,\tau)d\xi$$

then the left part of (22) is greater than or equal to

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 H^2(\xi,\tau)d\xi + \pi^2 P\int_0^1 H^2(\xi,\tau)d\xi.$$

The right part of (22) does not exceed

$$\eta P\Big(\int_0^1 H^2(\xi,\tau)\Big)^{\frac{1}{2}}\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}}.$$

Now for $E(\tau)=\Big(\int\limits_0^1 H^2(\xi,\tau)\Big)^{\frac{1}{2}}$ the following inequality is obtained

$$\frac{dE}{d\tau}+\pi^2 PE \leqslant \eta P\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}}.$$

Therefore,

$$\int_0^1 H^2(\xi,\tau)d\xi \leqslant \Big\{\Big(\int_0^1 H_0^2(\xi)d\xi\Big)^{\frac{1}{2}} + \eta P\int_0^\tau e^{\pi^2P^2\tau}\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)- \\ -C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}} d\tau\Big\}^2 e^{-2\pi^2P\tau} \equiv G_1(\tau)e^{-\pi^2P\tau} \tag{23}$$

for any $\tau\in[0,\tau_0]$. Now recall that functions $A(\xi,\tau)$, $C(\xi,\tau)$ satisfy estimates (19) or (21), where $A^s(\xi)=0$, $C^s(\xi)=0$.

Function $H(\xi,\tau)$ also satisfies the following identity

$$\int_0^1 H^2(\xi,\tau)d\xi + \frac{P}{2}\frac{d}{d\tau}\int_0^1 H_\xi^2(\xi,\tau)d\xi = -\eta P\int_0^1 H_\tau(\xi,\tau)d\xi\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\xi d\epsilon.$$

Using the elementary inequality $ab\leqslant \epsilon_1 a^2/2 + b^2/(2\epsilon_1)$ when $\epsilon_1=(2\eta P)^{-1}$, one can obtain from the previous identity that

$$\int_0^1 H_\xi^2(\xi,\tau)d\xi \leqslant 2\eta^2P^2\int_0^\tau\int_0^1\Big[\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\epsilon\Big]^2 d\xi d\tau + \\ +\int_0^1 H_{0\xi}^2(\xi)d\xi \equiv G_2(\tau) \tag{24}$$

Since $H(0,\tau)=0$ then

$$H^2(\xi,\tau)=2\int_0^\xi H(\xi,\tau)H_\xi(\xi,\tau)d\xi\leqslant$$
$$\leqslant 2\Big(\int_0^1 H^2(\xi,\tau)d\xi\Big)^{\frac12}\Big(\int_0^1 H_\xi^2(\xi,\tau)d\xi\Big)^{\frac12}\leqslant 2\sqrt{G_1(\tau)G_2(\tau)}e^{-\pi^2P\tau}$$

due to inequalities (23), (24) and

$$\mid H(\xi,\tau)\mid\leqslant\sqrt{2}(G_1(\tau)G_2(\tau))^{\frac14}e^{-\frac{\pi^2P}{2}\tau} \tag{25}$$

for any $\xi\in[0,1]$, $\tau\in[0,\tau_0]$.

Similar estimate holds for $F(\xi,\tau)$ if $A(\xi,\tau)-C(\xi,\tau)$ is replaced with $A(\xi,\tau)+C(\xi,\tau)$ in expressions $G_1(\tau)$, $G_2(\tau)$, and they are denoted by $G_3(\tau)$ and $G_4(\tau)$. Therefore

$$\mid F(\xi,\tau)\mid\leqslant\sqrt{2}(G_3(\tau)G_4(\tau))^{\frac14}e^{-\frac{\pi^2P}{2}\tau}. \tag{26}$$

Let us start first with the evaluation of $N_2(\tau)$. Multiplying the equation for $H(\xi,\tau)$ by $\xi-\xi^2$, integrating over the interval $[0,1]$ and using the boundary conditions, one can obtain

$$N_2(\tau)=6\int_0^1(\xi-\xi^2)H_\tau(\xi,\tau)d\xi+6\int_0^1(\xi-\xi^2)\int_0^\xi(A(\varepsilon,\tau)-C(\varepsilon,\tau))d\varepsilon d\xi, \tag{27}$$

since $\int\limits_0^1(\xi-\xi^2)H_{\xi\xi}(\xi,\tau)d\xi=0$. To evaluate $N_2(\tau)$ it is necessary to obtain an estimate of $\mid H_\tau(\xi,\tau)\mid$ at $\xi\in[0,1]$, $\tau\in[0,\tau_0]$. If

$$\begin{gathered}\mid A_j(\tau)\mid\leqslant d_2e^{-\mu\tau},\quad \mid C_j(\tau)\mid\leqslant d_2e^{-\mu\tau},\\ \mid A_{j\tau}(\tau)\mid\leqslant d_4e^{-\mu\tau},\quad \mid C_{j\tau}(\tau)\mid\leqslant d_4e^{-\mu\tau},\end{gathered} \tag{28}$$

$d_4>0$ then

$$\begin{gathered}\mid A(\xi,\tau)\mid\leqslant d_3e^{-\mu_1\tau},\quad \mid C(\xi,\tau)\mid\leqslant d_3e^{-\mu_1\tau},\\ \mid A_\tau(\xi,\tau)\mid\leqslant d_5e^{-\mu_1\tau},\quad \mid C_\tau(\xi,\tau)\mid\leqslant d_5e^{-\mu_1\tau}\end{gathered} \tag{29}$$

for any $\xi\in[0,1]$, $\tau\in[0,\tau_0]$. The first two equations of system (15) provide estimates of derivatives

$$\mid A_{\xi\xi}(\xi,\tau)\mid\leqslant d_5e^{-\mu_1\tau},\quad \mid C_{\xi\xi}(\xi,\tau)\mid\leqslant d_5e^{-\mu_1\tau}. \tag{30}$$

To obtain estimates of derivatives (29), (30) it is enough to differentiate with respect to τ the corresponding initial boundary value problems, and use the results presented in [13]. Similarly, differentiating with respect to τ the last equation of system (15), a problem on $H_\tau(\xi,\tau)$ is obtained. It is similar to the problem on $H(\xi,\tau)$ when $A(\xi,\tau)-C(\xi,\tau)$ is replaced with $A_\tau(\xi,\tau)-C_\tau(\xi,\tau)$ and $N_2(\tau)$ is replaced with $N_{2\tau}(\tau)$. Therefore, there is an estimate (see (25))

$$\mid H_\tau(\xi,\tau)\mid\leqslant\sqrt{2}(G_3(\tau)G_4(\tau))^{\frac14}e^{-\frac{\pi^2P}{2}\tau}, \tag{31}$$

$\xi\in[0,1]$, $\tau\in[0,\tau_0]$, where $H_0(\xi)$ is replaced with $H_\tau(\xi,0)$ in relation for $G_3(\tau)$ (see (23)). Then

$$H_\tau(\xi,0)=\frac{1}{P}H_{0\xi\xi}(\xi)-\eta P\int_0^\xi(A_0(\xi)-C_0(\xi))d\xi+N_2(0). \tag{32}$$

The value of $N_2(0)$ can be found from another representation of $N_2(\tau)$:

$$N_2(\tau) = \frac{1}{P}(H_\xi(0,\tau) - H_\xi(1,\tau)) + \eta P\int_0^1\int_0^\epsilon (A(\xi,\tau) - C(\xi,\tau))d\xi d\epsilon.$$

Thus

$$N_2(0) = \frac{1}{P}(H_{0\xi}(0) - H_{0\xi}(1)) + \eta P\int_0^1\int_0^\epsilon (A_0(\xi) - C_0(\xi))d\xi d\epsilon.$$

Considering (27) and using inequalities (31), (29), the following estimate is obtained

$$\mid N_2(\tau)\mid \leqslant \frac{3}{\sqrt{2}}[(G_3(\tau)G_4(\tau))^{\frac{1}{4}}e^{-\frac{\pi^2 P}{2}\tau} + 4d_3 e^{-\mu_1\tau}],\quad \tau\in[0,\tau_0]. \tag{33}$$

A similar assessment takes place for $H_\tau(\xi,\tau)$, $N_1(\tau)$

$$\begin{gathered}\mid F_\tau(\xi,\tau)\mid \leqslant \sqrt{2}(G_5(\tau)G_6(\tau))^{\frac{1}{4}}e^{-\frac{\pi^2 P}{2}\tau},\\ \mid N_1(\tau)\mid \leqslant \frac{3}{\sqrt{2}}[(G_5(\tau)G_6(\tau))^{\frac{1}{4}}e^{-\frac{\pi^2 P}{2}\tau} + 4d_3 e^{-\mu_1\tau}],\quad \tau\in[0,\tau_0].\end{gathered} \tag{34}$$

where $G_5(\tau)$ and $G_6(\tau)$ follow from $G_3(\tau)$ and $G_4(\tau)$ when the term $A(\xi,\tau)+C(\xi,\tau)$ is replaced with $A_\tau(\xi,\tau)+C_\tau(\xi,\tau)$, and $F_0(\xi)$ is replaced with $F_\tau(\xi,0)$. Moreover (see (32))

$$F_\tau(\xi,0) = \frac{1}{P}F_{0\xi\xi}(\xi) - \eta P\int_0^\xi (A_0(\xi)+C_0(\xi))d\xi + N_1(0),$$

$$N_1(0) = \frac{1}{P}(F_{0\xi}(0) - F_{0\xi}(1)) + \eta P\int_0^1\int_0^\epsilon (A_0(\xi)+C_0(\xi))d\xi d\epsilon.$$

Thus, if $A_j(\tau),\ C_j(\tau)\in C^1[0,\tau_0]$ and inequalities (28) are satisfied then solution of inverse initial boundary value problem (15), (10) and (14) satisfies a priori estimates (25), (26), (30)–(34). In addition, similarly to estimates (30), $F_{\xi\xi}(\xi,\tau)$, $H_{\xi\xi}(\xi,\tau)$ are bounded for any $\xi\in[0,1]$, $\tau\in[0,\tau_0]$.

Remark 5. *If $A_j(\tau),\ C_j(\tau)\in C^1[0,\tau_0]$, $A_0(\xi),\ C_0(\xi)\in C^2[0,1]$ then it follows from the maximum principle for parabolic equations that*

$$\mid A(\xi,\tau)\mid \leqslant \max\Big[\max_{\xi\in[0,1]}\mid A_0(\xi)\mid,\ \max_{\tau\in[0,\tau_0]}\mid A_j(\tau)\mid\Big],$$

$$\mid C(\xi,\tau)\mid \leqslant \max\Big[\max_{\xi\in[0,1]}\mid C_0(\xi)\mid,\ \max_{\tau\in[0,\tau_0]}\mid C_j(\tau)\mid\Big],$$

$$\mid A_\tau(\xi,\tau)\mid \leqslant \max\Big[\max_{\xi\in[0,1]}\mid A_{0\xi\xi}(\xi)\mid,\ \max_{\tau\in[0,\tau_0]}\mid A_{j\tau}(\tau)\mid\Big],$$

$$\mid C_\tau(\xi,\tau)\mid \leqslant \max\Big[\max_{\xi\in[0,1]}\mid C_{0\xi\xi}(\xi)\mid,\ \max_{\tau\in[0,\tau_0]}\mid C_{j\tau}(\tau)\mid\Big].$$

Therefore, the boundedness of $\mid F(\xi,\tau)\mid$, $\mid H(\xi,\tau)\mid$, $\mid F_\tau(\xi,\tau)\mid$, $\mid H_\tau(\xi,\tau)\mid$, $\mid F_{\xi\xi}(\xi,\tau)\mid$, $\mid H_{\xi\xi}(\xi,\tau)\mid$, $\mid N_1(\tau)\mid$, $\mid N_2(\tau)\mid$, with $\xi\in[0,1]$, $\tau\in[0,\tau_0]$ takes place for weaker conditions on functions $A_j(\tau)$, $C_j(\tau)$.

Relations for $G_1(\tau)$, $G_3(\tau)$, $G_5(\tau)$ contain integrals of exponent $e^{\pi^2 P\tau}$. Therefore, the use of a priori estimates for the behaviour of the solution at $\tau\gg 1$ requires the fulfilment of conditions (28), so that there are estimates (29) with some constant $\mu>0$. Let us assume that $A_j(\tau)$, $C_j(\tau)$, $A_{j\tau}(\tau)$, $A_{j\tau}(\tau)$ are defined and continuously differentiable for all $\tau\geqslant 0$. If $\mu_1 = P\pi^2+\gamma$,

$\gamma > 0$ then the specified integrals in relations for $G_1(\tau)$, $G_3(\tau)$, $G_5(\tau)$ and in the right-hand sides of inequalities (25), (26), (30)–(34) converge exponentially to zero.

Let us assume that inequalities (28) and estimates for derivatives (29) are satisfied. Considering the differences $F(\xi,\tau) - F^s(\xi)$, $H(\xi,\tau) - H^s(\xi)$, $N_j(\tau) - N_j^s$, $j = 1, 2$, let us ensure that they satisfy the same initial boundary value problems as $F(\xi,\tau)$, $H(\xi,\tau)$, $N_j(\tau)$. The difference is only in the initial conditions. They are replaced with $F_0(\xi) - F^s(\xi)$, $H_0(\xi) - H^s(\xi)$, $N_j(0) - N_j^s$, respectively. Therefore, the estimates follow from given above inequalities ($\mu_1 = \pi^2 + \gamma$)

$$(|\,F(\xi,\tau) - F^s(\xi)\,|, \quad |\,H(\xi,\tau) - H^s(\xi)\,|, \quad |\,F_\tau(\xi,\tau)\,|, \; |\,H_\tau(\xi,\tau)\,|,$$
$$|\,N_j(\tau) - N_j^s\,|) \leqslant De^{-\frac{\pi^2}{2}\tau}$$

with some constant $D > 0$.

Therefore, stationary solution (15) is exponentially stable under the given above conditions.

Solution of non-stationary problem by the Laplace method

Non-stationary solution of problem (10)–(12), (15) is found using the integral Laplace transform [14]. In our case, the method reduces the solution of non-stationary partial differential problem to the solution of a system of ordinary differential equations (ODEs).

Applying the Laplace transform to the initial boundary value problem

$$A_\tau = A_{\xi\xi},$$
$$A(\xi,0) = A_0(\xi),$$
$$A(0,\tau) = A_1(\tau), \quad A(1,\tau) = A_2(\tau),$$

the following system of ODEs for the Laplace images is obtained

$$\begin{gathered} \hat{A}_{\xi\xi} - s\hat{A} = -A_0(\xi), \\ \hat{A}(0,s) = \hat{A}_1(s), \quad \hat{A}(1,s) = \hat{A}_2(s). \end{gathered} \tag{35}$$

Taking into account (35), one can find $\hat{A}(\xi,s)$

$$\begin{gathered} \hat{A}(\xi,s) = \frac{\operatorname{sh}(\sqrt{s}\xi)}{\operatorname{sh}(\sqrt{s})}\hat{A}_2(s) + \frac{\operatorname{sh}(\sqrt{s}(1-\xi))}{\operatorname{sh}(\sqrt{s})}\hat{A}_1(s) + \\ + \frac{1}{\sqrt{s}}\left[\frac{\operatorname{sh}(\sqrt{s}\xi)}{\operatorname{sh}(\sqrt{s})}\int_0^1 A_0(\xi)\operatorname{sh}(\sqrt{s}(1-\xi))d\xi - \int_0^\xi A_0(\varepsilon)\operatorname{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right]. \end{gathered} \tag{36}$$

Similarly, function $\hat{C}(z,s)$ is defined as

$$\begin{gathered} \hat{C}(\xi,s) = \frac{\operatorname{sh}(\sqrt{s}\xi)}{\operatorname{sh}(\sqrt{s})}\hat{C}_2(s) + \frac{\operatorname{sh}(\sqrt{s}(1-\xi))}{\operatorname{sh}(\sqrt{s})}\hat{C}_1(s) + \\ + \frac{1}{\sqrt{s}}\left[\frac{\operatorname{sh}(\sqrt{s}\xi)}{\operatorname{sh}(\sqrt{s})}\int_0^1 C_0(\xi)\operatorname{sh}(\sqrt{s}(1-\xi))d\xi - \int_0^\xi C_0(\varepsilon)\operatorname{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right]. \end{gathered} \tag{37}$$

Therefore, $\hat{A}(\xi,s)$ and $\hat{C}(\xi,s)$ are known functions. Similarly, function $\hat{\Theta}(\xi,s)$ is

$$\begin{gathered} \hat{\Theta}(\xi,s) = \frac{2}{\sqrt{s}}\left(\frac{\operatorname{sh}(\sqrt{s}\xi)}{\operatorname{sh}(\sqrt{s})}\int_0^1 (\hat{A}(\xi,s) + \hat{C}(\xi,s))\operatorname{sh}(\sqrt{s}(1-\xi))d\xi - \right. \\ \left. - \int_0^\xi (\hat{A}(\varepsilon,s) + \hat{C}(\varepsilon,s))\operatorname{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right). \end{gathered} \tag{38}$$

Equation for function $F(\xi,\tau)$ in Laplace images has the form

$$\hat{F}_{\xi\xi}-\frac{s}{P}\hat{F}=\eta\int_0^{\xi}(\hat{A}(\varepsilon,s)+\hat{C}(\varepsilon,s))d\varepsilon-\frac{1}{P}\hat{N}_1(s)-F_0(\xi),\qquad \hat{F}(0,s)=\hat{F}(1,s)=0. \tag{39}$$

Then solution of problem (39) is

$$\hat{F}(\xi,s)=\frac{\sqrt{P}\eta}{\sqrt{s}}\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\times$$
$$\times\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon+\frac{P\eta}{s}\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\times$$
$$\times\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon-\frac{\mathrm{ch}(\sqrt{s/P}\xi)-1}{s}(\hat{N}_1(s)-PF_0(\xi))+P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi+$$
$$+\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P}}\frac{\mathrm{ch}(\sqrt{s/P})-1}{s}(\hat{N}_1(s)-PF_0(\xi))-P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P}}\frac{\mathrm{sh}(\sqrt{s/P})}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi. \tag{40}$$

Let us find $\hat{N}_1(s)$ from (12). Introducing

$$r=(\mathrm{sh}\sqrt{s/P}/\sqrt{s/P}-1)/P\sqrt{P}-\big((ch\sqrt{s/P}-1)/\sqrt{s/P}\big)^2/\sqrt{sP}\,\mathrm{sh}\sqrt{s/P},$$

one can obtain

$$\hat{N}_1(s)=\frac{\eta\sqrt{P}}{r\sqrt{s}}\Bigg[\int_0^1\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi-$$
$$-\frac{\eta\sqrt{P}}{r\sqrt{s}}\frac{\mathrm{ch}(\sqrt{s/P})-1}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^{\xi}(\hat{A}(\epsilon,s)+\hat{C}(\epsilon,s))\,\mathrm{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi- \tag{41}$$
$$-PF_0(\xi)+P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi\Bigg].$$

Similarly, find function $\hat{H}(\xi,s)$

$$\hat{H}(\xi,s)=\frac{\sqrt{P}\eta}{\sqrt{s}}\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\times$$
$$\times\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon+\frac{P\eta}{s}\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\times$$
$$\times\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon-\frac{\mathrm{ch}(\sqrt{s/P}\xi)-1}{s}(\hat{N}_1(s)-PH_0(\xi))+P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi+$$
$$+\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\frac{\mathrm{ch}(\sqrt{s/P})-1}{s}(\hat{N}_2(s)-PH_0(\xi))-P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\frac{\mathrm{sh}(\sqrt{s/P})}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi. \tag{42}$$

Function $\hat{N}_2(s)$ is defined from (12) as follows

$$\hat{N}_2(s)=\frac{\eta\sqrt{P}}{r\sqrt{s}}\Bigg[\int_0^1\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi-$$
$$-\frac{\eta\sqrt{P}}{r\sqrt{s}}\frac{\mathrm{ch}(\sqrt{s/P})-1}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^{\xi}(\hat{A}(\epsilon,s)-\hat{C}(\epsilon,s))\,\mathrm{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi- \tag{43}$$
$$-PH_0(\xi)+P\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi\Bigg].$$

Conditions for tendency of non-stationary solution to a given stationary solution

Suppose there are limits

$$\lim_{\tau\to\infty} A_j(\tau) = A_j^0, \quad \lim_{\tau\to\infty} C_j(\tau) = C_j^0, \; j = 1, 2, \tag{44}$$

and derivatives $A_j^{'}(\tau)$, $C_j^{'}(\tau)$ have Laplace images. Then [14]

$$\lim_{s\to 0} s\hat{A}_j(s) = \lim_{\tau\to\infty} A_j(\tau) = C_j^0, \quad \lim_{s\to 0} s\hat{C}_j(s) = \lim_{\tau\to\infty} C_j(\tau) = C_j^0. \tag{45}$$

Next, asymptotic expressions when $t \to 0$ for functions $\mathrm{sh}(t)$ and $\mathrm{ch}(t)$ are used: $\mathrm{sh}(t) \sim t + t^3/6, \; \mathrm{ch}(t) \sim 1 + t^2/2$.

The proof is given for function $\hat{\Theta}(\xi, s)$. The following relation is obtained for $s \to 0$

$$\begin{aligned} s\hat{\Theta}(\xi, s) \sim \frac{2}{\sqrt{s}}\Bigg(\xi \int_0^1 (s\hat{A}(\xi, s) + s\hat{C}(\xi, s))\left[(\sqrt{s}(1-\xi)) + \frac{(\sqrt{s}(1-\xi))^3}{6}\right] d\xi - \\ - \int_0^{\xi} (s\hat{A}(\varepsilon, s) + s\hat{C}(\varepsilon, s))\left[(\sqrt{s}(\xi-\varepsilon)) + \frac{(\sqrt{s}(\xi-\varepsilon))^3}{6}\right] d\varepsilon\Bigg) \sim \\ \sim \left(\frac{1}{3}[A_2^0 + C_2^0 - (A_1^0 + C_1^0)] + A_1^0 + C_1^0\right)\xi - \\ - \left(\frac{1}{3}[A_2^0 + C_2^0 - (A_1^0 + C_1^0)]\xi^3 + [A_1^0 + C_1^0]\xi^2\right) = \Theta^s(\xi). \end{aligned}$$

Lemma 1. *Under conditions (44), (45) the non-stationary solution of problem (10),(11), (12), (15) approaches stationary solution (16) when dimensionless time τ increases.*

Finding the originals of required functions

Functions (36), (37) are Laplace images. The inverse Laplace transform is used to determine the originals.

It is assumed that $A_j(\tau)$, $C_j(\tau)$ have the form

$$A_j(\tau) = A_j^0 + \epsilon_{j1} \exp[-\gamma_{j1}\tau] \sin(\omega_1\tau), \quad C_j(\tau) = C_j^0 + \epsilon_{j2} \exp[-\gamma_{j2}\tau] \sin(\omega_2\tau).$$

Then, their images are easily found from the Laplace transform table [15]

$$\hat{A}_j(s) = \frac{A_j^0}{s} + \frac{\epsilon_{j1}\omega_1}{(s+\gamma_{j1})^2 + \omega_1^2}, \quad \hat{C}_j(s) = \frac{C_j^0}{s} + \frac{\epsilon_{j2}\omega_2}{(s+\gamma_{j2})^2 + \omega_2^2}, \tag{46}$$

where $\gamma_{j1} > 0$, $\gamma_{j2} > 0$, i.e., the boundary mode is stabilized with time according to Lemma 1. If one of the values of γ_{j1}, γ_{j2} is negative then there is no stabilization effect of the solution.

At this point, for simplicity, it is assumed that motion arises from the state of rest and $A_0(\xi) = C_0(\xi) = 0$. In this case, compatibility conditions (13) are violated since $A_1(0) \neq A_0(0) = 0$, $C_1(0) \neq C_0(0) = 0$, that is, there are discontinuities of the 1st kind. This is acceptable since the integral Laplace transform is applicable for functions that have a finite number of discontinuities of the 1st kind [15].

Expressions for $\hat{A}(\xi,s)$, $\hat{C}(\xi,s)$, $\hat{F}(\xi,s)$, $\hat{N}_1(s)$ are simplified as

$$\hat{A}(\xi,s)=\left(\frac{A_2^0}{s}+\frac{\epsilon_{12}\omega_1}{(s+\gamma_{12})^2+\omega_1^2}\right)\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}+\left(\frac{A_1^0}{s}+\frac{\epsilon_{11}\omega_1}{(s+\gamma_{11})^2+\omega_1^2}\right)\frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})},$$

$$\hat{C}(\xi,s)=\left(\frac{C_2^0}{s}+\frac{\epsilon_{22}\omega_2}{(s+\gamma_{22})^2+\omega_2^2}\right)\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}+\left(\frac{C_1^0}{s}+\frac{\epsilon_{21}\omega_2}{(s+\gamma_{21})^2+\omega_2^2}\right)\frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})},$$

$$\hat{F}(\xi,s)=\frac{\eta}{\sqrt{sP}}\Big[\int_0^\xi\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon-$$
$$-\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon\Big]-$$
$$-\frac{\mathrm{ch}(\sqrt{s/P}\xi)-1}{s}\hat{N}_1(s)+\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\frac{\mathrm{ch}(\sqrt{s/P})-1}{s}\hat{N}_1(s),$$

$$\hat{N}_1(s)=\frac{\eta}{r\sqrt{sP}}\Big[\int_0^1\int_0^\xi\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi-$$
$$-\frac{\mathrm{ch}(\sqrt{s/P})-1}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^\xi(\hat{A}(\epsilon,s)+\hat{C}(\epsilon,s))\,\mathrm{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi\Big].$$

Expressions for $\hat{H}(\xi,s)$, $\hat{N}_2(s)$ have the same form only terms $\hat{A}(\zeta,s)+\hat{C}(\zeta,s)$ are replaced by $\hat{A}(\zeta,s)-\hat{C}(\zeta,s)$.

Function $\hat{\Theta}(\xi,s)$ has the following form

$$\hat{\Theta}(\xi,s)=\frac{2}{\sqrt{s}}\Big(\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\int_0^1(\hat{A}(\eta,s)+\hat{C}(\eta,s))\,\mathrm{sh}(\sqrt{s}(1-\eta))d\eta-$$
$$-\int_0^\xi(\hat{A}(\eta,s)+\hat{C}(\eta,s))\,\mathrm{sh}(\sqrt{s}(\xi-\eta))d\eta\Big).$$

After numerical inversion of the Laplace transform functions $N_j(\tau)$ are obtain (see Fig. 3), where $A_2^0=1.3$, $A_1^0=1$, $C_2^0=2.7$, $C_1^0=2$, $\epsilon_{12}=1.2$, $\epsilon_{11}=1$, $\epsilon_{22}=1.6$, $\epsilon_{21}=1.8$, $\omega_1=0.1$, $\omega_2=0.2$, $\gamma_{12}=0.04$, $\gamma_{11}=0.03$, $\gamma_{22}=0.07$, $\gamma_{21}=0.06$, $\chi=0.00143$ m^2/sec, $\nu=0.01006$ m^2/sec, $\beta=1.82\cdot10^{-4}$ 1/deg, $l=10^{-4}$ m, $g=9,81$ m/sec^2.

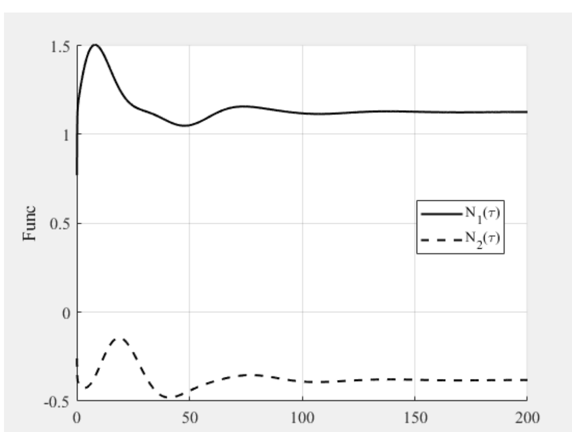

Fig. 3. Functions $N_j(\tau)$ versus dimensionless time

Dimensionless velocities

$$time\bar{u}(\xi,\tau) = \frac{l}{\chi Re}u = (F+H)\bar{x}, \quad \bar{v}(\xi,\tau) = \frac{l}{\chi Re}u = (F-H)\bar{y},$$

are shown in Fig. 4, 5 ($\bar{x} = \bar{y} = 1$).

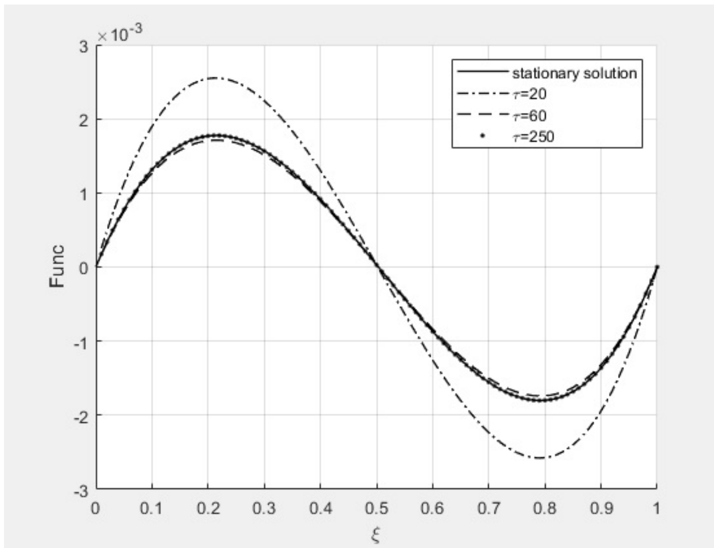

Fig. 4. Velocity $\bar{u}(\xi,\tau)$ as a function of dimensionless coordinate

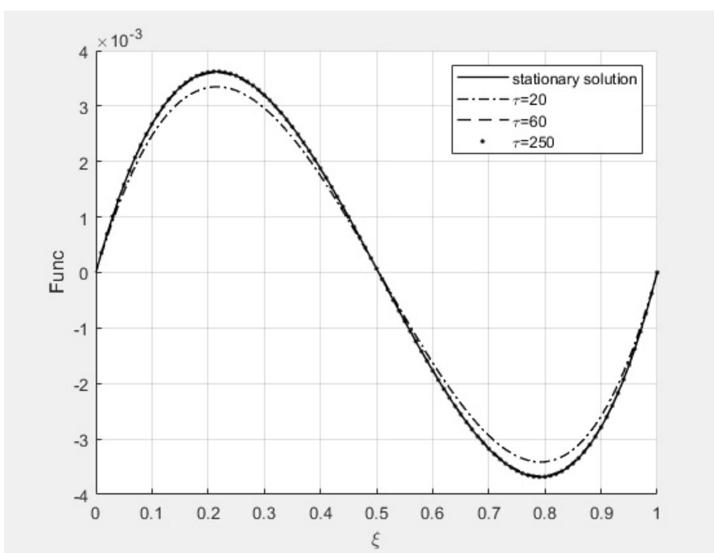

Fig. 5. Velocity $\bar{v}(\xi,\tau)$ as a function of dimensionless coordinate

Fig. 3 clearly shows that functions $N_j(\tau)$ approach constant values with increasing time. Figs. 4 and 5 show velocities along the x and y axes. One can see that distribution of velocities practically coincides with stationary distribution of velocities for large τ.

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Ползучее трехмерное конвективное движение в слое с полем скоростей специального вида

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Аннотация. Исследована задача о трехмерном движении теплопроводной жидкости в канале твердыми параллельными стенками, на которых поддерживается заданное распределение температуры. Температура в жидкостях квадратично зависит от горизонтальных координат, а поле скоростей имеет специальный вид. Возникающая начально-краевая задача для модели Обербека–Буссинеска является обратной и редуцирована к системе пяти интегродифференциальных уравнений. При малых числах Рейнольдса (ползущие движения) полученная система становится линейной. Для этой системы найдено стационарное решение, получены априорные оценки. На их основе установлены достаточные условия экспоненциальной сходимости гладкого нестационарного решения к стационарному режиму. В изображениях по Лапласу решение обратной задачи построено в виде квадратур, при более слабых условиях на температурный режим на стенках слоя. Приведены результаты расчетов, на основе численного обращения преобразования Лапласа, поведения поля скоростей для конкретной жидкой среды.

Ключевые слова: модель Обербека–Буссинеска, трехмерное движение, обратная задача, априорные оценки, устойчивость, преобразование Лапласа.

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Almost Inner Derivations of Some Leibniz Algebras

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Abstract. The present paper is devoted to almost inner derivations of thin and solvable Leibniz algebras. Namely, we consider a thin Lie algebra, solvable Lie algebra with nilradical natural graded filifform Lie algebra, natural graded thin Leibniz algebra, thin non-Lie Leibniz algebra and solvable Leibniz algebra with nilradical nul-filiform algebra. We prove that any almost inner derivations of all these algebras are inner derivations.

Keywords: Lie algebra, Leibniz algebra, solvable algebra, nilradical, thin Lie algebra, thin Leibniz algebra, derivation, inner derivation, almost inner derivation

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Introduction

Almost inner derivations of Lie algebras were introduced by C.S.Gordon and E.N.Wilson [13] in the study of isospectral deformations of compact manifolds. Gordon and Wilson wanted to construct not only finite families of isospectral nonisometric manifolds, but rather continuous families. They constructed isospectral but nonisometric compact Riemannian manifolds of the form G/Γ, with a simply connected exponential solvable Lie group G, and a discrete cocompact subgroup Γ of G. For this construction, almost inner automorphisms and almost inner derivations were crucial.

Gordon and Wilson considered not only almost inner derivations, but they studied almost inner automorphisms of Lie groups. The concepts of "almost inner" automorphisms and derivations, almost homomorphisms or almost conjugate subgroups arise in many contexts in algebra, number theory and geometry. There are several other studies of related concepts, for example, local derivations, which are a generalization of almost inner derivations and automorphisms [3,4].

In [7] authors study almost inner derivations of some nilpotent Lie algebras. The authors of this work proved the basic properties of almost inner derivations, calculated all almost inner derivations of Lie algebras for small dimensions.They also introduced the concept of fixed basis vectors for nilpotent Lie algebras defined by graphs and studied free nilpotent Lie algebras of the nilindex 2 and 3. In [8], almost inner derivations of Lie algebras over a field of characteristic zero has been studied and these derivations has been determined for free nilpotent Lie algebras, almost abelian Lie algebras, Lie algebras whose solvable radical is abelian and for several classes of filiform nilpotent Lie algebras. A family of n-dimensional characteristically nilpotent filiform Lie algebras f_n has been found for all $n \geqslant 13$, all derivations of which are almost inner. The almost inner derivations of Lie algebras considered over two different fields $K \supseteq k$ for a finite-dimensional field extension were compared.

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Motivated by the work [7], we studied almost inner derivations of some nilpotent Leibniz algebras [2] and in this work the almost inner derivations for Leibniz algebras were introduced and it was proved that on a filiform non-Lie Leibniz algebra there exists an almost inner derivation that is not an inner derivation.

In work [1] it is proved that any derivation complex maximal solvable extension of Lie algebras is inner [Theorem 4.1]. Moreover, it is proved that any non-maximal solvable extension of a nilpotent Lie algebra admits an outer derivation [Proposition 4.3]. Therefore, in this paper almost inner derivations of solvable Lie algebras with the nilradical naturally graded filiform Lie algebra and almost inner derivations of thin Lie algebras will be considered. In addition, almost inner derivations of natural graded thin Leibniz algebras, non-Lie thin Leibniz algebras and solvable Leibniz algebras with nilradical nul-filiform algebra will be studied.

1. Preliminaries

Definition 1.1. *An algebra $\mathfrak{g}$ over field $\mathbb{F}$ is called a Lie algebra if its multiplication satisfies:*
1) $[x,x]=0$,
2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$,
for all $x,y,z\in\mathfrak{g}$.

The product $[x,y]$ is called the bracket of x and y. Identity 2) is called the Jacobi identity.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For Lie algebra $\mathfrak{g}$ we consider the following central and derived series:

$$\mathfrak{g}^1=\mathfrak{g},\quad \mathfrak{g}^i=[\mathfrak{g}^{i-1},\mathfrak{g}],\quad i\geqslant 1,$$
$$\mathfrak{g}^{[1]}=\mathfrak{g},\quad \mathfrak{g}^{[k]}=[\mathfrak{g}^{[k-1]},\mathfrak{g}^{[k-1]}],\quad k\geqslant 1.$$

A Lie algebra $\mathfrak{g}$ is *nilpotent* (*solvable*) if there exists $m\geqslant 1$ such that $\mathfrak{g}^m=0$ ($\mathfrak{g}^{[m]}=0$).

Definition 1.2. *A derivation of Lie algebra $\mathfrak{g}$ is a linear map $D:\mathfrak{g}\to\mathfrak{g}$ which satisfies the Leibniz law, that is,*

$$D([x,y])=[D(x),y]+[x,D(y)]$$

for all $x,y\in\mathfrak{g}$.

The set of all derivations of $\mathfrak{g}$ with respect to the commutation operation is a Lie algebra and it is denoted by $Der(\mathfrak{g})$. For all $a\in\mathfrak{g}$, the map ad_a on $\mathfrak{g}$ defined as $ad_a(x)=[a,x]$, $x\in\mathfrak{g}$ is a derivation and derivations of this form are called *inner derivation*. The set of all inner derivations of $\mathfrak{g}$, denoted $InDer(\mathfrak{g})$.

Definition 1.3. *A derivation $D\in Der(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is said to be almost inner, if $D(x)\in[\mathfrak{g},x]$ for all $x\in\mathfrak{g}$. The space of all almost inner derivations of $\mathfrak{g}$ is denoted by $AID(\mathfrak{g})$.*

We now give the definition and necessary facts of the Leibniz algebra.

Definition 1.4. *An algebra $\mathfrak{L}$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x,y,z\in\mathfrak{L}$, the Leibniz identity*

$$[x,[y,z]]=[[x,y],z]-[[x,z],y]$$

is satisfied, where $[-,-]$ is the multiplication in $\mathfrak{L}$.

The definitions of nilpotency, solvability and derivation for Leibniz algebras are introduced in a similar way as the definition of nilpotency, solvability and derivation of Lie algebras.

Let $\mathfrak{L}$ be a Leibniz algebras. For each $a\in\mathfrak{L}$, the operator $R_x:\mathfrak{L}\to\mathfrak{L}$ which is called the *right multiplication*, such that $R_x(y)=[y,x]$, $y\in\mathfrak{L}$, is a derivation. This derivation is called an *inner derivation* of $\mathfrak{L}$, and we denote the space of all inner derivations by $InDer(\mathfrak{L})$.

Now let us give the definitions of the almost inner derivations for the Leibniz algebras.

Definition 1.5 ([2]). *The derivation $D \in Der(\mathfrak{L})$ of the Leibniz algebra $\mathfrak{L}$ is called almost inner derivation, if $D(x) \in [x, \mathfrak{L}]$ holds for all $x \in \mathfrak{L}$; in other words, there exists $a_x \in \mathfrak{L}$ such that $D(x) = [x, a_x]$. The space of all almost inner derivations of $\mathfrak{L}$ is denoted by $AID(\mathfrak{L})$.*

2. Almost inner derivations of thin Lie algebras

In this section, we will consider almost inner derivations of thin Lie algebras. Let's consider the following so-called thin Lie algebra $\mathfrak{g}$ with a basis $\{e_i : i \in \mathbb{N}\}$, which is defined by the following table of multiplications of the basic elements:

$$M_1 : \quad [e_1, e_i] = e_{i+1}, \ i \geqslant 2, \tag{1}$$

$$M_2 : \begin{cases} [e_1, e_j] = e_{j+1}, & j \geqslant 2, \\ [e_2, e_i] = e_{i+2}, & i \geqslant 3, \end{cases} \tag{2}$$

and other products of the basic elements being zero [12].

Note that the algebras M_1 and M_2 are an infinite-dimensional analog of the filiform Lie algebras L_n and Q_n which are given in [10]. In papers [7] and [8] it was proved that every almost inner derivation of the algebras L_n and Q_n is inner.

The derivations of thin Lie algebras M_1 has the following form [6]:

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad D(e_2) = \sum_{i=1}^{n} \beta_i e_i, \, D(e_j) = ((j-2)\alpha_1 + \beta_2)e_j + \sum_{i=1}^{n} \beta_{i+2} e_{i+j}, \, j \geqslant 3,$$

where $\alpha_i, \beta_i \in \mathbb{C}, \ i = 1, \ldots, n$, and $n \in \mathbb{N}$.

The following theorem is one of the main results in this section.

Theorem 2.1. *Let $\mathfrak{g}$ be the thin Lie algebra. Then any almost inner derivation on thin Lie algebras is inner.*

Proof. First, consider the thin Lie algebra $\mathfrak{g} = M_1$ with multiplication table (1) and inner derivation of this algebra. Let $x = \sum\limits_{i=1}^{n} x_i e_i \in \mathfrak{g}$, $n \in \mathbb{N}$. For basis e_i define $ad_x(e_i)$:

$$ad_x(e_1) = [x, e_1] = \left[\sum_{i=1}^{n} x_i e_i, e_1\right] = -\sum_{i=2}^{n} x_i e_{i+1};$$
$$ad_x(e_j) = [x, e_j] = \left[\sum_{i=1}^{n} x_i e_i, e_j\right] = x_1 e_{j+1}, \ j \geqslant 2.$$

In the next step, we study an almost inner derivation of a thin Lie algebra $\mathfrak{g}$. Let $D \in AID(\mathfrak{g})$. For basis $e_i \in \mathfrak{g}$ exists $a_{e_i} \in \mathfrak{g}$ such that $D(e_i) = [a_{e_i}, e_i]$, for all $i \geqslant 1$. Then

$$D(e_1) = [a_{e_1}, e_1] = \left[\sum_{i=1}^{n} a_{1,i} e_i, e_1\right] = -\sum_{i=2}^{n} a_{1,i} e_{i+1},$$
$$D(e_j) = [a_{e_j}, e_j] = \left[\sum_{i=1}^{n} a_{j,i} e_i, e_j\right] = a_{j,1} e_{j+1}, \ j \geqslant 2.$$

Now we check the conditions of derivation:

$$D(e_3) = D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] = [e_1, a_{2,1} e_3] = a_{2,1} e_4.$$

On the other hand $D(e_3) = a_{3,1}e_4$. From, here we get $a_{2,1} = a_{3,1}$.

For $i \geqslant 3$ consider

$$D(e_i) = D([e_1, e_{i-1}]) = [e_1, D(e_{i-1})] = [e_1, a_{i-1,1}e_i] = a_{i-1,1}e_{i+1}.$$

On the other hand $D(e_i) = a_{i,1}e_{i+1},\ i \geqslant 3$. From here we have

$$a_{i,1} = a_{i-1,1},\ i \geqslant 3.$$

Hence

$$D(e_1) = -\sum_{i=2}^{n} a_{1,i}e_{i+1}, \quad D(e_j) = a_{2,1}e_{j+1},\ j \geqslant 2.$$

For arbitrary element $x \in \mathfrak{g}$ we take element $a = a_{2,1}e_1 - \sum\limits_{k=2}^{n} a_{1,k}e_{k+1} \in \mathfrak{g}$ such that $D(x) = ad_a(x)$, and this means that almost inner derivations D is inner.

Now, we investigate the case $\mathfrak{g} = M_2$.

Let $x = \sum\limits_{i=1}^{n} x_ie_i \in \mathfrak{g}$, $n \in \mathbb{N}$. For basis e_i define $ad_x(e_i)$:

$$\begin{aligned} ad_x(e_1) = [x, e_1] &= \left[\sum_{i=1}^{n} x_ie_i, e_1\right] = -\sum_{i=2}^{n} x_ie_{i+1};\\ ad_x(e_2) = [x, e_2] &= \left[\sum_{i=1}^{n} x_ie_i, e_2\right] = x_1e_3 - \sum_{k=3}^{n} x_ke_{k+2},\ n \in \mathbb{N};\\ ad_x(e_j) = [x, e_j] &= \left[\sum_{i=1}^{n} x_ie_i, e_j\right] = x_1e_{j+1} + x_2e_{j+2},\ \ j \geqslant 3. \end{aligned}$$

Let $D \in AID(\mathfrak{g})$. For basis e_i exists a_{e_i} such that $D(e_i) = [a_{e_i}, e_i]$, for all $i \geqslant 1$. Then

$$\begin{aligned} D(e_1) = [a_{e_1}, e_1] &= \left[\sum_{i=1}^{n} a_{1,i}e_i, e_1\right] = -\sum_{i=2}^{n} a_{1,i}e_{i+1},\\ D(e_2) = [a_{e_2}, e_2] &= \left[\sum_{i=1}^{n} a_{2,i}e_i, e_2\right] = a_{2,1}e_3 - \sum_{k=3}^{n} a_{2,k}e_{k+2},\ n \in \mathbb{N},\\ D(e_i) = [a_{e_i}, e_i] &= \left[\sum_{k=1}^{n} a_{i,k}e_k, e_i\right] = a_{i,1}e_{i+1} + a_{i,2}e_{i+2},\ i \geqslant 3. \end{aligned}$$

According to the definition of derivation

$$\begin{aligned} D(e_3) &= D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] =\\ &= \left[-\sum_{i=2}^{n} a_{1,i}e_{i+1}, e_2\right] + \left[e_1, a_{2,1}e_3 - \sum_{k=3}^{n} a_{2,k}e_{k+2}\right] =\\ &= a_{2,1}e_4 + a_{1,2}e_5 + \sum_{k=3}^{n} (a_{1,k} - a_{2,k})e_{k+3}. \end{aligned}$$

On the other hand $D(e_3) = a_{3,1}e_4 + a_{3,2}e_5$. Comparing the coefficients at the basis elements, we obtain

$$\begin{cases} a_{2,1} = a_{3,1},\\ a_{1,2} = a_{3,2},\\ a_{1,k} = a_{2,k},\ 3 \leqslant k \leqslant n. \end{cases} \tag{3}$$

Hence

$$D(e_2) = a_{2,1}e_3 - \sum_{k=3}^{n} a_{1,k}e_{k+2},\ n \in \mathbb{N},$$

$$D(e_3) = a_{2,1}e_4 + a_{1,2}e_5.$$

For $i \geqslant 4$ consider the following:

$$D(e_i) = D([e_1, e_{i-1}]) = [e_1, D(e_{i-1})] = a_{i-1,1}e_{i+1} + a_{i-1,2}e_{i+2}.$$

On the other hand $D(e_i) = a_{i,1}e_{i+1} + a_{i,2}e_{i+2}$. Hence for $i \geqslant 4$ it follows that

$$\begin{cases} a_{i-1,1} = a_{i,1}, \\ a_{i-1,2} = a_{i,2}. \end{cases} \tag{4}$$

Combining (3) and (4) we get

$$D(e_1) = -\sum_{k=2}^{n} a_{1,k}e_{k+1},\ n \in \mathbb{N},$$
$$D(e_2) = a_{2,1}e_3 - \sum_{k=3}^{n} a_{1,k}e_{k+2},\ \ n \in \mathbb{N},$$
$$D(e_i) = a_{2,1}e_{i+1} + a_{1,2}e_{i+2},\ \ i \geqslant 3.$$

For every element $x \in \mathfrak{g}$ we take element $a = a_{2,1}e_1 + a_{1,2}e_3 + \sum\limits_{k=3}^{n} a_{1,k}e_k \in \mathfrak{g}$ such that $D(x) = ad_a(x)$, and this means that almost inner derivations D is inner. □

3. Almost inner derivation of naturally graded complex thin Leibniz algebras

In this section, we will consider almost inner derivation of naturally graded complex thin Leibniz algebras. In [14], the following theorem is given, which classifies the naturally graded complex thin Leibniz algebras.

Theorem 3.1 ([14])**.** *Up to isomorphism, there are three naturally graded complex thin Leibniz algebras, namely,*

$$\begin{array}{llll} L_1: & [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & i \geqslant 2, \\ L_2: & [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & i \geqslant 3, \\ L_3: & [e_i, e_1] = e_{i+1}, & [e_1, e_i] = -e_{i+1}, & i \geqslant 2, \end{array}$$

where $\{e_1, e_2, e_3, \dots\}$ are bases of the algebras L_1, L_2, L_3 and other products vanish.

The following lemma holds.

Lemma 3.1. *The derivations of naturally graded complex thin Leibniz algebras have the following forms:*

$$L_1: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_i) = ((i-1)\alpha_1 + \alpha_2)e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2},\quad i \geqslant 2,\ n \in \mathbb{N};$$

$$L_2: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_2) = \sum_{k=2}^{n} \beta_k e_k,$$
$$D(e_i) = (i-1)\alpha_1 e_i + \alpha_3 e_{i+1} + \sum_{k=4}^{n} \alpha_k e_{k+i-2},\quad i \geqslant 3,\quad n \in \mathbb{N};$$

$$L_3: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_2) = \sum_{k=1}^{n} \beta_k e_k,$$
$$D(e_i) = ((i-2))\alpha_1 + \beta_2)e_i + \sum_{k=3}^{n} \beta_k e_{k+i-2},\quad i \geqslant 3,\ n \in \mathbb{N},$$

where $\alpha_i, \beta_i \in \mathbb{C}$, $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$.

Proof. Let $D(e_1) = \sum\limits_{k=1}^{n} \alpha_k e_k$, $D(e_2) = \sum\limits_{k=1}^{n} \beta_k e_k$, $n \in \mathbb{N}$.

Using the definition of derivation of algebra L_1 from Theorem 3.1 we obtain the following:

$$\begin{aligned} D(e_3) &= D([e_1, e_1]) = [D(e_1), e_1] + [e_1, D(e_1)] = \sum_{k=1}^{n} \alpha_k [e_k, e_1] + \sum_{k=2}^{n} \alpha_k [e_2, e_k] = \\ &= (2\alpha_1 + \alpha_2) e_3 + \sum_{k=3}^{n} \alpha_k e_{k+2}. \end{aligned}$$

On the other hand

$$\begin{aligned} D(e_3) &= D([e_2, e_1]) = [D(e_2), e_1] + [e_2, D(e_1)] = \sum_{k=1}^{n} \beta_k [e_k, e_1] + \sum_{k=1}^{n} \alpha_k [e_2, e_k] = \\ &= (\beta_1 + \beta_2 + \alpha_1) e_3 + \sum_{k=3}^{n} \beta_k e_{k+1}. \end{aligned}$$

Comparing coefficients from basis elements we have

$$\begin{cases} \alpha_1 + \alpha_2 = \beta_1 + \beta_2, \\ \alpha_k = \beta_k, \ k \geqslant 3. \end{cases} \tag{5}$$

Consider the following:

$$0 = D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] = \sum_{k=1}^{n} \alpha_k [e_k, e_2] + \sum_{k=1}^{n} \beta_k [e_1, e_k] = \beta_2 e_3.$$

From this, we get $\beta_1 = 0$. Then from equality (5) we obtain $\beta_2 = \alpha_1 + \alpha_2$. Hence,

$$D(e_2) = (\alpha_1 + \alpha_2) e_2 + \sum_{k=3}^{n} \alpha_k e_k, \ D(e_3) = (2\alpha_1 + \alpha_2) e_3 + \sum_{k=3}^{n} \alpha_k e_{k+1}.$$

Consider the following:

$$D(e_4) = D([e_3, e_1]) = [D(e_3), e_1] + [e_3, D(e_1)] = (3\alpha_1 + \alpha_2) e_4 + \sum_{k=3}^{n} \alpha_k e_{k+2}.$$

Continuing this process we have

$$D(e_i) = D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = ((i-1)\alpha_1 + \alpha_2) e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2}.$$

Thus, derivations of algebra L_1 has the following form:

$$D(e_1) = \sum_{k=1}^{n} \alpha_k e_k, \ D(e_i) = ((i-1)\alpha_1 + \alpha_2) e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2}, \quad i \geqslant 2, \ n \in \mathbb{N};$$

Derivations of algebras L_2 and L_3 are obtained in the same way. $\square$

Note that in Theorem 3.1 the algebra L_3 is a thin Lie algebra, i.e., algebra with multiplication (3.1). Therefore, we will study almost inner derivations of thin Leibniz algebras L_1 and L_2.

The following theorem is one of the main results in this paper.

Theorem 3.2. *Let $\mathcal{L}$ be the naturally graded complex thin Leibniz algebra. Then any almost inner derivation on naturally graded complex thin Leibniz algebras is inner.*

Proof. Let $\mathcal{L} = L_1$ and $D \in AID(\mathcal{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathcal{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' = D - R_{a_{e_1}}$, then we have $D'(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

By definition of almost inner derivation for basis e_2 exists $a_{e_2} \in \mathcal{L}$ such that

$$D'(e_2) = [e_2, a_{e_2}] = [e_2, a_{2,1}e_1] = a_{2,1}e_3.$$

Then

$$0 = D'(e_3) = D'([e_2, e_1]) = [D'(e_2), e_1] = a_{2,1}e_4.$$

From this we get $D'(e_2) = 0$.

The next step consider the almost inner derivations of naturally graded thin Leibniz algebras $\mathcal{L} = L_2$. Let $D \in AID(\mathcal{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathcal{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' = D - R_{a_{e_1}}$, then we have $D'(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

By definition of almost inner derivation for basis e_2 exists $a_{e_2} \in \mathcal{L}$ such that

$$D'(e_2) = [e_2, a_{e_2}] = 0.$$

□

4. Almost inner derivation of complex non-Lie thin Leibniz algebras

In this section, we will consider almost inner derivation of complex non-Lie thin Leibniz algebras. We present the following theorem.

Theorem 4.1 ([14]). *Every complex non-Lie thin Leibniz algebra is isomorphic to one of the following two nonisomorphic non-Lie thin Leibniz algebras:*

$$\begin{aligned}
F_1^\infty: \quad & [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad i \geqslant 2,\\
& [e_1, e_2] = \sum_{k=1}^{n} \alpha_{p_k} e_{p_k},\\
& [e_i, e_2] = \sum_{k=1}^{n} \alpha_{p_k} e_{p_k+i-2}, \qquad i \geqslant 2, \ n \in \mathbb{N},\\
F_2^\infty: \quad & [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad i \geqslant 3,\\
& [e_1, e_2] = \sum_{s=1}^{m} \beta_{t_s} e_{t_s},\\
& [e_i, e_2] = \sum_{s=1}^{m} \beta_{t_s} e_{t_s+i-2}, \qquad i \geqslant 3, \ m \in \mathbb{N},
\end{aligned}$$

where $4 \leqslant p_1 < p_2 < \cdots < p_n$ and $4 \leqslant t_1 < t_2 < \cdots < t_m$, and the other products vanish.

The following theorem is one of main the results of this section.

Theorem 4.2. *Let $\mathfrak{L}$ be the complex non-Lie thin Leibniz algebra. Then any almost inner derivation on complex non-Lie thin Leibniz algebras is inner.*

Proof. Let $\mathfrak{L} = F_1^\infty$ is a complex non-Lie thin Leibniz algebra and $D \in AID(\mathfrak{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathfrak{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' \in AID(\mathfrak{L})$ and $D' = D - R_{a_{e_1}}$, then we get $D'(e_1) = (D - R_{a_{e_1}})(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

Let $D'(e_2) = \sum\limits_{k=1}^{n} b_k e_k, \ n \in \mathbb{N}$. By derivation conditions we have the following:

$$0 = D'(e_3) = D'([e_2, e_1]) = [D'(e_2), e_1] = \left[\sum_{k=1}^{n} b_k e_k, \, e_1\right] = (b_1 + b_2)e_3 + \sum_{k=4}^{n} b_k e_{k+1}.$$

It follows from the latter that

$$b_1 = -b_2, \ b_i = 0, \ 3 \leqslant i \leqslant n.$$

Then $D'(e_2) = b_1 e_1 - b_1 e_2$.

Since $4 \leqslant p_1 < p_2 < \cdots < p_n$, then

$$0 = D'([e_1, e_2]) = [e_1, D'(e_2)] = [e_1, b_1 e_1 - b_1 e_2] = b_1 e_3 - b_1 \sum_{k=1}^{n} \alpha_{p_k} e_{p_k}.$$

From this we get $b_1 = 0$. Hence, $D'(e_2) = 0$.

Let $\mathfrak{L} = F_2^\infty$. Then by definition AID for e_1 there exists $a_{e_1} \in \mathfrak{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' \in AID(\mathfrak{L})$ and $D' = D - R_{a_{e_1}}$, then we get $D'(e_1) = (D - R_{a_{e_1}})(e_1) = 0$. Then

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0;$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

Let $D'(e_2) = \sum\limits_{j=1}^{n} b_j e_j, \ n \in \mathbb{N}$. Consider

$$0 = D'([e_2, e_1]) = [D'(e_2), e_1] = \left[\sum_{j=1}^{n} b_j e_j, e_1\right] = b_1 e_3 + \sum_{j=3}^{n} b_j e_{j+1}, \ n \in \mathbb{N}.$$

From the last equality we have $b_1 = 0, \ b_j = 0, \ 3 \leqslant j \leqslant n$. Hence $D'(e_2) = b_2 e_2$. Since $D'(e_i) = 0, \ i \geqslant 3$, then considering equality

$$0 = D'([e_1, e_2]) = [e_1, D'(e_2)] = b_2 \sum_{s=1}^{m} \beta_{t_s} e_{t_s},$$

we obtain

$$b_2 \cdot \beta_{t_s} = 0, \quad 1 \leqslant s \leqslant m. \tag{6}$$

In algebra F_2^∞ at least one of the parameters $\beta_{t_s} (1 \leqslant s \leqslant m)$ is nonzero, otherwise if all are $\beta_{t_s} = 0 (1 \leqslant s \leqslant m)$, then algebra coincides with algebra of naturally graded thin Leibniz algebras L_2. So there will always be $\beta_{t_{s_0}} \neq 0$, then we have $b_2 = 0$, as a consequence $D'(e_2) = 0$. $\square$

5. Almost inner derivations of solvable Lie algebra whose nilradical is natural graded filifform Lie algebra

In this section we consider almost inner derivations of solvable Lie algebra whose nilradical is natural graded filifform Lie algebra. The multiplication table of natural graded filifform Lie algebra has the next form:

$$\mathfrak{n}_{n,1},\ (n \geqslant 4): \quad [e_i, e_1] = -[e_1, e_i] = e_{i+1},\ 2 \leqslant i \leqslant n-1.$$

Theorem 5.1 ([15]). *There are three of solvable Lie algebras of dimension $(n+1)$ whose nilradical is isomorphic to $\mathfrak{n}_{n,1}$ $(n \geqslant 4)$. The isomorphism classes in the basis $\{e_1, e_2, \dots, e_n, x\}$ are represented by the following algebras:*

$$S_{n+1}(\alpha,\beta) = \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = ((i-2)\alpha + \beta)e_i, & 2 \leqslant i \leqslant n, \\ [e_1, x] = -[x, e_1] = \alpha e_1. \end{cases}$$

The mutually non-isomorphic algebras:

1) $S_{n+1,n}(\beta) := S_{n+1}(1,\beta)$ depending on the value of β, in this case there are three different classes: a) $S_{n+1}(1,0)$, b) $S_{n+1}(1,n-2)$, c) $S_{n+1}(1,\beta)$, $\beta \notin \{0, n-2\}$;

2) $S_{n+1,2} := S_{n+1}(0,1)$;

3) $S_{n+1,3}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = (i-1)e_i, & 2 \leqslant i \leqslant n, \\ [e_1, x] = -[x, e_1] = e_1 + e_2. \end{cases}$

4) $S_{n+1,4}(\alpha_3, \alpha_4, \dots, \alpha_{n-1}): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = e_i + \sum\limits_{l=i+2}^{n} \alpha_{l+1-i} e_l, & 2 \leqslant i \leqslant n, \end{cases}$ *where at least one $\alpha_i \neq 0$ and the first non-vanishing parameter $\{\alpha_3, \alpha_4, \dots, \alpha_{n-1}\}$ can be assumed to be equal to 1.*

The following theorem is the main result in this section.

Theorem 5.2. *Let $\mathfrak{g}$ is solvable Lie algebra with nilradical $\mathfrak{n}_{n,1}$. Then any almost inner derivation solvable Lie algebra with nilradical $\mathfrak{n}_{n,1}$ is inner.*

Proof. Consider the following cases:

Case 1. Let $\mathfrak{g} = S_{n+1}(1,0)$ be the solvable Lie algebra and let $a = \sum\limits_{i=1}^{n} a_i e_i + a_x x \in \mathfrak{g}$. For basis e_i, x $(i = 1, \dots, n)$ define $ad_a(e_i)$, $ad_a(x)$:

$$ad_a(e_1) = [a, e_1] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_1\right] = -a_x e_1 + \sum_{k=2}^{n-1} a_k e_{k+1},$$
$$ad_a(e_2) = [a, e_2] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_2\right] = -a_1 e_3,$$
$$ad_a(e_i) = [a, e_i] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_i\right] = -(i-2)a_x e_i - a_1 e_{i+1}, \quad 3 \leqslant i \leqslant n,$$
$$ad_a(x) = [a, x] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, x\right] = a_1 e_1 + \sum_{k=3}^{n} (k-2) a_k e_k.$$

Let $D \in AID(\mathfrak{g})$. For basis e_i and x exists b_{e_i} and b_x respectively such that $D(e_i) = [b_{e_i}, e_i]$ $(1 \leqslant i \geqslant n)$ and $D(x) = [b_x, x]$. Then

$$D(e_1) = [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \delta_1 x, e_1\right] = -\delta_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1},$$
$$D(e_2) = [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \delta_2 x, e_2\right] = -b_{2,1}e_3.$$

By multiplication of algebra $S_{n+1}(1,0)$ for all $3 \leqslant i \leqslant n$ we obtain:

$$D(e_i) = D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = -(i-2)\delta_1 e_i - b_{2,1}e_{i+1}.$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$.

Consider the following:

$$D([e_1, x]) = [D(e_1), x] + [e_1, D(x)] = (\delta_x - \delta_1)e_1 + \sum_{k=2}^{n-1}((k-1)b_{1,k} - b_{x,k})e_{k+1}.$$

On the other hand $D([e_1, x]) = D(e_1) = -\delta_1 e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_{k+1}$. Comparing coefficients we have:

$$\begin{cases} \delta_x = 0, \\ b_{x,2} = 0, \\ b_{x,j} = (j-2)b_{1,j}, \quad 3 \leqslant j \leqslant n-1. \end{cases}$$

Hence $D(x) = b_{x,1}e_1 + \sum\limits_{k=3}^{n-1}(k-2)b_{1,k}e_k + b_{x,n}e_n$.

Now consider

$$0 = D([e_2, x]) = [D(e_2), x] + [e_2, D(e_x)] = (-b_{2,1} + b_{x,1})e_3.$$

From this we get $b_{x,1} = b_{2,1}$. Then

$$D(e_1) = -\delta_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1},$$
$$D(e_2) = -b_{2,1}e_3,$$
$$D(e_i) = -(i-2)\delta_1 e_i - b_{2,1}e_{i+1},$$
$$D(x) = b_{2,1}e_1 + \sum_{k=3}^{n-1}(k-2)b_{1,k}e_k + b_{x,n}e_n.$$

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = (b_{2,1} + \delta_1)e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

Case 2. Let $\mathfrak{g} = S_{n+1}(1, n-2)$. Analogously as Case 1 we have

$$ad_a(e_1) = [a, e_1] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_1\right] = -a_x e_1 + \sum_{i=2}^{n-1} a_i e_{i+1},$$
$$ad_a(e_j) = [a, e_j] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_j\right] = -(n+i-4)a_x e_j - a_1 e_{j+1}, \quad 2 \leqslant j \leqslant n,$$
$$ad_a(x) = [a, x] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, x\right] = a_1 e_1 + \sum_{k=2}^{n}(n+k-4)a_k e_k$$

and

$$\begin{aligned}
D(e_1) &= [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \gamma_1 x, e_1\right] = -\gamma_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1},\\
D(e_2) &= [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \gamma_2 x, e_2\right] = -(n-2)\gamma_2 e_2 - b_{2,1}e_3,\\
D(e_i) &= D([e_{i-1}, e_1]) = -((i-2)\gamma_1 + (n-2)\gamma_2)e_i - b_{2,1}e_{i+1}, \quad 3 \leqslant i \leqslant n.
\end{aligned}$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$. Consider the following

$$D([e_1, x]) = [D(e_1), x] + [e_1, D(x)] = (\gamma_x - \gamma_1)e_1 + \sum_{k=2}^{n-1}((n+k-3)b_{1,k} - b_{x,k})e_{k+1}.$$

On the other hand $D([e_1, x]) = D(e_1) = -\gamma_1 e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_{k+1}$. From this we have

$$\begin{cases} \gamma_x = 0, \\ b_{x,k} = (n+k-4)b_{1,k}, \quad 2 \leqslant k \leqslant n-1. \end{cases}$$

Hence $D(x) = b_{x,1}e_1 + \sum\limits_{k=2}^{n-1}(n+k-4)b_{1,k}e_k + b_{x,n}e_n$.

Consider the next equality

$$\begin{aligned}
(n-2)(-(n-2)\gamma_2 e_2 - b_{21}e_3) &= D([e_2, x]) = [D(e_2, x)] + [e_2, D(x)] = \\
&= -(n-2)^2\gamma_2^2 e_2 + (b_{x,1} - (n-1)b_{21})e_3.
\end{aligned}$$

From this we get

$$\begin{cases} (n-2)^2\gamma_2 = (n-2)^2\gamma_2 \\ b_{x,1} - (n-1)b_{2,1} = -(n-2)b_{2,1} \end{cases} \Rightarrow \begin{cases} \gamma_2 = 0, n \neq 2 \\ b_{x,1} = b_{2,1} \end{cases}.$$

Hence $D(x) = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1}(n+k-4)b_{1,k}e_k + b_{x,n}e_n$.

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n + (\gamma_1 + \gamma_2)x \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

Case 3. Let $\mathfrak{g} = S_{n+1}(1, \beta)$. Similar as Case 1 we get

$$\begin{aligned}
ad_a(e_1) &= [a, e_1] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_1\right] = -a_x e_1 + \sum_{i=2}^{n-1} a_i e_{i+1},\\
ad_a(e_j) &= [a, e_j] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_j\right] = -(j-2+\beta)a_x e_j - a_1 e_{j+1}, \quad 2 \leqslant j \leqslant n,\\
ad_a(x) &= [a, x] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, x\right] = a_1 e_1 + \sum_{k=2}^{n}(k-2+\beta)a_k e_k
\end{aligned}$$

and

$$\begin{aligned}
D(e_1) &= [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \gamma_1 x, e_1\right] = -\gamma_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1}.\\
D(e_2) &= [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \gamma_2 x, e_2\right] = -\gamma_2\beta e_2 - b_{2,1}e_3,\\
D(e_i) &= D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = -((i-2)\gamma_1 + \beta\gamma_2)e_i - b_{2,1}e_{i+1}, \ 3 \leqslant i \leqslant n.
\end{aligned}$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$. Now we check the conditions of derivation:

From $D([e_1, x])$ we have

$$\begin{cases} \gamma_x = 0, \\ b_{x,k} = (k-1+\beta)b_{1,k}, \quad 2 \leqslant k \leqslant n-1. \end{cases}$$

From $D([e_2, x])$ we obtain $b_{x,1} = b_{2,1}$. Hence $D(x) = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1}(k-1+\beta)b_{1,k}e_k + b_{x,n}e_n$.

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n + (\gamma_1+\gamma_2)x \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

For the remaining algebras $S_{n+1,2},\ S_{n+1,3},\ S_{n+1,4}(\alpha_3, \ldots, \alpha_{n-1})$ is proved in a similar way. □

6. Almost inner derivations of solvable Leibniz algebra whose nilradical is null filiform algebra

Recall the definition of null-filiform Leibniz algebras.

Definition 6.1 ([5]). *An n-dimensional Leibniz algebra is said to be null-filiform if* $\dim L^i = n+1-i,\ 1 \leqslant i \leqslant n+1$.

Theorem 6.1 ([5]). *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:*

$$NF_n: \quad [e_i, e_1] = e_{i+1},\ 1 \leqslant i \leqslant n-1,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra NF_n.

From this theorem it is easy to see that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra. Note that this notion has no sense in Lie algebras case, because they are at least two-generated.

We present the following well-known results that we will use to study the main result.

Theorem 6.2 ([11]). *Let R be a solvable Leibniz algebra whose nilradical is NF_n. Then there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant n-1, \\ [x, e_1] = e_1, \\ [e_i, x] = -ie_i, \quad 1 \leqslant i \leqslant n. \end{cases} \tag{7}$$

Theorem 6.3 ([11]). *Let R be a solvable Leibniz algebra such that $R = NF_k \oplus NF_s + Q$, where $NF_k \oplus NF_s$ is the nilradical of R and $\dim Q = 1$. Let us assume that $\{e_1, e_2, \ldots, e_k\}$ is a basis of NF_k, $\{f_1, f_2, \ldots, f_s\}$ is a basis of NF_s and $\{x\}$ is a basis of Q. Then the algebra R is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R(\alpha): \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leqslant i \leqslant s-1, \\ [x, e_1] = e_1, & [x, f_1] = \alpha f_1, \quad \alpha \neq 0, \\ [e_i, x] = -ie_i, \quad 1 \leqslant i \leqslant k, & [f_i, x] = -i\alpha f_i, \quad 1 \leqslant i \leqslant s. \end{cases} \tag{8}$$

$$R(\beta_2,\beta_3,\dots,\beta_s,\gamma):\begin{cases}[e_i,e_1]=e_{i+1}, & 1\leqslant i\leqslant k-1,\\ [f_i,f_1]=f_{i+1}, & 1\leqslant i\leqslant s-1,\\ [x,e_1]=e_1,\\ [f_i,x]=\sum\limits_{j=i+1}^{s}\beta_{j-i+1}f_j, & 1\leqslant i\leqslant s,\\ [e_i,x]=-ie_i, & 1\leqslant i\leqslant k,\\ [x,x]=\gamma f_s.\end{cases} \tag{9}$$

in the second family of algebras the first non-zero element of the set $(\beta_2,\beta_3,\dots,\beta_s,\gamma)$ can be assumed equal to 1.

Theorem 6.4 ([11]). *Let L be a solvable Leibniz algebra such that $L=NF_{n_1}\oplus NF_{n_2}\oplus\dots\oplus NF_{n_s}\dot{+}Q$, where $NF_{n_1}\oplus NF_{n_2}\oplus\cdots\oplus NF_{n_s}$ is nilradical of L and $dimQ=1$. There exists $p,q\in\mathbb{N}$ with $p\neq 0$ and $p+q=s$, a basis $\{e_1^i,e_2^i,\dots,e_{n_i}^i\}$ of NF_{n_i}, for $1\leqslant i\leqslant p$, a basis $\{f_1^k,f_2^k,\dots,f_{n_k}^k\}$ of NF_{p+k}, for $1\leqslant k\leqslant q$, and a basis $\{x\}$ of Q such that the multiplication table of the algebra is given by*

$$R_{p,q}=\begin{cases}[e_i^j,e_1^j]=e_{i+1}^j,\ 1\leqslant i\leqslant n_j-1,\ [f_i^k,f_1^k]=f_{i+1}^k,\ 1\leqslant i\leqslant n_k-1,\\ [x,e_1^j]=\delta^j e_1^j,\ \delta^j\neq 0,\ [f_i^k,x]=\sum\limits_{m=i+1}^{n_k}\beta_{m-i+1}^k f_m^k,\ 1\leqslant i\leqslant n_k,\\ [e_i^j,x]=-i\delta^j e_i^j,\ 1\leqslant i\leqslant n_j,\ [x,x]=\sum\limits_{m=1}^{k}\gamma^m f_{n_m}.\end{cases} \tag{10}$$

6.1. Almost inner derivations of solvable Leibniz algebra whose nilradical is NF_n

In the subsection consider almost inner derivations on solvable Leibniz algebra whose nilradical is NF_n.

Let $\mathcal{L}$ solvable Leibniz algebra whose nilradical is NF_n with multiplication the form (7). Then we have the next is one of the main results in this section.

Theorem 6.5. *Let $\mathcal{L}$ solvable Leibniz algebra with nilradical NF_n. Then any almost inner derivations solvable Leibniz algebra $\mathcal{L}$ is inner*

Proof. The solvable algebra $\mathcal{L}$ is a two-generated algebra, i.e. generated by e_1, x. Let $D\in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for basis e_1 there exists b_{e_1} such that $D(e_1)=R_{b_{e_1}}$. Let $D'\in AID(\mathcal{L})$ and let $D'=D-R_{b_{e_1}}$, then we get $D'(e_1)=0$. Then by multiplication (7) we have

$$D'(e_i)=D'([e_{i-1},e_1])=[D'(e_{i-1}),e_1]+[e_{i-1},D'(e_1)]=0,\quad 2\leqslant i\leqslant n.$$

Let $D'(x)=\sum\limits_{i=1}^{n}a_ie_i+a_{n+1}x$. Consider

$$\begin{aligned}0\ &=\ D'(e_1)=D'([x,e_1])=[D'(x),e_1]+[x,D'(e_1)]=\left[\sum_{i=1}^{n}a_ie_i+a_{n+1}x,e_1\right]=\\ &=\ a_{n+1}e_1+a_1e_2+a_2e_3+\cdots+a_{n-1}e_n.\end{aligned}$$

Hence we have

$$a_1=a_2=\cdots=a_{n-1}=a_{n+1}=0$$

and $D'(x)=a_ne_n$.

On the other hand by definition of almost inner derivations for basis x exists $\xi_x \in \mathcal{L}$, such that $D'(x) = [x, \xi_x]$. Further

$$a_n e_n = D'(x) = [x, \xi_x] = [x, \xi_{x,1}e_1 + \xi_{x,2}e_2 + \cdots + \xi_{x,n}e_n + \xi_{x,n+1}x] = \xi_{x,1}e_1.$$

Hence we get $a_n = \xi_{x,1} = 0$. Then $D'(x) = 0$. □

6.2. Almost inner derivations of solvable Leibniz algebra whose nilradical is $NF_k \oplus NF_s$

In this subsection consider almost inner derivations on solvable Leibniz algebra whose nilradical is $NF_k \oplus NF_s$. Let $\mathcal{L} = R(\alpha)$ first solvable Leibniz algebra in Theorem 6.2 with table multiplication (8). Then we get the following results. The following theorem is one the results in this section.

Theorem 6.6. *Let $\mathcal{L} = R(\alpha)$ solvable Leibniz algebra with nilradical $NF_k \oplus NF_s$. Then any almost inner derivation solvable Leibniz algebra $\mathcal{L}$ is inner.*

Proof. The solvable algebra $\mathcal{L}$ is a three-generated algebra, i.e.generated by e_1, f_1, x. Let $D \in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for element e_1 there exists b_{e_1} such that $D(e_1) = R_{b_{e_1}}$. Let $D' \in AID(\mathcal{L})$ and let $D' = D - R_{b_{e_1}}$, then we get $D'(e_1) = 0$. Then by multiplication (8) we have

$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}, e_1)] + [e_{i-1}, D'(e_1)] = 0, \quad 2 \leqslant i \leqslant k.$$

Let $D'(x) = \sum\limits_{i=1}^{k} \epsilon_{x,i}e_i + \sum\limits_{j=1}^{s} \phi_{x,j}f_j + a_x x$. Consider

$$\begin{aligned} 0 &= D'(x) = D'([x, e_1]) = [D'(x), e_1] = \left[\sum_{i=1}^{k} \epsilon_{x,i}e_i + \sum_{j=1}^{s} \phi_{x_j}f_j + a_x x, e_1 \right] = \\ &= a_x e_1 + \epsilon_{x,1}e_2 + \epsilon_{x,2}e_3 + ... + \epsilon_{x,k-1}e_k. \end{aligned}$$

We have

$$\epsilon_{x,1} = \cdots = \epsilon_{x,k-1} = a_x = 0.$$

Hence $D'(x) = \epsilon_{x,k}e_k + \sum\limits_{j=1}^{s} \phi_{x,j}f_j$. By definition AID (Almost Inner Derivation) for basis x exists the element $b_x \in \mathcal{L}$ such that $D'(x) = [x, b_x]$. Then we obtain

$$\epsilon_{x,k}e_k + \sum_{j=1}^{s} \phi_{x,j}f_j = D'(x) = [x, b_x] = [x, \epsilon_{b_x,1}e_1 + \phi_{b_x,1}f_1] = \epsilon_{b_x,1}e_1 + \alpha\phi_{b_x,1}f_1.$$

Hence

$$\begin{cases} \epsilon_{b_{x,1}} = \epsilon_{x,k} = 0 \\ \phi_{x,1} = \alpha\phi_{b_{x,1}} \\ \phi_{x,j} = 0, \quad 2 \leqslant j \leqslant s \end{cases}.$$

Then $D'(x) = \phi_{x,1}f_1$.

Let $D'(f_1) = \sum\limits_{i=1}^{k} \epsilon_{f_1,i}e_i + \sum\limits_{j=1}^{s} \phi_{f_1,j}f_j + a_{f_1}x$. By definition AID for basis f_1 exists the element $b_{f_1} \in \mathcal{L}$ such that

$$D'(f_1) = [f_1, b_{f_1}] = [f_1, \phi_{b_{f_1},1}f_1 + a_{b_{f_1},x}x] = -\alpha a_{b_{f_1}}f_1 + \phi_{b_{f_1},1}f_2.$$

Comparing the coefficients at the basis elements we get

$$\begin{cases} \epsilon_{f_1,i} = 0, \quad 1 \leqslant i \leqslant k \\ \phi_{f_1,1} = -\alpha a_{b_{f_1}} \\ \phi_{f_1,2} = \phi_{f_1,1} \\ \phi_{f_1,j} = 0, \quad 3 \leqslant j \leqslant s \\ a_{f_1} = 0 \end{cases}.$$

Hence $D'(f_1) = \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2$.

Consider the following:

1)

$$\begin{aligned} D'(f_2) &= D'([f_1, f_1]) = [D'(f_1), f_1] + [f_1, D'(f_1)] = \\ &= [\phi_{f_1,1} f_1 + \phi_{f_1,2} f_2, f_1] + [f_1, \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2] = \\ &= 2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3. \end{aligned}$$

On other hand by definition AID for basis f_2 exists b_{f_2} such that

$$2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3 = D'(f_2) = [f_2, b_{f_2}] = [f_2, \phi_{b_{f_2},1} f_1 + a_{b_{f_2}} x] = -2\alpha a_{b_{f_2}} f_2 + \phi_{b_{f_2},1} f_3.$$

From here we get

$$\phi_{f_1,1} = -\alpha a_{b_{f_2}}, \quad \phi_{f_1,2} = \phi_{b_{f_2},1}. \tag{11}$$

2)

$$\begin{aligned} D'(f_3) = D'([f_2, f_1]) &= [D'(f_2), f_1] + [f_2, D'(f_1)] = \\ &= [2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3, f_1] + [f_1, \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2] = \\ &= 3\phi_{f_1,1} f_3 + \phi_{f_1,2} f_4. \end{aligned}$$

On other hand by definition AID for f_3 exists b_{f_3} such that

$$3\phi_{f_1,1} f_3 + \phi_{f_1,2} f_4 = D'(f_3) = [f_3, b_{f_3}] = [f_3, \phi_{b_{f_3},1} f_1 + a_{b_{f_3}} x] = -3\alpha a_{b_{f_3}} f_3 + \phi_{b_{f_3},1} f_4.$$

From here we have

$$\phi_{f_1,1} = -\alpha a_{b_{f_3}}, \quad \phi_{f_1,2} = \phi_{b_{f_3},1}. \tag{12}$$

Continuing this process we obtain

$$D'(f_j) = D'([f_{j-1}, f_1]) = j\phi_{f_1,1} f_j + \phi_{f_1,2} f_{j+1}, \quad 4 \leqslant j \leqslant s$$

and by definition AID for $4 \leqslant j \leqslant s$:

$$D'(f_j) = [f_j, b_{f_j}] = -j\alpha a_{b_{f_j}} f_j + \phi_{b_{f_j},1} f_{j+1},$$

and we have that

$$\phi_{f_1,1} = -\alpha a_{b_{f_j}}, \quad \phi_{f_1,2} = \phi_{b_{f_j},1}, \quad 4 \leqslant j \leqslant s.$$

So, we have that $b := b_{f_1} = b_{f_2} = \cdots = b_{f_s}$, $1 \leqslant j \leqslant s$, i.e.

$$D'(f_j) = [f_j, b], \ 1 \leqslant j \leqslant s.$$

Let $T \in AID(\mathcal{L})$, then for basis f_i, $1 \leqslant i \leqslant s$ exists element $b \in \mathcal{L}$ such that $T(f_i) = [f_i, b]$.

Since $D' = D - R_{b_{e_1}}$, then

$$D'(f_1) = D(f_1) - R_{b_{e_1}}(f_1) = [f_1, b] - [f_1, b_{e_1}] = [f_1, b] - [f_1, b] = 0.$$

Thus, according to the multiplication (8) for all $2 \leqslant i \leqslant s$ we have

$$D'(f_i) = D'([f_{i-1}, f_1]) = [D'(f_{i-1}), f_1] + [f_{i-1}, D'(f_1)] = 0$$

and $D'(x) = \phi_{x,1} f_1$.

Now from the following equality

$$0 = \alpha D'(f_1) = D'([x, f_1]) = [D'(x), f_1] = \phi_{x,1} f_2.$$

we get that $\phi_{x,1} = 0$. Then $D'(x) = 0$. □

Let $\mathcal{L} = R(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ solvable Leibniz algebra with product table (9). The following result holds. The following theorem is one the results in this section.

Theorem 6.7. *Let $\mathcal{L} = R(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ solvable Leibniz algebra with nilradical $NF_k \oplus NF_s$. Then any almost inner derivations solvable Leibniz algebra $\mathcal{L}$ is inner.*

Proof. The solvable algebra $\mathcal{L}$ is a three-generated algebra, i.e. generated by e_1, f_1, x. Let $D \in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for basis f_1 there exists b_{f_1} such that $D(f_1) = R_{b_{f_1}}(f_1)$. Let $D' \in AID(\mathcal{L})$ and let $D' = D - R_{b_{f_1}}$, then we get $D'(f_1) = 0$. Then by multiplication (8) we have

$$D'(f_i) = D'([f_{i-1}, f_1]) = [D'(f_{i-1}, f_1)] + [f_{i-1}, D'(f_1)] = 0, \quad 2 \leqslant i \leqslant s.$$

Let $D'(x) = \sum\limits_{i=1}^{s} \epsilon_{x,i} e_i + \sum\limits_{j=1}^{s} \phi_{x,j} f_j + a_x x$. By the definition of AID for basis x exists $b_x \in \mathcal{L}$ such that

$$D'(x) = [x, b_x] = [x, \epsilon_{b_x,1} e_1 + a_{b_x} x] = \epsilon_{b_x,1} e_1 + a_{b_x} \gamma f_s,$$

and we have that

$$\begin{cases} \epsilon_{x,1} = \epsilon_{b_x,1} \\ \epsilon_{x,i} = 0, \quad 2 \leqslant i \leqslant k \\ \phi_{x,i} = 0, \quad 1 \leqslant i \leqslant s-1 \\ \phi_{x,s} = a_{b_x} \gamma \\ a_x = 0 \end{cases}.$$

Then $D'(x) = \epsilon_{x,1} e_1 + \phi_{x,s} f_s = \epsilon_{b_x,1} e_1 + a_{b_x} \gamma f_s$.

Let $D'(e_1) = \sum\limits_{i=1}^{k} \epsilon_{e_1,i} e_i + \sum\limits_{j=1}^{s} \phi_{e_1,j} f_j + a_{e_1} x$. By the definition of AID for basis e_1 exists element $b_{e_1} \in \mathcal{L}$ such that

$$D'(e_1) = [e_1, b_{e_1}] = [e_1, \epsilon_{b_{e_1}} e_1 + a_{b_{e_1}} x] = -a_{b_{e_1}} e_1 + \epsilon_{b_{e_1},1} e_2.$$

Comparing we get

$$\begin{cases} \epsilon_{e_1,1} = -a_{b_{e_1}} \\ \epsilon_{e_1,2} = \epsilon_{b_{e_1},1} \\ \epsilon_{e_1,i} = 0, \quad 3 \leqslant i \leqslant k \\ \phi_{e_1,i} = 0, \quad 1 \leqslant i \leqslant s, \\ a_{e_1} = 0 \end{cases}.$$

Then $D'(e_1) = \epsilon_{e_1,1} e_1 + \epsilon_{e_1,2} e_2 = -a_{b_{e_1}} e_1 + \epsilon_{b_{e_1},1} e_2$. Further for all $2 \leqslant i \leqslant k$ we have

$$\begin{aligned} D'(e_i) &= D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] + [e_{i-1}, D'(e_1)] = \\ &= [(i-1)\epsilon_{e_1,1} e_{i-1} + e_{e_1,2} e_i, e_1] + [e_{i-1}, \epsilon_{e_1,1} e_1 + \epsilon_{e_1,2} e_2] = \\ &= i\epsilon_{e_1,1} e_i + \epsilon_{e_1,2} e_{i+1}. \end{aligned}$$

Hence $D'(e_j) = j\epsilon_{e_1,1}e_j + \epsilon_{e_1,2}e_{j+1}, \quad 1 \leqslant j \leqslant k$. By the definition of AID for e_i exists elements b_{e_i} such that

$$D'(e_i) = [e_i, b_{e_i}] = [e_i, \epsilon_{e_i,1}e_1 + a_{e_i}x] = -ia_{e_i}e_i + \epsilon_{e_i,1}e_{i+1}, \quad 1 \leqslant i \leqslant k.$$

So, for all $1 \leqslant i \leqslant k$ we have

$$\begin{cases} \epsilon_{e_1,1} = -a_{e_i} \\ \epsilon_{e_2,1} = \epsilon_{e_i,1} \end{cases}.$$

From the last equality we obtain $b_{e_1} = b_{e_2} = \cdots = b_{e_k} =: b, \quad 1 \leqslant i \leqslant k$, i.e. for any $T \in AID(\mathcal{L})$ such that $T(e_i) = [e_i, b], \ 1 \leqslant i \leqslant k$.

Since $D' = D - R_{b_{f_1}}$, then

$$D'(e_1) = (D - R_{b_{f_1}})(e_1) = D(e_1) - R_{b_{f_1}}(e_1) = [e_1, b] - [e_1, b_{f_1}] = [e_1, b] - [e_1, b] = 0.$$

Hence $D'(e_i) = 0, \quad 2 \leqslant i \leqslant k$.

Now consider the following:

$$0 = D'(e_1) = D'([x, e_1]) = [D'(x), e_1] = [\epsilon_{x,1}e_1 + \phi_{x,s}f_s, e_1] = \epsilon_{x,1}e_2.$$

From here we have $\epsilon_{x,1} = 0$. Hence $D'(x) = \phi_{x,s}f_s = a_{b_x}\gamma f_s$.

Consider the following cases.

Case 1. Let $\gamma = 0$. Then $D'(x) = 0$ and $AID(\mathcal{L}) = InDer(\mathcal{L})$.

Case 2. Let $\gamma \neq 0$. Then by the definition of AID for $e_1 + x$ exists $b_{e_1+x} \in \mathcal{L}$ such that

$$\begin{aligned} D'(e_1 + x) &= [e_1 + x, b_{e_1+x}] = \Big[e_1 + x, \sum_{i=1}^{k} \epsilon_{e_1+x,i}e_i + \sum_{j=1}^{s} \phi_{e_1+x,j}f_j + a_{e_1+x}x\Big] = \\ &= (\epsilon_{e_1+x,1} - a_{e_1+x})e_1 + \epsilon_{e_1+x,1}e_2 + \alpha\phi_{e_1+x,1}f_1. \end{aligned}$$

On the other hand

$$a_{b_x}\gamma f_s = D'(x) = D'(x) + D'(e_1) = D'(e_1 + x) = [e_1 + x, b_{e_1+x}].$$

Comparing the coefficients at the basic elements, we obtain the following

$$\begin{cases} \epsilon_{e_1+x,1} = a_{e_1+x} \\ \epsilon_{e_1+x,1} = 0 \\ \alpha\phi_{e_1+x,1} = 0 \\ \gamma a_{b_x} = 0 \end{cases}.$$

The last equation implies $a_{b_x} = 0$, hence $D'(x) = 0$. $\square$

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Почти внутренние дифференцирования некоторых алгебр Лейбница

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Аннотация. Настоящая работа посвящена почти внутренним дифференцированиям тонких и разрешимых алгебр Лейбница. А именно мы рассматриваем тонкую алгебру Ли, разрешимую алгебру Ли с нильрадикалом естественной градуированной филиформной алгеброй Ли, натуральную градуированную тонкую алгебру Лейбница, тонкую нелиевскую алгебру Лейбница и разрешимую алгебру Лейбница с нильрадикалом нуль-филиформная алгебра. Доказано, что любые почти внутренние дифференцирования всех этих алгебр являются внутренними дифференцированиями.

Ключевые слова: алгебра Ли, алгебра Лейбница, разрешимая алгебра, нильрадикал, тонкая алгебра Ли, тонкая алгебра Лейбница, дифференцирования, внутренние дифференцирования, почти внутренние дифференцирования.

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Modeling of Electric Field Impact on a Cholesteric Liquid Crystal Layer

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Abstract. A new mathematical model is proposed to describe the spatial static state of a cholesteric liquid crystal. The model is constructed with the assumption of elastic resistance of a liquid crystal under weak mechanical action or under disturbance of electric field. Along with rotational degrees of freedom displacements of the centres of mass of the liquid crystal molecules relative to initial positions are taken into account. Using numerical calculations, the effect of deformation of cholesteric spirals in a thin layer under the action of electric field of a capacitor is analysed.

Keywords: cholesteric liquid crystal, statics, electric field, Fréedericksz effect.

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Introduction

It is common to divide natural and artificial liquid crystals into three classes. These classes include nematics, smectics and cholesterics. The centres of mass of molecules are randomly distributed in space in nematics but the direction vectors of molecules lie in the same plane. Smectics differ from nematics in a layered structure with abrupt/sharp boundaries of change in the orientation of molecules when moving from layer to layer. Cholesterics have a helical structure. The essential difference between these classes from the point of view of mathematical modelling is that under certain assumptions regarding external actions, two-dimensional models can be used to analyse nematics and smectics while two-dimensional models are not applicable to cholesterics. To simulate the deformation of liquid crystals in the cholesteric phase under the action of homogeneously distributed volumetric and surface forces and moments of forces it is necessary to use three-dimensional equations.

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Cholesterics are structurally similar to nematics. The molecules in cholesterics are arranged in thin layers in such a way that their long axes are parallel to each other, that is, a layer-by-layer orientation order is observed. But the presence of asymmetric (chiral) atoms in the molecules causes the molecules of the next layer to rotate through a small angle forming a helical structure (see Fig. 1). If we move along the helix axis then after a certain number of layers the orientation of the molecules becomes the same as in the first layer. One of the main characteristics of a cholesteric liquid crystal (ChLC) is the pitch of the cholesteric helix p_0, i.e., the distance over which liquid crystal molecules rotate in space by the angle $2\,\pi$. Another important characteristic of a liquid crystal is the director (vector) $\vec{n}$ which determines the direction of the preferred orientation of the long axes of LC molecules.

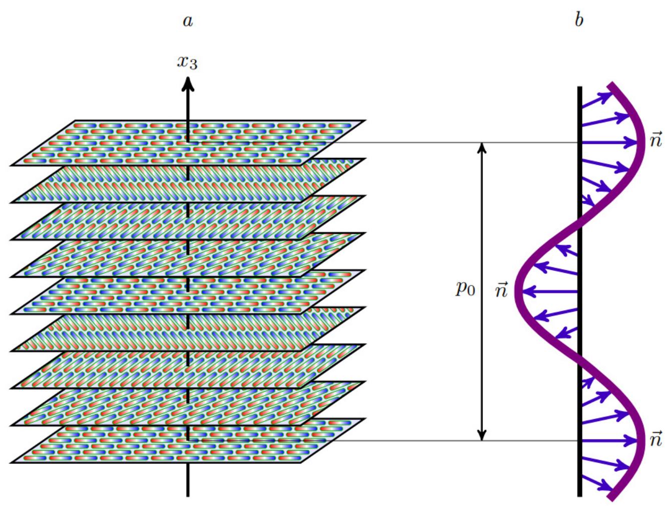

Fig. 1. Packing of rod-shaped molecules in cholesterics (a) and spiral arrangement of director $\vec{n}$ (b)

The interaction of a cholesteric liquid crystal with bounding surfaces leads to the formation of various structures depending on the boundary conditions and the ratio of the helix pitch and the thickness of the drop or layer [1]. Various orientations of near-boundary molecules are provided at the stage of preliminary preparation of liquid crystal with the help of special technological processes. Orientation structures in cholesteric droplets and their optical textures were studied, for example, in [2,3]. Oriented ChLCs have a wide area of practical application as highly sensitive sensors based on colour changes, thermal indicators, reflectors, notch filters, polarizes and optical rotators, lasers, microlenses, etc. Detailed information about the current state of researche on physical properties of cholesteric liquid crystals and technical devices based on them can be found in [4–6].

The theory of Eriksen–Leslie is used for mathematical modelling of liquid crystals (see, for example, [7]). It is applicable for solving static and non-stationary problems without restrictions on the flow structure. However, the complexity of non-linear equations of this theory is a significant obstacle to the development and justification of methods and algorithms for numerical implementation. Therefore, it is appropriate to apply approximate models that are based on simplifying hypotheses to solve specific problems.

We develop one of the approaches to model the behaviour of liquid crystals under the action of weak thermomechanical and electromagnetic perturbations. The model of acoustic approximation for the description of dynamic processes in liquid crystals was proposed [8]. Algorithms for numerical implementation of this model were developed and computations were performed for the layer of nematic liquid crystal (NLC) under the action of inhomogeneous electric field [9,10]. Computational algorithms for solving two-dimensional static problems were described [11]. The purpose of this paper is to create a simplified mathematical model of spatial deformation of a liquid crystal that is suitable for describing the cholesteric phase.

1. Mathematical model

The distribution of director in the liquid crystal relative to the Cartesian coordinate system x_1, x_2, x_3 with basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is given by a field of normals with orientation angles θ and ψ:

$$\vec{n} = \cos\theta\cos\psi\,\vec{e}_1 + \sin\theta\cos\psi\,\vec{e}_2 + \sin\psi\,\vec{e}_3\,.$$

In the initial state of the ChLC layer $\psi = \psi_0$ and $\theta = \Delta\theta\, x_3/h$ that corresponds to helical structure with a given helix twist angle $\Delta\theta$ over the layer thickness h ($\psi_0 = 0$ in Fig. 2).

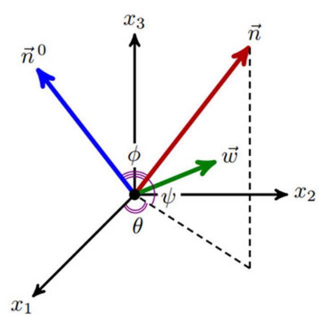

Fig. 2. Kinematic scheme of the rotational motion of director

Deformation caused by inhomogeneous external action at the boundary or inside the layer can lead to arbitrary change in both angles θ and ψ. Wherein a spatial stress-strain state is realized that is described on the basis of simplified equations of the Cosserat continuum under the assumption on hydrostatic state of a medium in the liquid phase. In this case, the stress tensor is represented by the components $\sigma_{jk} = -\,p\,\delta_{jk} + \tau_{jk}$, where p is the hydrostatic pressure, $\tau_{jk} = -\,\tau_{kj}$ are the components of antisymmetric tensor of tangential stresses, δ_{jk} is the Kronecker delta. Tangential stresses in a medium are due to the rotational degrees of freedom of the particles. In addition to tangential stresses the rotation of particles leads to the occurrence of couple stresses μ_{jk} which are the components of asymmetric tensor. Differential equations of equilibrium for an element of the medium take the following form

$$\frac{\partial p}{\partial x_k} - \frac{\partial \tau_{jk}}{\partial x_j} = f_k\,, \qquad \frac{\partial \mu_{jk}}{\partial x_j} + \varepsilon_{ijk}\,\tau_{ij} = -\,m_k\,. \tag{1}$$

Here f_k и m_k are the projections of vectors of external body force and moment of force, ε_{ijk} is the Levi – Civita symbol. Einstein's summation rule over repeated indices is accepted. Everywhere below the commonly accepted notations and operations of tensor analysis are used.

The governing equations of the model are obtained using the Castigliano variational principle. According to this principle the actual equilibrium state of the medium minimizes the potential energy integral on the set of admissible states that satisfies equilibrium equations (1) and boundary conditions in stresses. These conditions are

$$-\,p\,\nu_k + \nu_j\,\tau_{jk} = \sigma^0_k \quad \text{on} \quad S_\sigma\,, \qquad \nu_j\,\mu_{jk} = \mu^0_k \quad \text{on} \quad S_\mu\,, \tag{2}$$

where S_σ and S_μ are the parts of boundary S of domain V (layer, in a particular case), ν_k are projections of the outer normal vector to the boundary, σ^0_k and μ^0_k are the surface stresses given on S_σ and S_μ. The energy integral takes the form

$$J = \frac{1}{2}\int_V \Big(\frac{1}{\kappa}\,p^2 + \frac{1}{\alpha}\,\tau_{jk}\,\tau_{jk} + \frac{1}{\gamma}\,\mu_{jk}\,\mu_{jk}\Big)dV + \int_{S_u} u^0_k\,\big(p\,\nu_k - \nu_j\,\tau_{jk}\big)\,dS - \int_{S_w} w^0_k\,\nu_j\,\mu_{jk}\;dS\,.$$

Here κ, α and γ are phenomenological parameters of the medium: κ is the bulk compression modulus, α is the modulus of elastic resistance to relative rotation of particles, γ is the modulus of elastic resistance to curvature change; u^0_k and w^0_k are the displacements and rotations of particles that are set on the remaining parts of the boundary $S_u = S\setminus S_\sigma$ and $S_w = S\setminus S_\mu$, respectively.

The kinematic characteristics in the state of equilibrium (components of the displacement vector and the rotation vector in the case of the Cosserat continuum) are the Lagrange multipliers that corresponds to the constraints in the form of equilibrium equations. Therefore, the Lagrangian in the problem of conditional minimization under consideration can be represented as follows

$$L = J + \int_V \Big(-\,u_k\,\frac{\partial p}{\partial x_k} + u_k\,\frac{\partial \tau_{jk}}{\partial x_j} + w_k\,\frac{\partial \mu_{jk}}{\partial x_j} + \varepsilon_{ijk}\,w_i\,\tau_{jk}\Big)dV\,.$$

Equating to zero the variation of Lagrangian $\delta_p\,L = 0$, we obtain

$$\int_V \Big(\frac{1}{\kappa}\,p\,\delta p - u_k\,\frac{\partial\,\delta p}{\partial x_k}\Big)dV + \int_{S_u} u^0_k\,\nu_k\,\delta p\;dS = 0.$$

After applying Green's formula, we have

$$\int_V \Big(\frac{1}{\kappa}\,p + \frac{\partial u_k}{\partial x_k}\Big)\delta p\;dV + \int_{S_u} \big(u^0_k - u_k\big)\,\nu_k\,\delta p\;dS = 0.$$

Since variation δp is arbitrary we obtain equation and boundary condition

$$p = -\,\kappa\,\frac{\partial u_k}{\partial x_k}\,, \qquad \big(u_k - u^0_k\big)\,\nu_k = 0 \quad \text{on} \quad S_u\,. \tag{3}$$

Similarly, the equality $\delta_{\tau_{jk}}L = 0$ implies that

$$\int_V \Big(\frac{1}{\alpha}\,\tau_{jk} - \frac{\partial u_k}{\partial x_j} + \varepsilon_{ijk}\,w_i\Big)\delta\tau_{jk}\;dV + \int_{S_u} \big(u_k - u^0_k\big)\,\nu_j\,\delta\tau_{jk}\;dS = 0\,.$$

Hence, taking into account boundary condition (3) and the antisymmetry of variation $\delta\tau_{jk} = -\,\delta\tau_{kj}$, the following equations and boundary condition are obtained

$$\tau_{jk} = \frac{\alpha}{2}\Big(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} - 2\,\varepsilon_{ijk}\,w_i\Big), \qquad u_k = u^0_k \quad \text{on} \quad S_u\,. \tag{4}$$

The equality $\delta_{\mu_{jk}}L = 0$ leads to equations and boundary condition

$$\mu_{jk} = \gamma\,\frac{\partial w_k}{\partial x_j}\,, \qquad w_k = w^0_k \quad \text{on} \quad S_w\,. \tag{5}$$

With an appropriate choice of phenomenological parameters of the medium system of equations and boundary conditions (1)–(5) is a closed mathematical model of the spatial deformation of the liquid crystal. To reduce it to a compact vector form the antisymmetric tangential stress tensor

$$\begin{pmatrix} 0 & -\tau_{21} & \tau_{13} \\ \tau_{21} & 0 & -\tau_{32} \\ -\tau_{13} & \tau_{23} & 0 \end{pmatrix},$$

is identified with the pseudovector

$$\vec{\tau}^{\,\times} = \tau_{32}\,\vec{e}_1 + \tau_{13}\,\vec{e}_2 + \tau_{21}\,\vec{e}_3 = -\,\varepsilon_{ijk}\,\tau_{jk}\,\vec{e}_i\,.$$

Then $\tau_{jk} = -\,\varepsilon_{ijk}\,\tau_i^{\times},\ \ \nu_j\,\tau_{jk}\,\vec{e}_k = -\,\varepsilon_{ijk}\,\nu_j\,\tau_i^{\times}\vec{e}_k = \varepsilon_{ijk}\,\nu_j\,\tau_k^{\times}\vec{e}_i = \vec{\nu}\times\vec{\tau}^{\,\times}$,

$$\frac{\partial\tau_{jk}}{\partial x_j}\,\vec{e}_k = -\,\varepsilon_{ijk}\,\frac{\partial\tau_i^{\times}}{\partial x_j}\,\vec{e}_k = \varepsilon_{ijk}\,\frac{\partial\tau_k^{\times}}{\partial x_j}\,\vec{e}_i = \nabla\times\vec{\tau}^{\,\times}, \qquad \nabla\times\vec{u} = \varepsilon_{ijk}\,\frac{\partial u_k}{\partial x_j}\,\vec{e}_i\,.$$

Using these relations, differential equations included in (1)–(5) are transformed to the following form

$$\begin{gathered} \nabla p - \nabla\times\vec{\tau}^{\,\times} = \vec{f}, \qquad p = -\,\kappa\,\nabla\cdot\vec{u}\,, \qquad \vec{\tau}^{\,\times} = \alpha\Big(\vec{w} - \frac{1}{2}\,\nabla\times\vec{u}\Big), \\ -\,\nabla\cdot\boldsymbol{\mu} + 2\,\vec{\tau}^{\,\times} = \vec{m}\,, \qquad \boldsymbol{\mu} = \gamma\,\nabla\vec{w}\,. \end{gathered} \tag{6}$$

Boundary conditions for displacements and rotation angles obtained from the Castigliano variational principle have the following vector form

$$\vec{u} = \vec{u}^{\,0} \quad \text{on} \quad S_u\,, \qquad \vec{w} = \vec{w}^{\,0} \quad \text{on} \quad S_w\,. \tag{7}$$

Boundary conditions (2) for stresses and couple stresses are as follows

$$-\,p\,\vec{\nu} + \vec{\nu}\times\vec{\tau}^{\,\times} = \vec{\sigma}^{\,0} \quad \text{on} \quad S_\sigma\,, \qquad \vec{\nu}\cdot\boldsymbol{\mu} = \vec{\mu}^{\,0} \quad \text{on} \quad S_\mu\,. \tag{8}$$

Equations (6) with boundary conditions (7), (8) can be used to model the deformation of liquid crystal occupying an arbitrary domain under sufficiently general external actions of mechanical, temperature or electromagnetic fields inside the domain and on its boundary.

Let us consider the case of a non-magnetic liquid crystal (dielectric) when the bulk forces and moments of forces are caused by the action of an inhomogeneous electric field.

2. The action of electric field

The inhomogeneity of electric field is directly connected with the previously unknown orientation of LC molecules in a deformed state. Orientation, in its turn, depends on the electric field direction. The electric field $\vec{E}$ is defined in terms of the spatial distribution of the electric potential φ: $\vec{E} = -\nabla\varphi$. In the absence of bulk electric charges inside domain V, the equation for the potential takes the form:

$$\nabla\cdot\vec{D} = 0\,, \quad \vec{D} = \boldsymbol{\varepsilon}\cdot\vec{E} \quad \Longrightarrow \quad \nabla\cdot\big(\boldsymbol{\varepsilon}\cdot\nabla\varphi\big) = 0\,. \tag{9}$$

Here $\vec{D}$ is the electric induction vector, $\boldsymbol{\varepsilon}$ is the dielectric permittivity tensor. It is defined as

$$\boldsymbol{\varepsilon} = \varepsilon^{\perp}\,\boldsymbol{I} + \Delta\varepsilon\,\vec{n}\,\vec{n}\,,$$

where $\varepsilon^{\|}$ and $\varepsilon^{\perp}$ are permittivities along and across molecules, respectively, $\boldsymbol{I}$ is the unit tensor, $\Delta\varepsilon = \varepsilon^{\|} - \varepsilon^{\perp}$.

The spatial distribution of the director $\vec{n}$ depends on the electric field indirectly through the molecular rotation vector $\vec{w}$. When rotating through an infinitesimal angle, one can write

$$\vec{n} = \vec{n}^{\,0} + \vec{w} \times \vec{n}^{\,0}. \tag{10}$$

This relation has a simple geometric interpretation (see Fig. 2). The projection $w_n = \vec{w} \cdot \vec{n}^{\,0}$ of the rotation vector onto the initial direction of the director describes the rotation of a medium around the $\vec{n}^{\,0}$ axis. Such rotation has no effect on the distribution of mechanical stresses and electric field since individual LC molecules are represented as rectilinear rigid needles of nanoscale length with negligible thickness. Vector $\vec{w} - w_n\,\vec{n}^{\,0}$ is orthogonal to the direction $\vec{n}^{\,0}$ and it describes the rotation of director from the initial position to the current one. Therefore, $\vec{w} - w_n\,\vec{n}^{\,0} = \vec{n}^{\,0} \times \vec{n}$. It is consistent with relation (10):

$$\vec{n}^{\,0} \times \vec{n} = \vec{n}^{\,0} \times \vec{n}^{\,0} + \vec{n}^{\,0} \times \left(\vec{w} \times \vec{n}^{\,0}\right) = \vec{w}\left(\vec{n}^{\,0} \cdot \vec{n}^{\,0}\right) - \vec{n}^{\,0}\left(\vec{n}^{\,0} \cdot \vec{w}\right) = \vec{w} - w_n\,\vec{n}^{\,0}.$$

For finite rotations relation (10) is not applicable because condition $\vec{n}^{\,2} = 1$ is violated. In this case, $\vec{n} = \boldsymbol{R} \cdot \vec{n}^{\,0}$, where $\boldsymbol{R}$ is the rotation tensor that is defined in terms of the unit vector of the rotation axis

$$\vec{q} = \frac{\vec{w}}{|\vec{w}|} = \frac{\vec{n}^{\,0} \times \vec{n}}{|\vec{n}^{\,0} \times \vec{n}|}$$

and the rotation angle ϕ as follows

$$\boldsymbol{R} = \boldsymbol{I} + \sin\phi\,\boldsymbol{Q} + (1 - \cos\phi)\,\boldsymbol{Q}^2\,, \qquad \boldsymbol{Q} = \begin{pmatrix} 0 & -\,q_3 & q_2 \\ q_3 & 0 & -\,q_1 \\ -\,q_2 & q_1 & 0 \end{pmatrix}.$$

If rotation occurs in the positive direction of vector $\vec{w}$ then $\phi = |\vec{w}|$. If rotation takes place in the negative direction then $\phi = -|\vec{w}|$.

Contrary to traditional mathematical models of LC deformation in this model the director $\vec{n}$, which is required to calculate the dielectric permittivity tensor, does not belong to the main required functions. It is determined using rotation vector $\vec{w}$ by relation (10) or by the more precise relation $\vec{n} = \boldsymbol{R} \cdot \vec{n}^{\,0}$.

Let us note that differential equation (9) is not sufficient to uniquely define the electric potential in V since the electric potential must be determined in the entire space including the exterior of V. If there are no bulk electric charges in the surrounding space and if it is filled with air or other rarefied gas with dielectric permittivity close to unity then the potential in it satisfies the Laplace equation $\nabla^2\varphi = 0$. Moreover, it tends to zero at infinity. At the same time, conditions for continuity of the electric potential and the component of the electric induction vector normal to the interface are satisfied at the interface between the dielectric and the gas. It is also necessary to add to equation (9) boundary conditions on boundary S or on its part simulating the occurrence of non-zero electric field.

When potential is given the vector of bulk forces caused by the inhomogeneity of electric field is determined as follows

$$\vec{f} = \left(\vec{P} \cdot \nabla\right)\vec{E}\,, \qquad \vec{P} = \varepsilon_0\,\boldsymbol{\chi} \cdot \vec{E}\,, \qquad \boldsymbol{\chi} = \boldsymbol{\varepsilon} - \boldsymbol{I}\,,$$

where $\boldsymbol{\chi}$ is the dielectric susceptibility tensor, $\vec{P}$ is the polarization vector, $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{F/m}$ is the electrical constant. In expanded form it becomes

$$\vec{f} = \varepsilon_0 \Big((\varepsilon^{\perp} - 1)\, \nabla\varphi \cdot \nabla + \Delta\varepsilon\, (\vec{n} \cdot \nabla\varphi)\, \vec{n} \cdot \nabla \Big) \nabla\varphi\,. \tag{11}$$

To determine the vector of moment of bulk forces the following relation is used

$$\vec{m} = \vec{P} \times \vec{E} \implies \vec{m} = \varepsilon_0\, \Delta\varepsilon\, (\vec{n} \cdot \nabla\varphi)\, \vec{n} \times \nabla\varphi \tag{12}$$

It follows from (11) and (12) that LC molecules are subjected to bulk moments of forces in an arbitrary electric field, not excluding the case when field vector $\vec{E}$ is constant everywhere in V, while bulk forces appear only with a non-uniform distribution of this vector.

3. One-dimensional problem

Let us consider LC layer of thickness h infinite in the plane x_1, x_2 between extended capacitor plates. Initial distribution of molecular orientation angles inside the layer is known: $\theta_0(x_3) = \Delta\theta\, x_3/h$, $\psi_0 = \psi_0(x_3)$. It corresponds to the cholesteric phase with the turn of spirals across the layer at an angle $\Delta\theta$. Molecules are reoriented when charges appear on the capacitor plates under the action of electric field.

Components of the dielectric permittivity tensor $\varepsilon_{jk} = \varepsilon^{\perp}\,\delta_{jk} + \Delta\varepsilon\, n_j\, n_k$ in the considered Cartesian coordinate system are

$$\begin{aligned}
&\varepsilon_{11} = \varepsilon^{\perp} + \Delta\varepsilon\, \cos^2\theta\, \cos^2\psi\,, \qquad && \varepsilon_{12} = \frac{1}{2}\,\Delta\varepsilon\, \sin 2\theta\, \cos^2\psi\,,\\
&\varepsilon_{22} = \varepsilon^{\perp} + \Delta\varepsilon\, \sin^2\theta\, \cos^2\psi\,, && \varepsilon_{23} = \frac{1}{2}\,\Delta\varepsilon\, \sin\theta\, \sin 2\psi\,,\\
&\varepsilon_{33} = \varepsilon^{\parallel} \sin^2\psi + \varepsilon^{\perp}\cos^2\psi\,, && \varepsilon_{13} = \frac{1}{2}\,\Delta\varepsilon\, \cos\theta\, \sin 2\psi\,.
\end{aligned}$$

In addition to reorientation, the layer is deformed under the action of electromagnetic forces. Taking into account the symmetry of the problem, we have

$$\vec{E} = -\,\varphi'\,\vec{e}_3\,, \quad \vec{P} = -\,\varepsilon_0\,\varphi' \Big(\varepsilon_{13}\,\vec{e}_1 + \varepsilon_{23}\,\vec{e}_2 + (\varepsilon_{33} - 1)\,\vec{e}_3\Big), \quad f_1 = f_2 = 0\,, \quad f_3 = \varepsilon_0\,(\varepsilon_{33} - 1)\,\varphi'\varphi'',$$

and the prime denotes the derivative with respect to x_3. The rotation of molecules is due to the action of moments of forces. Non-zero projections of the vector of moments are

$$m_1 = \frac{\varepsilon_0\,\Delta\varepsilon}{2}\,(\varphi')^2 \sin\theta\, \sin 2\psi\,, \qquad m_2 = -\,\frac{\varepsilon_0\,\Delta\varepsilon}{2}\,(\varphi')^2 \cos\theta\, \sin 2\psi\,.$$

Vector $\vec{m}$ at each point of the layer is turned out to be directed perpendicular to the plane passing through the director $\vec{n}$ and the axis x_3. This follows from the equality to zero of the scalar products $\vec{m} \cdot \vec{n} = \vec{m} \cdot \vec{e}_3 = 0$. Thus, the reorientation of molecules occurs only due to the change in angle ψ while angle $\theta = \theta_0(x_3)$ remains unchanged.

The differential equations of equilibrium for the layer take the form

$$-\,p' = f_3\,, \qquad \mu_{31}' - 2\,\tau_1^{\times} = -\,m_1\,, \qquad \mu_{13}' - 2\,\tau_2^{\times} = -\,m_2\,.$$

Non-zero projections of the rotation vector are

$$w_1 = \Delta\psi\, \sin\theta\,, \qquad w_2 = -\,\Delta\psi\, \cos\theta \qquad (\Delta\psi = \psi - \psi_0)\,.$$

Constitutive equations for pressure, moment stresses and tangential stresses are

$$p = -\kappa\, u_3', \qquad \mu_{31} = \gamma\, w_1', \qquad \mu_{32} = \gamma\, w_2', \qquad \tau_1^{\times} = \alpha\, w_1, \qquad \tau_2^{\times} = \alpha\, w_2.$$

They allow one to transform the equilibrium equations to the following system of equations for displacement u_3 and rotation angle ψ

$$\kappa\, u_3'' = \varepsilon_0\, (\varepsilon_{33} - 1)\, \varphi' \varphi'', \qquad -2\,\gamma\, \big(\psi'' - \psi_0''\big) + 4\,\alpha\, (\psi - \psi_0) = \varepsilon_0\, \Delta\varepsilon\, (\varphi')^2 \sin 2\psi\,. \tag{13}$$

Equation (9) for the electric potential is integrated as follows

$$(\varepsilon_{33}\, \varphi')' = 0 \quad \Longrightarrow \quad \varphi' = \frac{C_1}{\varepsilon_{33}}\,.$$

The next condition is used to determine constant C_1

$$C_1 \int_0^h \frac{dx_3}{\varepsilon_{33}} = \Delta\varphi \qquad \big(\varepsilon_{33} = \varepsilon^{\|} \sin^2\psi + \varepsilon^{\perp} \cos^2\psi\big)\,, \tag{14}$$

where $\Delta\varphi$ is the difference of potentials on the capacitor plates.

After substituting expression for φ' and integrating the first equation (13), the system is transformed into

$$\kappa\, u_3' = -\,\varepsilon_0\, C_1^2\, \frac{1 - 2\,\varepsilon_{33}}{2\,\varepsilon_{33}^2} + C_2\,, \qquad -2\,\gamma\, \Delta\psi'' + 4\,\alpha\, \Delta\psi = \varepsilon_0\, C_1^2\, \Delta\varepsilon\, \frac{\sin 2(\psi_0 + \Delta\psi)}{\varepsilon_{33}^2}\,. \tag{15}$$

The boundary conditions $\Delta\psi(0) = \Delta\psi(h) = 0$ are added to the equation for the angle of rotation. Such problem is solved numerically. The distribution $\psi^0(x_3) = \psi_0(x_3)$ is taken as the initial distribution of angles. According to the given distribution $\psi^n(x_3)$, the approximate value of constant C_1^n is calculated using (14). New approximation $\psi^{n+1}(x_3)$ is determined using three-point sweep method based on the iterative algorithm

$$-2\,\gamma\, \frac{\Delta\psi_{j+1}^{n+1} - 2\,\Delta\psi_j^{n+1} + \Delta\psi_{j-1}^{n+1}}{\Delta x_3^2} + 4\,\alpha\, \Delta\psi_j^{n+1} = \varepsilon_0\, C_1^2\, \Delta\varepsilon\, \frac{\sin 2(\psi_{0\,j} + \Delta\psi_j^n)}{(\varepsilon_{33\,j}^n)^2}\,. \tag{16}$$

The process is stopped when the condition $||\Delta\psi^{n+1} - \Delta\psi^n|| / ||\Delta\psi^n|| < \delta$ is fulfilled, where $||\Delta\psi||$ is a uniform difference norm, δ is a given calculation error.

After finding constant C_1 using the trapezoid rule, potential φ is calculated from relation

$$\varphi(x_3) = C_1 \int_{h/2}^{x_3} \frac{dx_3}{\varepsilon_{33}}\,.$$

The equation for displacement is integrated numerically for boundary conditions of two types: $u_3(0) = u_3(h) = 0$ and $u_3(0) = 0$, $u_3'(h) = 0$. In the first case, constant C_2 is determined as

$$C_2 = \frac{\varepsilon_0\, C_1^2}{2\,h} \int_0^h \frac{1 - 2\,\varepsilon_{33}}{\varepsilon_{33}^2}\, dx_3,$$

and in the second case as

$$C_2 = \varepsilon_0\, C_1^2\, \frac{1 - 2\,\varepsilon_{33}}{2\,\varepsilon_{33}^2}\,\bigg|_{x_3 = h}\,.$$

When initial angle $\psi_0 = 0$ is equal to zero equation (15) for the rotation angle describes the Fréedericksz effect of the loss of equilibrium of LC molecules in electric field. As a result of linearisation of the equation, the problem is reduced to the boundary value problem

$$\gamma\, h^2\, \psi'' = \left(2\,\alpha\, h^2 \psi - \varepsilon_0\, \Delta\varepsilon\, \Delta\varphi^2\right) \psi\,, \qquad \psi(0) = \psi(h) = 0\,.$$

After substituting the solution $\psi = A\, \sin \pi x_3/h$, where A is an arbitrary constant, we obtain the formula for the difference of potentials at which the trivial solution becomes unstable

$$\Delta\varphi_0 = \sqrt{\frac{\pi^2\, \gamma + 2\, h^2\, \alpha}{\varepsilon_0\, \Delta\varepsilon}}\,. \tag{17}$$

In comparison with the classical formula for the Fréedericksz transition threshold, which takes into account only moment interactions, it contains a correction accounting the resistance to rotation of particles due to tangential stresses and it shows that such resistance prevents the loss of stability.

Formula (17) is used for verification of the algorithm and program. According to the results of computations of the liquid crystal with parameters $\varepsilon^{\|} = 16.7$, $\varepsilon^{\perp} = 7$, $\alpha = 2.45\,\mathrm{Pa}$, $\gamma = 6{\cdot}10^{-12}\,\mathrm{N}$ and $\kappa = 3.12\,\mathrm{GPa}$ the value of potential difference $\Delta\varphi = 1.27\,\mathrm{V}$ is obtained which is close to the threshold value corresponding to the transition of the layer into unstable state. At smaller values of $\Delta\varphi$ the orientation of molecules calculated by scheme (16) with $\psi_0 = 0$ remains unchanged and $\psi = 0$. The electric potential is distributed linearly over the layer: $\varphi = (x_3/h - 0.5)\,\Delta\varphi$. For larger values of the difference of potentials the transition occurs from initial unstable state to a stable one which is characterized by inhomogeneous distribution of angle ψ and non-linear distribution of potential φ over the layer. There are two stable states that differ in the sign of the molecular orientation angle. The positive or negative sign is realized in computations. It depends on the small perturbation of the initial angle ψ_0.

Let us note that the sequence of approximations of the orientation angle in the numerical implementation of scheme (16) is rapidly converges (number of iterations is about 10) if the resulting value of angle ψ at the layer centre is away from 90°, i.e., differs from the orientation angle of the electric field. When the value of angle ψ approaches 90° the convergence of the iterative process slows down with the transition to the divergent regime. In addition, the expansion of non-linear right-hand side (16) according to the Newton method does not allow one to expand the range of admissible setting of potential difference $\Delta\varphi$ in which the approximations converge but, on the contrary, leads to a significant narrowing this range.

4. Numerical results

The results of computations for the layer of thickness $h = 4$ μm with potential difference $\Delta\varphi = 1.28$ V (it is close to the threshold value) are shown in Figs. 3–6. The curves of red, green, blue and violet colours in Fig. 3 demonstrate diagrams of the distribution of the orientation angle over the layer for initial values $\psi_0 \approx 0$, $\psi_0 = 5°$, $10°$ and $15°$. Deviations of potential from the linear distribution $\delta\varphi(x_3) = \varphi(x_3) - (x_3/h - 0.5)\,\Delta\varphi$ corresponding to these values are shown in Fig. 4. Results of computations demonstrate that potential distribution for small values of initial angle ψ_0 is close to linear distribution but it changes significantly with a slight change in this parameter.

Figs. 5 and 6 show diagrams of strain distribution $\epsilon_{33} = u_3'$ for two types of boundary conditions on the sides of the layer (on capacitor plates). In both cases, the level of strains is

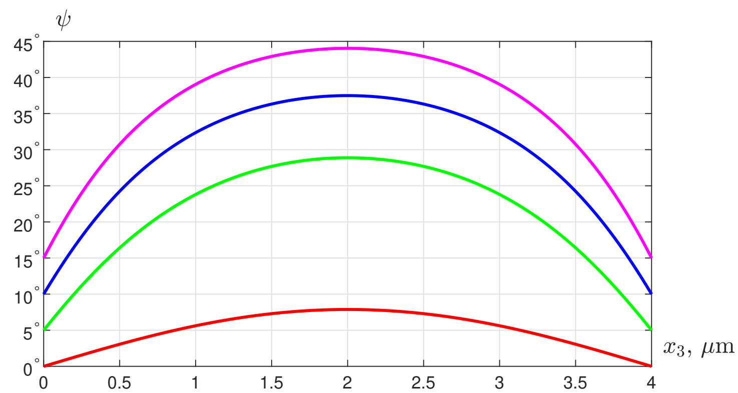

Fig. 3. Distribution of the rotation angles of molecules over the LC layer

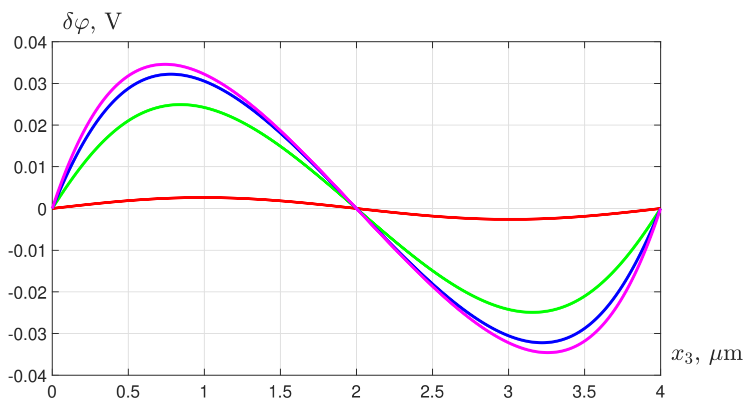

Fig. 4. Deviation of the electric potential from linear distribution

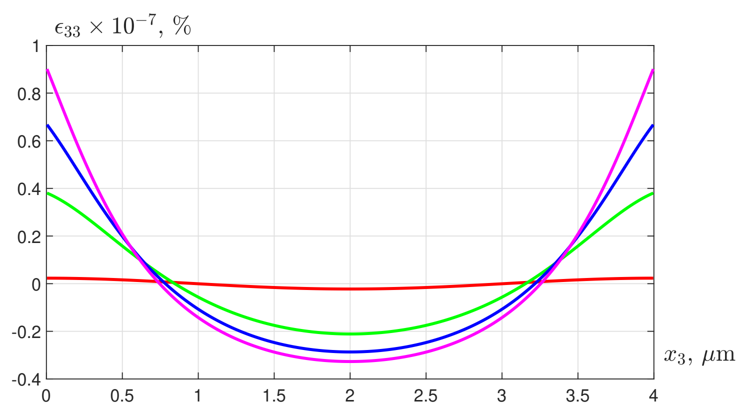

Fig. 5. Strain distribution over the LC layer for fixed sides

negligible (about 10^{-7} %) since the electric field in the problem under consideration is practically uniform. Its inhomogeneity is determined by a slight change in the LC dielectric permittivity due to relative rotation of molecules. Nevertheless, the following characteristic qualitative features

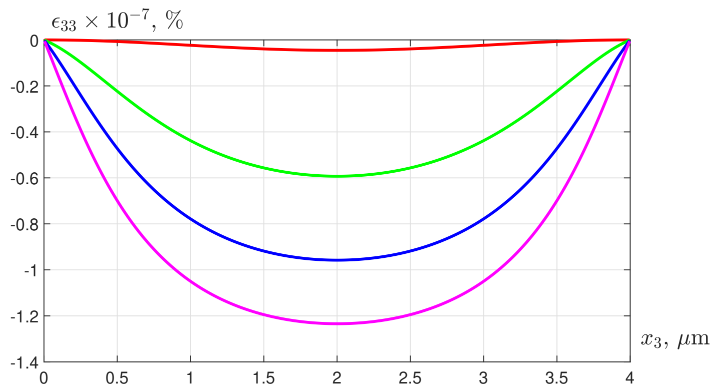

Fig. 6. Strain distribution over the LC layer for free surface

can be noted. If both sides are fixed, the layer is stretched near boundaries and compressed in the centre. Thus, the pitch of cholesteric helices of the liquid crystal is increased in comparison with the initial pitch near capacitor plates, and it is decreased in the middle part of the layer. Under the condition of a free surface, the layer is compressed everywhere but the pitch of helices is decreased in the centre, and it remains practically the same as in the initial undeformed state near the sides.

Results of computations presented in Fig. 7 correspond to the LC layer that consists of two sublayers of equal thickness. The initial orientation angles are $\psi_0 = 0$ (in the lower sublayer) and $\psi_1 = 5°, 10°, 15°, 20°$ (in the upper sublayer). Considering results of computations, one can see that jump in the orientation angle of molecules at the interface between sublayers after the application of constant electric field remains the same as it was set in the initial state. This follows directly from the analysis of equation (15) for the rotation angle. The right-hand side of the equation is discontinuous function with discontinuity of the first kind at the interface between sublayers.

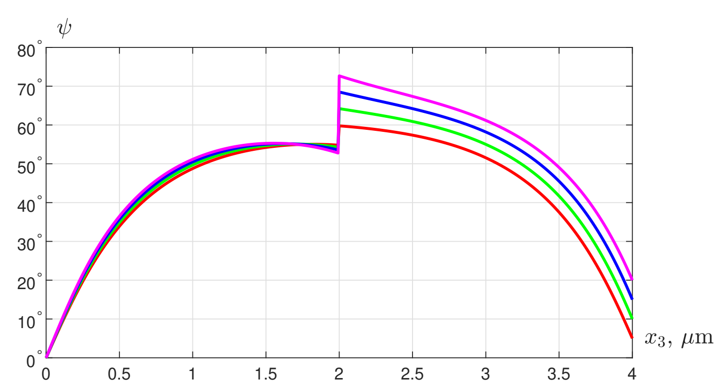

Fig. 7. Distribution of the rotation angles of molecules over the LC layer consisting of two sublayers

Performed computations demonstrate the applicability of the proposed mathematical model for calculating liquid crystals of a layered smectic phase.

Conclusion

To describe the static deformed state of the liquid crystal under the action of weak external perturbations a simplified mathematical model is proposed. The liquid crystal is considered as structurally inhomogeneous continuum with translational and rotational degrees of freedom of the micro-structure particles (LC molecules). This model is applicable to the analysis of cholesteric liquid crystals with spatial helical orientation of molecules. To demonstrate implementation of the model the problem of deformation of a cholesteric liquid crystal layer in the electric field of a capacitor was considered. The state of the liquid crystal in the vicinity of the Fréedericksz transition was studied numerically. Distributions of the orientation angle, electric potential and strain over the layer were obtained for various initial orientation angles. Analysis of the results of computations demonstrates that predominant compression of cholesteric spirals under the electric field action (its inhomogeneity over the layer is determined by the change in the dielectric permittivity due to the rotation of molecules) occurs in the middle part of the ChLC layer.

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Моделирование действия электрического поля на жидкокристаллический слой холестерика

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Аннотация. В рамках предположения об упругом сопротивлении холестерического жидкого кристалла слабым механическим воздействиям или возмущениям электрическим полем строится новая математическая модель для описания пространственного статического состояния. Наряду с вращательными степенями свободы учитываются смещения центров масс молекул жидкого кристалла относительно начального положения. С помощью численных расчетов в задаче для тонкого слоя анализируется эффект деформации холестерических спиралей под действием электрического поля конденсатора.

Ключевые слова: холестерический жидкий кристалл, статика, электрическое поле, эффект Фредерикса.

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On the Convergence Exponent of the Special Integral of the Tarry Problem for a Quadratic Polynomial

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Abstract. In this paper it is considered the summation problem for trigonometric integrals with quadratic phase. This problem was considered in the papers [7–9] in particular cases. Our results generalize the results of those papers to multidimensional trigonometrical integrals.

Keywords: trigonometrical integral, exponent, sums, phase, polynomial.

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Introduction

Let $P(x, s) \in \mathbb{R}[x]$ (where $x \in \mathbb{R}^k$) be a polynomial with real coefficients $s \in \mathbb{R}^N$. We consider the trigonometric integral given by

$$T(s) = \int_Q \exp(iP(x, s))dx, \tag{1}$$

where $Q \subset \mathbb{R}^k$ is a compact set.

Problems related to such kind of integrals arise in mathematical physics (see [1]), harmonic analysis (see [2–5]), analytic number theory (see [6–11]) and so on. Surely, the given references are not complete.

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One of the well known problems related to the trigonometric integrals is the issue on convergence of the special integral of the Tarry problem, which is given by the following:

$$\theta = \int_{\mathbb{R}^N} |T(s)|^p ds, \text{ with } Q = [0,1]^k. \tag{2}$$

The integral θ arises as the coefficient of asymptotic representation for a number of integer solutions of a Diophantine system [2, 6, 7]. Therefore, it is important to find a minimal value of the parameter p, where the special integral is convergent, which is also essential in the Fourier restriction problem in harmonic analysis [3].

Definition. *A real number γ is called to be a convergence exponent of the special integral if for every $p > \gamma$ the integral (2) is convergent and for every $p < \gamma$ it is divergent. In other words $\gamma = \inf\{p : T \in L_p(\mathbb{R}^N)\}$.*

It should be noted that the convergence exponent essentially depends on the form of the polynomials $P(x, s)$. Thus the main problem can be formulated as:

Problem: Find the number γ.

This problem was considered by I. M. Vinogradov [11] in connection with the problems of analytic number theory. He obtained an upper bound for the number γ in the case $k = 1$. This bound was improved in [10].

The exact value of γ was indicated in [6] for the case $k = 1$. It is interesting to note that in one-dimensional case depending on form of the polynomial $P(x, s)$ the exact value of γ can be expressed by the sum of exponents of the non-trivial terms of the polynomial $P(x, s)$. Moreover, it was proved un upper bound for the number γ in multidimensional cases.

It should be noted that, in [12] a lower bound was found for the number γ. Moreover, it was found the number γ provided that the coefficients of the polynomial vary in some subspace of $\mathbb{R}^N$. Similar problems were considered in the works [13, 14, 15].

In [7] a lower bound was obtained for γ and also, it was investigated analogical problem for more subtle object trigonometric sums in the case $k = 2$. In [7] and [9] a similar problem was considered in the case $k = 2$. Moreover, in [7], it is shown that if P is a homogeneous quadratic polynomial and $k = 2$, then $\gamma = 4$ in the case when $Q = [0,1]^2$, more precisely, the special integral θ is convergent if $p > 4$ and divergent if $p \leqslant 4$.

It was interesting to extend the results proved by L. G .Arkhipova, V. N. Chubarikov related to trigonometric integrals to multidimensional case.

In this paper we study the problem in the classical setting. In other words, P is a quadratic polynomial function and $Q = [0,1]^k$ is the unit cube and also for the case when Q is a compact domain. Analogical problem was considered by J. Makenhaupt [2], who obtain the number γ in the case when the polynomial $P(x, s)$ satisfies some "non-degeneracy" condition.

It should be noted that the condition of J. Makenhaupt does not hold for the general case (see [2]). Actually, J. Mokenhaupt used an interesting approach. He computed the multidimensional trigonometric integral, for which the amplitude function is the gauss function. Then he be able to get the sharp value of the convergence exponent for some cases. It should be noted that using the gauss functions to investigate behavior of oscillatory integrals goes back to E. M. Stein [1]. We obtain the exact value of γ, whenever P is a homogeneous polynomial of degree two.

We use the idea of J. Makenhaupt and then we able to investigate the obtained integrals. We observe that the integral over $\mathbb{R}^N$ can be written as an iterated integral over the orbit of the orthogonal group and then over the corresponding fundamental domain. It is interesting that the integrant in the trigonometric integrals with quadratic phase with special amplitude function, more precisely gauss functions, is invariant under action of the orthogonal group. Thus, our approach is natural in this case. Unfortunately, it seems such approach does not work for trigonometric integrals with more general polynomial phase functions.

The paper is organized as follows in the next Section 1 we formulate our main results. In the next Section 2 we give some auxiliary results on integrals. In particular, we obtain transformation of the volume form under the natural action of the orthogonal group. Then we give a proof of our main results in the next Section 3. Finally, we give some results related to two-dimensional integrals in the last Section 4.

1. Formulation of the main results

Let P be the polynomial given by

$$P(x, A, b) = (Ax, x) + (b, x),$$

where $A = (a_{lm})^k{}_{l,m=1}$ is a symmetric $k \times k$ matrix with real entries, $b := (b_1, b_2, \ldots, b_k) \in \mathbb{R}^k$ and $(\cdot, \cdot)$ is the inner product of the corresponding vectors. Consider the trigonometric integral

$$T(A, b) = \int_{\mathbb{R}^k} \exp{(iP(x, A, b))}\chi_Q(x)dx,$$

where Q is a compact set and $\chi_Q(x)$ is its characteristic function.

Consider the integral

$$\theta = \int_{\mathbb{R}^N} |T(A, b)|^p db\, da,$$

where $db = db_1 db_2 \ldots db_k$ and $da = \prod\limits_{1 \leqslant l \leqslant m \leqslant k} da_{lm}$.

The following is true:

Theorem 1.1. *Let Q be a compact set, then the integral θ converges, whenever $p > 2k + 2$ and if Q contains an interior point x^0 and there exists a line l passing through point x^0 such that the boundary of the set $\{l \cap Q\}$ contains only a finite number of points, then the integral diverges provided $p \leqslant 2k + 2$. In particular, if $Q = [0, 1]^k$, then $\gamma = 2k + 2$.*

1. The case when P is a homogeneous polynomial of the second order

Now suppose that $P(x, A) = (Ax, x)$. In [9] it has been proved that if Q is a quadratic polynomial in $\mathbb{R}^2$, then for $p > 4$ the θ integral converges and when $p \leqslant 4$ the θ integral diverges. In this paper we extend those results to the case when Q is a polyhedron in $\mathbb{R}^k$.

By polyhedron we mean a finite union of nondegenerate simplexes [5].

Theorem 1.2. *If $P(x, A) = (Ax, x)$ and Q is a polyhedron, then for $p > 2k$ the integral θ converges. If $Q = [0, 1]^k$, then for $p \leqslant 2k$ the integral θ diverges.*

Remark 1. In this case, we cannot apply the results of [3] as the corresponding set $\{x_i x_j\}_{i \leqslant j=1}^n$ is not a smooth surface.

Remark 2. Depending on the set Q, the exponent p may be smaller than $2k$. For example, if $k = 2$ and Q is a sufficiently small square centered at $(1, 1)$, then it can be proved that for $p > 3$ the integral θ converges.

2. Preliminaries

Consider the following integral

$$T_\infty(A,b)=\int_{\mathbb{R}^k}\exp\left(iP(x,A,b)-(x,x)\right)dx.$$

It is easy to check that this integral, whose calculation details are given in [2], is absolutely and uniformly converges with respect to the parameters A and b.

Lemma 2.1. *The following equality holds*

$$T_\infty(A,b)=(2\pi)^{\frac{k}{2}}(det(I-iA))^{-\frac{1}{2}}\exp\left(-\frac{((I-iA)^{-1}b,b)}{4}\right),$$

where the square root is determined in the following way

$$(det(I-iA))^{-\frac{1}{2}}:=(1-i\lambda_1)^{-\frac{1}{2}}\cdot(1-i\lambda_2)^{-\frac{1}{2}}\cdot\ldots\cdot(1-i\lambda_k)^{-\frac{1}{2}},$$

with $\lambda_1,\ldots,\lambda_k$ being eigenvalues of A. The branch cut of the multiply-valued function $z^{-\frac{1}{2}}$ is taken on the complex plane by cutting the negative part of the real axis and $1^{-\frac{1}{2}}=1$.

Lemma 2.1 is proved by reducing A to the diagonal form. Consequently, the calculation of the integral is reduced to a one-dimensional integral and it is explicitly calculated (see. [1]).

Obviously, the following equations are satisfied:

$$\left|\exp\left(-\frac{((I-iA)^{-1}b,b)}{4}\right)\right|^p=\exp\left(-\frac{((I+A^2)^{-1}b,b)p}{4}\right),$$

$$\int\limits_{\mathbb{R}^k}\exp\left(-\frac{((I+A^2)^{-1}b,b)p}{4}\right)db=\frac{(8\pi)^{\frac{k}{2}}(det(I+A^2))^{\frac{1}{2}}}{p^{\frac{k}{2}}}.$$

Let us introduce the following notation:

$$\theta_\infty=\int_{\mathbb{R}^N}|T_\infty(A,b)|^p\,db\,da,$$

where $N=\dfrac{k(k+2)}{2}$.

Proposition 1. *The integral θ_∞ converges when $p>2k+2$ and diverges when $p\leqslant 2k+2$.*

Due to Lemma 2.1, the proof of the Proposition 1 comes by studying the following integral

$$\theta_\infty=c(p)\int\limits_{\mathbb{R}^{N-k}}\frac{da}{(det(I+A^2))^{\frac{p-2}{4}}},\tag{3}$$

where $c(p)$ is some positive number.

As the determinant is an invariant of the orthogonal group, it is convenient to integrate it first by the orbits of the orthogonal group and then by the quotient space, e.g. over fundamental domain with respect to action of the orthogonal group.

Let M be the set of symmetric matrices with real entries and $G=SO_k$ be a special subgroup of orthogonal matrices. This group naturally acts in the space M as $g(A)=g^tAg$, where $g\in SO_k$ and $A\in M$.

It is known that for any real symmetric matrix A, there exists $g\in G$ such that $g(A) = diag(\lambda_1, \dots, \lambda_k)$, where $diag(\lambda_1, \dots, \lambda_k)$ is a diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_k$. In other words for any matrix A there exists $g\in G$ such that $A = g^t \Lambda g$, where $\Lambda = diag(\lambda_1, \dots, \lambda_k)$. Hence it is possible to define a surjective smooth map

$$\Phi : \mathbb{R}^k \times G \mapsto M$$

which is defined by the formula $\Phi(\Lambda, g) = g^t \Lambda g$.

Let $da = da_{11} \wedge \dots \wedge da_{kk}$ be the standard volume form in the space M. We can define the image of this form under the map Φ, denoted by $\Phi^* da \in \wedge^{N-k}(\mathbb{R}^k \times SO_k)$.

Lemma 2.2. *The following equality holds*

$$\Phi^* da = \prod_{1\leqslant l<m\leqslant k} (\lambda_m - \lambda_l) d\lambda_1 \wedge \dots \wedge d\lambda_k \wedge \omega,$$

where ω is the volume form on the orthogonal group SO_k.

Lemma 2.2 can be proved by using the zero sets of the Jacobian of the map Φ. Note that the equality $\prod\limits_{1\leqslant l<m\leqslant k} (\lambda_m - \lambda_l)^2 = \rho_A(\lambda)$ holds, where $\rho_A(\lambda)$ is the discriminant of the characteristic polynomial of the matrix A.

By Lemma 2.1 the integral (3) can be rewritten as

$$\int\limits_{\mathbb{R}^{N-k}} \frac{da}{(det(I + A^2))^{\frac{p-2}{4}}} = \int\limits_{\mathbb{R}^k} \frac{\prod\limits_{1\leqslant l<m\leqslant k} |\lambda_m - \lambda_l|}{\prod\limits_{1\leqslant l\leqslant k} (1 + {\lambda_l}^2)^{\frac{p-2}{4}}} d\lambda_1 \wedge \dots \wedge d\lambda_k \int\limits_{SO_k} \omega.$$

From the last equality, it follows that the convergence of the integral (3) comes from the investigation of the convergence of the following integral

$$\int\limits_{\mathbb{R}^k} \frac{\prod\limits_{1\leqslant l<m\leqslant k} |\lambda_m - \lambda_l|}{\prod\limits_{1\leqslant l\leqslant k} (1 + {\lambda_l}^2)^{\frac{p-2}{4}}} d\lambda_1 \wedge \dots \wedge d\lambda_k.$$

Note that this integral converges when $p > 2k + 2$ and diverges when $p \leqslant 2k + 2$ and this proves the Proposition 1. □

3. Proofs of the main results

Proof of the Theorem 1.1. The upper bound for γ follows from the main Theorem 1.1 of paper [3]. Consider the following subset $\Omega(a_{11})$ in $\mathbb{R}^{N-1}$:

$$|a_{12}| + |a_{13}| + \dots + |a_{1k}| < c_1 a_{11}, \quad -\frac{1}{2} < \frac{b_1}{a_{11}} < -\frac{1}{4}, \quad |a_{lj} - \frac{a_{1l}a_{1j}}{a_{11}}| \leqslant c_2, \quad |b_l - \frac{2b_l a_{1l}}{a_{11}}| \leqslant c_2,$$

where $l = 2, \dots, n$ and c_1, c_2 are sufficiently small fixed positive numbers and $a_{11} > 1$.

Lemma 3.1. *There is a positive number c such that, the following equality holds:*

$$\mu(\Omega(a_{11})) = c \cdot a_{11}^k,$$

for the Lebesgue measure of μ of the set $\Omega(a_{11})$.

Proof. Consider the following maps:

$$\xi_{1l}(A, b_1, \dots, b_k) = a_{1l},$$

$$\xi^1(A, b_1, \dots, b_k) = b_1,$$

$$\xi^l(A, b_1, \dots, b_k) = b_l - \frac{2b_1 a_{1l}}{a_{11}},$$

$$\xi_{lj}(A, b_1, \dots, b_k) = a_{lj} - \frac{a_{1l}a_{1j}}{a_{11}},$$

$$j \leqslant l, \quad j, l = 2, 3, \dots, k.$$

Jacobian of this map is equal to ± 1.

Denote by $\Omega(\xi_{11})$ the image of the map. Since the Jacobian is ± 1, then we have

$$\mu(\Omega(a_{11})) = \mu(\Omega(\xi_{11})).$$

It is easy to verify that for the set $\Omega(\xi_{11})$ with

$$\mid \xi_{12} \mid + \mid \xi_{13} \mid + \dots + \mid \xi_{1k} \mid < c_1 \cdot a_{11},$$

$$-\frac{1}{2} < \frac{\xi^1}{\xi_{11}} < -\frac{1}{4},$$

$$\mid \xi^l \mid \leqslant c_2, \quad \mid \xi_{lj} \mid \leqslant c_2, \quad j \leqslant l, \quad j, l = 2, 3, \dots, k$$

we have

$$\mu(\Omega(\xi_{11}) = c \cdot \xi_{11}^k = c \cdot a_{11}^k.$$

Hence,

$$\mu(\Omega(a_{11})) = c \cdot a_{11}^k.$$

□

Lemma 3.2. *There exists a positive number L such that when $a_{11} > L$ and $(A, b) \in \Omega(a_{11})$ for the integral $T(A, b)$ the following asymptotic equality holds*

$$T(A, b) = \frac{c(A, b)}{a_{11}^{\frac{1}{2}}} + O\left(\frac{1}{a_{11}}\right) \quad \text{as} \quad a_{11} \to +\infty.$$

Moreover, there exists a positive number δ such that for any $(A, b) \in \Omega(a_{11})$, the following inequality holds:

$$|c(A, b)| > \delta.$$

Proof. Lemma 3.2 is proved by the method of stationary phases. Note that for the sufficiently small c_1, c_2 and for the sufficiently large L, the phase has oscillation only in the x_1 direction on the set $(A, b) \in \Omega(a_{11})$. Consequently, for fixed values of $x_2, \dots, x_n \in [0, 1]$, the non-degenerated critical point $x_1(A, b, x_2, \dots, x_n)$ lies in $(0, 1)$. □

Finally, for integral θ we have the following lower bound:

$$\theta \geqslant \int_L^\infty \int_{\Omega(a_{11})} |T(A, b)|^p db\, da \geqslant \delta c \int_L^\infty a_{11}^{k - \frac{p}{2}} da_{11}.$$

Thus, when $p \leqslant 2k + 2$ the last integral diverges, which proves the Theorem 1.1. □

Proof of the Theorem 1.2. We use the classical Young inequality.

Let $f\in L_p(\mathbb{R}^k)$ and $g\in L_r(\mathbb{R}^k)$ be arbitrary functions. The following inequality holds:

$$\|f*g\|_{L_q} \leqslant \|f\|_{L_p}\|g\|_{L_r},$$

where $f*g$ is a convolution of the functions f and g. Moreover, constants $1\leqslant p, q, r\leqslant\infty$ are related by

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}.$$

Let Q be a compact polyhedron in $\mathbb{R}^k$ and

$$h(b)=\int_{\mathbb{R}^k} e^{|x|^2}\chi_Q(x)e^{-2\pi i(b,x)}dx.$$

Lemma 3.3. *The following relation $h\in L_{1+0}(\mathbb{R}^k)$ holds true, where $L_{1+0}(\mathbb{R}^k) := \cap_{p>1}L_p(\mathbb{R}^k)$.*

Proof. Note that, for any $\varepsilon > 0$, $\hat{\chi}_Q \in L_{1+\varepsilon}(\mathbb{R}^k)$ (see. [4]). Then the statement of Lemma 2.1 easily follows from the Young's inequality.

Now let us return to the proof of Theorem 1.2. According to the Plancherel theorem we have:

$$T(A)=\int_Q e^{i(Ax,x)}dx=\int_{\mathbb{R}^k} e^{i(Ax,x)}\chi_Q(x)dx=\int_{\mathbb{R}^k} e^{i(Ax,x)-|x|^2}e^{|x|^2}\chi_Q(x)dx=\int_{\mathbb{R}^k}\widehat{f}(A,b)\overline{g}(b)db,$$

where $\widehat{f}(A,b)=\int\limits_{\mathbb{R}^k} e^{i(Ax,x)-|x|^2-2\pi i(x,b)}dx$ and $\widehat{g}(b)=\int\limits_{\mathbb{R}^k} e^{|x|^2}e^{-2\pi i(x,b)}dx$.

Let $q > 1$ be a fixed number. Then, using the Hölder inequality, we have:

$$|T(A)| \leqslant \|\widehat{f}(A,\cdot)\|_{L_{q'}(\mathbb{R}^k)}\|g\|_{L_q(\mathbb{R}^k)},$$

where $\dfrac{1}{q}+\dfrac{1}{q'}=1$.

According to Lemma 2.1 , we have

$$|T(A)| \leqslant \frac{c_q}{(det(I+A^2))^{\frac{p}{4}-\frac{1}{2q'}}}.$$

Thus, if $p > 2k$, then we can choose $q' > 1$ such that $\dfrac{p}{4}-\dfrac{1}{2q'} > \dfrac{k}{2}$. It follows that if $\dfrac{p}{4}-\dfrac{1}{2q'} > \dfrac{k}{2}$, then $T \in L_p(\mathbb{R}^k)$.

It remains to prove the sharpness of the result. Consider the following subset $\Omega^+(a_{11})$ in $\mathbb{R}^{N-1}$, where $N=\dfrac{k(k+1)}{2}$.

$$a_{11} > 0, |a_{12}|+|a_{13}|+\cdots+|a_{1k}| < c_1a_{11}, \quad \left|a_{lj}-\frac{a_{1l}a_{1j}}{a_{11}}\right| \leqslant c_2, a_{1l} < 0$$

where $l \leqslant j=\overline{2,n}$, $l=2,\dots,n$ and c_1, c_2 are sufficiently small fixed positive numbers.

According to the Lemma 3.1 there exist positive numbers c_1 and c_2 such that the following equality holds for the Lebesgue measure of $\Omega^+(a_{11})$:

$$\mu(\Omega^+(a_{11}))=c\cdot a_{11}^{k-1}. \qquad \square$$

Lemma 3.4. *There exists a positive number L such that when $a_{11} > L$ and $(A, b) \in \Omega(a_{11})$ for the integral $T(A)$ the following asymptotic equality holds*

$$T(A) = \frac{c(A)}{a_{11}^{\frac{1}{2}}} + O\left(\frac{1}{a_{11}}\right) \quad as \quad a_{11} \to +\infty.$$

Moreover, there exists a positive number δ such that for any $(A, b) \in \Omega^+(a_{11})$ the inequality

$$|c(A)| > \delta > 0$$

holds true.

Lemma 3.4 is proved by the method of stationary phases. Note that if $\delta_2 > 0$ and $\delta_1 < 0$ are fixed numbers then the following relation holds true

$$\int\limits_{\delta_1\sqrt{\lambda}}^{\delta_2\sqrt{\lambda}} \cos y^2 dy = c(\delta_1, \delta_2, \lambda)$$

and there exist λ_0, $\varepsilon > 0$ such that the inequality $c(\delta_1, \delta_2, \lambda) \geqslant \varepsilon > 0$ holds for all $\lambda \geqslant \lambda_0$.

Indeed, we have the following relation

$$\lim_{\lambda \to +\infty} \int\limits_{\delta_1\sqrt{\lambda}}^{\delta_2\sqrt{\lambda}} \cos y^2 dy = \frac{\sqrt{2\pi}}{2}.$$

Note that, for sufficiently small c_1, c_2 at $A \in \Omega^+(a_{11})$ and for sufficiently large L, the phase has oscillations only in the x_1 direction. Also, for fixed values $x_2, \ldots, x_n \in [0, 1]$, the nondegenerate critical point $x_1(A, b, x_2, \ldots, x_n)$ lies inside $(0, 1)$.

Finally, for the integral θ, we have the following lower bound:

$$\theta \geqslant \int_L^\infty \int_{\Omega(a_{11})} |T(A)|^p da \geqslant \delta c \int_L^\infty a_{11}^{k - \frac{p}{2} - 1} da_{11}.$$

Thus, the last integral diverges, whenever $p \leqslant 2k$. The Theorem 1.2 is proved. □

4. Two-dimensional case

Note that in the homogeneous case the results of [3] are not applicable. The proof of Theorem 1.2 essentially uses the property $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^k)$.

In Lebedev's paper, it is given an example of the domain $\partial D \in C^{1,\omega}$, where ω is the continuity module of the gradient φ that locally defines ∂D, such that $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^k)$. Therefore, we can assume that D is a compact domain with sufficiently smooth boundary.

The following is true

Theorem 4.1. *Let D be a compact domain such that $\widehat{\chi}_D \in L_q(\mathbb{R}^2)$ and $T(A) = \int\limits_D e^{i(Ax,x)} dx$. Then $T \in Lp(\mathbb{R}^3)$ for $p > 6 - \dfrac{2}{q}$. Moreover, if $\widehat{\chi}_D \in L_{1+0}(\mathbb{R}^2)$, then for any $p > 4$, the inclusion $T \in L_p(\mathbb{R}^3)$ is valid.*

Remark 3. From the results given in [4] it follows that there exists a domain D other than a polygon such that $\widehat{\chi}_Q \in L_{1+0}(\mathbb{R}^2)$.

Corollary 1. If $D \subset \mathbb{R}^2$ is a compact set such that $\partial D \subset C^1$, then for $p > 4.5$ the relation $T \in L_p(\mathbb{R}^3)$ holds.

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О показателях сходимости особого интеграла проблемы Терри для квадратичного многочлена

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Аннотация. В статье рассматривается проблема суммируемости для тригонометрических интегралов с квадратичной фазой. Аналогичная задача рассмотрена в работах [7–9] в частных случаях. Наши результаты обобщают результаты этих работ на кратные тригонометрические интегралы.

Ключевые слова: тригонометрический интеграл, экспонент, сумма, фаза, многочлен.

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On a Note on Apéry-like Series with an Application

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Abstract. The goal of this note is to use a hypergeometric series strategy to build many Apéry-like series. As an application, we obtain several results due to Sherman.

Keywords: Apéry-like series, factorials, hypergeometric function, summation formulas, Gauss summation theorem, contiguous results, binomial coefficients, combinatorial sums.

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1. Introduction and preliminaries

The following standard notations will be used throughout the paper:

$$\mathbb{N} := \{1, 2, 3, \dots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [12, p. 73, Eqn.(2)]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} . \frac{z^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the well-known Pochhammer's symbol (or the shifted or the raised factorial since $(1)_n = n!$) defined for any complex number $a(\neq 0)$ by

$$(a)_n = \begin{cases} a(a+1)\dots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} . \tag{2}$$

Using the fundamental relation $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3}$$

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where Γ is the well known Gamma function.

For more details about this function, and its convergence conditions (including absolute convergence), we refer standard texts [12, 14].

It's worth noting that anytime a generalized hypergeometric function reduces to the gamma function, the results are crucial from the standpoint of applications. Thus, classical summation theorems like as those of Gauss, Gauss second, Kummer, and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple, and Saalschütz for the series ${}_3F_2$, and others, are relevant.

During 1992–2011, the classical summation theorems listed above have been extended and generalised to their most general form. For this we refer interesting research papers by Lavoie et al. [6–8], Kim et al. [5] and Rakha and Rathie [13].

The following summation formula for the series ${}_2F_1$ which can be obtained from a very general summation formula established earlier by Rakha and Rathie [13, Theorem 2 (for $i = 2$), p. 828] is required in our current inquiry.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b-1) \end{matrix}; \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}-\frac{1}{2}\right)\left[\frac{\frac{1}{2}(a+b-1)}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})}+\frac{2}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\right]. \tag{4} $$

The result (4) is seen to be closely related to the following well-known and useful Gauss's second summation theorem [12, p. 69, Ex. 2; 14, p. 243, Eqn. (III.6)] viz.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})}. \tag{5} $$

On the other hand, in 1979, Apéry [1] proved irrationality of $\zeta(3)$ and in the same manner, the irrationality of $\zeta(2)$ by making use of the following well-known identities viz,

$$ \zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n!)^2}{(2n)!\; n^3} $$

and

$$ \zeta(2) = 3\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!\; n^2}. $$

Also, following Apéry's proof, a large number of a similar series

$$ \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}f(n) = \sum_{n=0}^{\infty}\frac{f(n)}{\binom{2n}{n}} $$

which was commonly referred to as Apéry-like series have been studied by van der Poortan [11], Leschiner [10], Lehmer [9], Zucker [16] and Borwein et al. [3]. Berndt and Joshi [2], in a review of chapter 9 of Ramanujan's second notebook have also recorded many of such similar formulas. In addition to this, if we denote

$$ S_k = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}n^k\; 2^n, \tag{6} $$

then in the year 2000, Sherman [15] establishedthe results S_k for $k = 0, 1, 2, \ldots, 10$ given here in the Tab. 1.

On the other hand, it is not difficult to see that

$$ n^k = \sum_{r=0}^{k}(-1)^r\left\{\begin{matrix} k \\ r \end{matrix}\right\}(-n)_r $$

Table 1. For S_k

k	0	1	2	3	4	5	6
S_k	$\frac{\pi}{2}+2$	$\pi+3$	$\frac{7\pi}{2}+11$	$\frac{35\pi}{2}+55$	$113\pi+355$	$\frac{1787\pi}{2}+2807$	$\frac{16717\pi}{2}+26259$

k	7	8	9	10
S_k	$90280\pi+283623$	$\frac{2211181\pi}{2}+34733315$	$\frac{30273047\pi}{2}+47552791$	$229093376\pi+719718067$

where $\left\{ {k \atop r} \right\}$ denotes the well-known Sterling numbers of the second kind [4] written here in slightly modified form as:

$$\left\{ {k \atop r} \right\} = \frac{1}{r!}\sum_{i=0}^{r}(-1)^r \binom{r}{i}(r-i)^k.$$

Thus this note aims to offer closed expressions for Apéry-like series of the form

$$\sum_{n=k}^{\infty} \frac{(n!)^3 2^n}{(2n)!(n-k)!}$$

for $k = 1, 2, \ldots, 10$ via a hypergeometric series approach. As an application, we recover the above results of Apéry-like series obtained earlier by Sherman [15].

2. Main results

In this section, we shall establish the results asserted in the following theorem.

Theorem 2.1. *For $k \in \mathbb{N}_0$, the following general result holds true.*

$$\sum_{n=k}^{\infty} \frac{(n!)^3 2^n}{(2n)!(n-k)!} = 2^{-k}\pi\Gamma^2(k+1)\left[\frac{k+\frac{1}{2}}{\Gamma^2(\frac{1}{2}k+1)} + \frac{2}{\Gamma^2(\frac{1}{2}k+\frac{1}{2})}\right]. \tag{7}$$

Proof. In order to establish the result (7) asserted in the Theorem 2.1, we proceed as follows. Denoting the left hand side of (7) by Δ_k, we have

$$\Delta_k = \sum_{n=k}^{\infty} \frac{(n!)^3\ 2^n}{(2n)!(n-k)!}.$$

Replacing n by n+k, we have

$$\Delta_k = \sum_{n=0}^{\infty} \frac{((n+k)!)^3\ 2^{n+k}}{(2n+2k)!\ n!}.$$

But it is easy to see that $(n+k)! = \Gamma(n+k+1) = \Gamma(k+1)\dfrac{\Gamma(n+k+1)}{\Gamma(k+1)} = \Gamma(k+1)(k+1)_n$ (using (3)) and using Duplication formula for the gamma function

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\ \Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we see that

$$\begin{aligned}(2n+2k)! &= \Gamma(2n+2k+1) = \\ &= 2^{2n+2k}\pi^{-\frac{1}{2}}\Gamma\Big(n+k+\frac{1}{2}\Big)\Gamma(n+k+1) = \\ &= 2^{2n+2k}\pi^{-\frac{1}{2}}\Gamma\Big(k+\frac{1}{2}\Big)\Gamma(k+1)\Big(k+\frac{1}{2}\Big)_n(k+1)_n \quad (using(3)).\end{aligned}$$

Thus we have after some algebra

$$\Delta_k = \frac{2^{-k}\pi^{\frac{1}{2}}\Gamma^2(k+1)}{\Gamma(k+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(k+1)_n(k+1)_n}{2^n(k+\frac{1}{2})_n\; n!}.$$

Summing up the series using (1), we have

$$\Delta_k = \frac{2^{-k}\pi^{\frac{1}{2}}\Gamma^2(k+1)}{\Gamma(k+\frac{1}{2})}\;{}_2F_1\left[\begin{matrix} k+1, & k+1 \\ & k+\frac{1}{2}\end{matrix};\frac{1}{2}\right].$$

We now observe that the ${}_2F_1$ appearing can be evaluated with the help of the result (4) by letting $a=b=k+1$, and we easily arrive at the right hand side of (7). This completes the proof of the result (7) asserted in the Theorem 2.1. □

3. Corollaries

In this section, we shall provide several interesting special cases of our main result asserted in the Theorem 2.1 since

$$\Delta_k = \sum_{n=k}^{\infty}\frac{(n!)^3 2^n}{(2n)!(n-k)!} = \sum_{n=k}^{\infty}\frac{(n!)^2 2^n}{(2n)!}(n-k+1)_k. \tag{8}$$

Fortunately, the results Δ_k for k= 0 and 1, we get the same results $\Delta_0 = S_0$ and $\Delta_1 = S_1$ due to Sherma [15] recorded in Section 1. The results Δ_k for k = 2 to 10 are recorded in the Tab. 2.

Table 2. For Δ_k

k	2	3	4	5	6
Δ_k	$\frac{5\pi}{2}+8$	$9\pi+28$	$\frac{81\pi}{2}+128$	$225\pi+704$	$\frac{2925\pi}{2}+4608$

k	7	8	9	10
Δ_k	$11025\pi+34560$	$\frac{187425\pi}{2}+294912$	$893025\pi+2801664$	$\frac{18753525\pi}{2}+29491200$

Application of these results will be given in the next section.

4. Application

As an application of our newly obtained results given in section 3, in this section, we shall obtain the results given in the table S_k.

(a) Derivation of the result S_k for $k=0$.

Denoting the left-hand side of the series given in (6) for $k = 0$ by S_0 and converting the factorials into the Pochhammer symbols, we have

$$S_0 = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(\frac{1}{2})_n 2^n n!}.$$

Summing up the series we have

$$S_0 = {}_2F_1 \left[\begin{matrix} 1, & 1 \\ & \frac{1}{2} \end{matrix} ; \frac{1}{2} \right].$$

This may be evaluated using the result (4) by letting $a = b = 1$, and we get the right-hand side of (6) for $k = 0$ right away.

(b) Derivation of the result S_k for $k = 1$.

Denoting the left-hand side of the series given in (6) for $k = 1$ by S_1, we have

$$S_1 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n 2^n.$$

Setting $n = m + 1$ and proceeding as above, we have

$$S_1 = {}_2F_1 \left[\begin{matrix} 2, & 2 \\ & \frac{3}{2} \end{matrix} ; \frac{1}{2} \right].$$

The result follows by using the result (4) by letting $a = b = 2$

(c) Derivation of the result S_k for $k = 2$.

Denoting the left-hand side of the series given in (6) for $k = 2$ by S_2, we have

$$S_2 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n^2 \, 2^n.$$

Expressing $n^2 = n(n-1) + n$ and separating into two series, we get

$$S_2 = \sum_{n=2}^{\infty} \frac{(n!)^2}{(2n)!} (n-1)_2 \, 2^n + \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n \, 2^n.$$

Finally, using the result given in the table Δ_k for $k = 2$ and S_k for $k = 1$, we get at once the right-hand side of S_k for $k = 2$.

In exactly the same manner, the results S_k for $k = 3, 4, \ldots, 10$ can be proven on similar lines by using the result (7) for $k = 3, 4, \ldots, 10$ and taking appropriate results of Δ_k given in the tabular form in Section 3 together with the result S_2 given in the tabular form in Section 2. So we left as an exercise to the interested reader.

5. Hypergeometric series representations of the result given in the equations (7) and (8)

It is interesting to mention here that the results given in the equations (7) and (8) can also be written in terms of generalized hypergeometric series. These are

$$ {}_2F_1 \left[\begin{matrix} 1, & 1 \\ & \frac{1}{2} \end{matrix} ; \frac{1}{2} \right] = \frac{\pi}{2} + 2 \tag{9}$$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ & \frac{3}{2} \end{matrix} ; \frac{1}{2}\right] = \pi + 3 \tag{10}$$

$$ {}_3F_2\left[\begin{matrix} 2, 2, 2 \\ \frac{3}{2}, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{7\pi}{2} + 11 \tag{11}$$

$$ {}_4F_3\left[\begin{matrix} 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{35\pi}{2} + 55 \tag{12}$$

$$ {}_5F_4\left[\begin{matrix} 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = 113\pi + 355 \tag{13}$$

$$ {}_6F_5\left[\begin{matrix} 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{1787\pi}{2} + 2807 \tag{14}$$

$$ {}_7F_6\left[\begin{matrix} 2, 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{16717\pi}{2} + 26259 \tag{15}$$

$$ {}_8F_7\left[\begin{matrix} 2, 2, 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = 90280\pi + 283623 \tag{16}$$

$$ {}_9F_8\left[\begin{matrix} 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{2211181\pi}{2} + 34733315 \tag{17}$$

$$ {}_{10}F_9\left[\begin{matrix} 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = \frac{30273047\pi}{2} + 47552791 \tag{18}$$

$$ {}_{11}F_{10}\left[\begin{matrix} 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \\ \frac{3}{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1 \end{matrix} ; \frac{1}{2}\right] = 229093376\pi + 719718067 \tag{19}$$

$$ {}_2F_1\left[\begin{matrix} 3, & 3 \\ & \frac{5}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{3}{4}\left(\frac{5\pi}{2} + 8\right) \tag{20}$$

$$ {}_2F_1\left[\begin{matrix} 4, & 4 \\ & \frac{7}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{5}{12}(9\pi + 28) \tag{21}$$

$$ {}_2F_1\left[\begin{matrix} 5, & 5 \\ & \frac{9}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{32}{192}\left(\frac{81\pi}{2} + 128\right) \tag{22}$$

$$ {}_2F_1\left[\begin{matrix} 6, & 6 \\ & \frac{11}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{21}{320}(225\pi + 704) \tag{23}$$

$$ {}_2F_1\left[\begin{matrix} 7, & 7 \\ & \frac{13}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{77}{3840}\left(\frac{2925\pi}{2} + 4608\right) \tag{24}$$

$$ {}_2F_1\left[\begin{matrix} 8, & 8 \\ & \frac{15}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{143}{26880}(11025\pi + 34560) \tag{25}$$

$$ {}_2F_1\left[\begin{matrix} 9, & 9 \\ & \frac{17}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{45045}{896}\left(\frac{187425\pi}{2} + 294912\right) \tag{26}$$

$$ {}_2F_1\left[\begin{matrix} 10, & 10 \\ & \frac{19}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{2431}{9289728}(893025\pi + 2801664) \tag{27}$$

$$ {}_2F_1\left[\begin{matrix} 11, & 11 \\ & \frac{21}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{46189}{928972800}\left(\frac{18753525\pi}{2} + 29491200\right) \tag{28}$$

Concluding remark

In this note we have established the closed expressions for the Apéry-like series of the form

$$\sum_{n=k}^{\infty} \frac{(n!)^3\ 2^n}{(2n)!(n-k)!} \tag{*}$$

for $k = 1, 2, \ldots, 10$ via a hypergeometric series approach. As an application, we obtained the Apéry-like series of the form

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^k\ 2^n}{(2n)!} \tag{**}$$

for $k = 1, 2, \ldots, 10$ established earlier by Sherman [15].

We conclude this note by remarking that the results (*) and (**) in the most general forms for $k \in \mathbb{N}_0$ are under investigations and will form a part of the subsequent paper in this direction.

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Заметка об Апери-подобном ряде с приложением

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Аннотация. Цель этой заметки — использовать стратегию гипергеометрических рядов для построения многих рядов, подобных Апери. В качестве приложения мы получаем несколько результатов, принадлежащих Шерману.

Ключевые слова: Апери-подобные ряды, факториалы, гипергеометрическая функция, формулы суммирования, теорема суммирования Гаусса, смежные результаты, биномиальные коэффициенты, комбинаторные суммы.

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Some New Fixed Point Results in b-metric Space with Rational Generalized Contractive Condition

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Abstract. In this paper, we improve and generalize several results in fixed point theory to the b-metric space. Where we confirm the existence of the fixed point for self mapping T satisfying some rational contractive conditions. Over-more, we establish the uniqueness of the fixed point in some cases and give dynamic information linking the fixed points between them in the other cases. Some illustrative examples are furnished, which demonstrate the validity of the hypotheses.

Keywords: metric space, b-metric space, Picard sequence, Fixed point, Rational contraction mapping.

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Introduction

The fixed point theory is an exceptional combination of analysis (pure and applied), topology and geometry. This theory stems from purely mathematical thought, and herein lies the difficulty of developing and expanding this field. On the other hand, we find that the application of these theorems as tool to study of non-linear natural phenomena gave amazing results that match reality in various fields that include biology, chemistry, economics, engineering, game theory and physics, which increased its aesthetic and importance, for more detait we refer reader to [10]. Despite the difficulty of purely mathematical study, the fixed point theory developed rapidly because of its applications in diverse fields, especially after the emergence of Banach's contraction [6], which is a basic result on fixed points for contraction type mappings, it was introduced by great Polish mathematician Stefan Banach in 1922. It has been generalized in various directions. These generalizations are made either by using contractive conditions or

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by imposing some additional conditions on the ambient spaces for more detail see references [3–5, 13, 19, 20, 22, 27].

There exist various generalizations of usual metric spaces. One of them is b-metric space or metric-type space. This concept was first introduced by Bakhtin [5].

The b-metric space has been studied topologically in many works, including: such as S. Czerwik [9], N. Bourbaki [8] which confirmed the fundamental difference between it and the metric space, for example the b-metric is not necessarily continuous unlike the metric distance.

In 1993, Czerwik [9] extended the results of metric spaces that generalized the famous Banach contraction principle for b-metric space. Later, several authors extended the fixed point theorem in b-metric space. For fixed point results and more examples in b-metric spaces, the readers may refer to [1, 2, 7, 9, 11–18, 21–26]. The aim of this paper is to present some fixed point results for mappings satisfying generalized contractive condition in a b-metric space.

1. Preliminary

In this section, we look back on some famous notions and definition of the b-metric spaces which will be used in the sequel.

Definition 1 ([9]). *Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold:*
(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.
The triple (X, d, s) is called a b-metric space with constant $s \geqslant 1$.

Remark 1. *It is obvious from the above definition that the class of b-metric spaces is larger than that of metric spaces, since a b-metric space is a metric space when $s = 1$ but the converse is not true.*

Remark 2. *In general, the b-metric is not usually continuous (see example 4 in [19]).*

Definition 2 ([21]). *Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called*
(a) convergent if and only if there exists $x \in X$ such that $\lim\limits_{n\to+\infty} d(x_n, x) = 0$ and in this case we write $\lim\limits_{n\to+\infty} x_n = x$;
(b) Cauchy if and only if $\lim\limits_{n,m\to+\infty} d(x_n, x_m) = 0$.

Before starting, we present the following simple lemma proven by A. Aghajani, M. Abbas and J. R. Roshan [3] which has a fundamental role in proving our results.

Lemma 1 ([3]). *Let (X, d, s) be a b-metric space such that $s \geqslant 1$ and $\{x_n\}$ be a convergent sequence in X to x. Then for each $y \in X$, we have*

$$\frac{1}{s} d(x, y) \leqslant \liminf_{n\to+\infty} d(x_n, y) \leqslant \limsup_{n\to+\infty} d(x_n, y) \leqslant s d(x, y). \tag{1}$$

2. Results

Firstly, we state and prove our first theorem that generalize and improve the result of Khojasteh et al [20].

Theorem 1. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number $a, b, c, f,\ e \in \mathbb{R}^+$ such that $s^2a \leqslant \min\{c, f\}$ or $s^2b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} d(x, y). \tag{2}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2a \leqslant \min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (2), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} d(x_{n-1}, x_n). \end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.

Since $s^2a \leqslant \min\{c, f\}$, then $0 \leqslant \theta_n < \dfrac{1}{s}$ for all $n \in \mathbb{N}$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$,

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0. \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \theta_n d(x_{n-1}, x_n) \leqslant \\ &\leqslant \theta_n\theta_{n-1} d(x_{n-2}, x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \theta_n\theta_{n-1}\cdots\theta_1 d(x_0, x_1) \leqslant \\ &\leqslant \theta_1^n d(x_0, x_1). \end{aligned}$$

Now for all n, $m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1}\theta_1^i d(x_0, x_1) \leqslant \\ &\leqslant \frac{(s\theta_1)^n - (s\theta_1)^m}{1 - s\theta_1} \times \frac{1}{s^{n-1}} d(x_0, x_1). \end{aligned}$$

By passing to the limits $n, m \to +\infty$ on a both side of previous inequality we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \tag{3}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = x_n$ and $y = x_{n-1}$ in inequality (2), we find

$$d(x_n, x_{n+1}) \leqslant \frac{bd(x_{n-1}, x_{n+1})}{cd(x_n, x_{n+1}) + fd(x_{n-1}, x_n) + e} d(x_{n-1}, x_n).$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X, d, s) is complete, there exit $\dot{x} \in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
By putting $x = \dot{x}$, $y = x_n$ in inequality (2), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(\dot{x}, x_{n+1}) + bd(x_n, T\dot{x})}{cd(\dot{x}, T\dot{x}) + fd(x_n, x_{n+1}) + e} d(\dot{x}, x_n). \tag{4}$$

By taking limit on both sides of (4), we have $\lim_{n\to+\infty} x_n = T\dot{x}$. Because of the uniqueness of the limit, we find $T\dot{x} = \dot{x}$.
Step 3: Suppose that T have two distinct fixed points $\dot{x}, \dot{y}$ in X and we find the distance between them.
By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (2), we find,

$$d(\dot{x}, \dot{y}) \leqslant \frac{ad(\dot{x}, \dot{y}) + bd(\dot{y}, \dot{x})}{cd(\dot{x}, \dot{x}) + fd(\dot{y}, \dot{y}) + e} d(\dot{x}, \dot{y}) \leqslant \frac{(a+b)d(\dot{x}, \dot{y})}{e} d(\dot{x}, \dot{y}).$$

Then $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$. This complete the proof of the theorem. □

Remark 3. *If we take $a = b = c = f = e = 1$ and $s = 1$ in Theorem 1, we returne to results of Khojasteh et al [20].*

The following example support our Theorem 1.

Example 1. *Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = (x-y)^2$ for all $x, y \in X$. $(X, d, 2)$ is a complete b-metric space. Let $T : X \to X$ be a self mapping given by $T(0) = 0$, $T(1) = 0$ and $T(2) = 2$.*
If $x = y$, the equation is obviously verified. Now, we treat the other cases.
If $x = 0$ and $y = 2$,

$$d(0, 2) \leqslant (d(0, 2) + 5d(0, 2))d(0, 2).$$

If $x = 1$ and $y = 2$,

$$d(0, 2) \leqslant \frac{d(1, 2) + 5d(0, 2)}{4d(1, 0) + 1} d(1, 0).$$

If $x = 2$ and $y = 0$,

$$d(0, 2) \leqslant (d(0, 2) + 5d(0, 2))d(0, 2).$$

If $x = 2$ and $y = 1$,

$$d(0, 2) \leqslant \frac{d(0, 2) + 5d(1, 2)}{4d(1, 0) + 1} d(1, 0).$$

That mean that the equation (2) is verified with constants $a = 1$, $b = 5$, $c = 4$, $e = 1$ and $f = 4$. Over more, all conditions of Theorem 1 was satisfied, then T has at least one fixed point in X. We remark that T has exactly two fixed point $0, 2$, over more, $d(0, 2) \geqslant \dfrac{1}{6}$.

If we take $s = 1$ in Theorem 1, we get the following corollary.

Corollary 1. *Let (X, d) be a complete metric space and let T be a self mapping in X. If there exist five positive real number $a, b, c, f,\ e \in \mathbb{R}^+$ such that $a \leqslant \min\{c, f\}$ or $b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} d(x, y). \tag{5}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then, $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$.*

This example illustrates and supports Theorem 1 and Corollary 1.

Example 2. *Let $X = \{0, 1, 2\}$ associated with a metric d such that $d(0, 1) = 0.75$, $d(0, 2) = 1$ and $d(1, 2) = 0.25$. Also, $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, x) = 0$ for all $x \in X$.*

Let T be a self mapping in X such that $T(0) = 2$, $T(1) = 1$ and $T(2) = 2$.

It is easy to conclude that (X, d) is a complete metric space and the inequality (5) was verified for all $x, y \in X$ with constant $a = b = \dfrac{1}{2}$, $c = f = 1$ and $e = \dfrac{1}{4}$. According to Corollary 1, we conclude that T has at least one fixed point. (exactly, it has two fixed point 1 and 2). Moreover, the distance between them is $d(1, 2) \geqslant \dfrac{1}{4}$.

Remark 4. *It should be noted that Khojasteh et al theorem [20] is not applicable in this example while the generalized Corollary 1 is applicable as shown in the example above, which proves the robustness of our results.*

Secondly, we state and prove our second theorem that generalize and improve the result of A. C. Aouine and A. Aliouche [4].

Theorem 2. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number $a, b, c, f,\ e \in \mathbb{R}^+$ such that $s^2a \leqslant \min\{c, f\}$ or $s^2b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} \max\{d(x, Tx), d(y, Ty)\}. \tag{6}$$

Then T has a unique fixed point $\dot{x} \in X$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2a \leqslant \min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (6), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} d(x_{n-1}, x_n). \end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.
Since $s^2 a \leqslant \min\{c, f\}$, then $0 \leqslant \theta_n < \dfrac{1}{s}$ for all $n \in \mathbb{N}$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$,

$$\begin{aligned}\theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d(x_n, x_{n+1}) &\leqslant \theta_n d(x_{n-1}, x_n) \leqslant \\ &\leqslant \theta_n \theta_{n-1} d(x_{n-2}, x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \theta_n \theta_{n-1} \cdots \theta_1 d(x_0, x_1) \leqslant \\ &\leqslant \theta_1^n d(x_0, x_1).\end{aligned}$$

Now, for all n, $m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned}d(x_n, x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} \theta_1^i d(x_0, x_1) \leqslant \\ &\leqslant \frac{(s\theta_1)^n - (s\theta_1)^m}{1 - s\theta_1} \times \frac{1}{s^{n-1}} d(x_0, x_1).\end{aligned}$$

By passing to the limits $n, m \to +\infty$ on a both side of previous inequality, we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \tag{7}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = x_n$ and $y = x_{n-1}$ in inequality (6), we find

$$\begin{aligned}d(x_n, x_{n+1}) &\leqslant \frac{bd(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{cd(x_n, x_{n+1}) + fd(x_{n-1}, x_n) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{\min\{c, f\}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{\min\{c, f\}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + e} d(x_{n-1}, x_n).\end{aligned}$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X, d, s) is complete, there exit $\dot{x} \in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
Suppose that $d(\dot{x}, T\dot{x}) > 0$

Case 1. If $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_n$, $y = \dot{x}$ in inequality (6), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(x_n, T\dot{x}) + bd(x_{n+1}, \dot{x})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e} \max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \tag{8}$$

On the other hand, we have

$$\begin{aligned} d(\dot{x}, T\dot{x}) &\leqslant sd(\dot{x}, x_{n+1}) + sd(x_{n+1}, T\dot{x}) \leqslant \\ &\leqslant sd(\dot{x}, x_{n+1}) + s\frac{ad(x_n, T\dot{x}) + bd(x_{n+1}, \dot{x})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e} \max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \end{aligned} \tag{9}$$

By taking limit superior on both sides of (9), we have

$$d(\dot{x}, T\dot{x}) \leqslant \frac{sa \limsup_{n\to+\infty} d(x_n, T\dot{x})}{fd(\dot{x}, T\dot{x}) + e} d(\dot{x}, T\dot{x}). \tag{10}$$

According to Lemma 1, we get

$$d(\dot{x}, T\dot{x}) \leqslant \frac{s^2 a}{fd(\dot{x}, T\dot{x}) + e} d(\dot{x}, T\dot{x})^2, \tag{11}$$

then

$$1 \leqslant \frac{s^2 a}{fd(\dot{x}, T\dot{x}) + e} d(\dot{x}, T\dot{x}). \tag{12}$$

Since, $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, then $s^2 a \leqslant f$, then $s^2 ad(\dot{x}, T\dot{x}) < fd(\dot{x}, T\dot{x}) + e$ which contradict inequality (12). Then $d(\dot{x}, T\dot{x}) = 0$ that mean $T\dot{x} = \dot{x}$.

Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = \dot{x}$, $y = x_n$ in inequality (6), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(\dot{x}, x_{n+1}) + bd(x_n, T\dot{x})}{cd(\dot{x}, T\dot{x}) + fd(x_n, x_{n+1}) + e} \max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \tag{13}$$

Similarly, as Case 1, we can deduce that $T\dot{x} = \dot{x}$.

Step 3: Suppose that T have two fixed points $\dot{x}, \dot{y}$ in X. By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (6), we find

$$d(\dot{x}, \dot{y}) \leqslant \frac{ad(\dot{x}, \dot{y}) + bd(\dot{y}, \dot{x})}{cd(\dot{x}, \dot{x}) + fd(\dot{y}, \dot{y}) + e} \max\{d(\dot{x}, \dot{x}), d(\dot{y}, \dot{y})\}. \tag{14}$$

Then $d(\dot{x}, \dot{y}) = 0$, that mean, $\dot{x} = \dot{y}$, and this completes the proof of the theorem. □

Remark 5. *If we take $a = b = c = f = e = 1$ and $s = 1$ in Theorem 2, we returne to results of A. C. Aouine and A. Aliouche [4].*

The following example illustrates and supports our Theorem 2.

Example 3. *Let $X = [0, 4.5]$ and $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$. $(X, d, 2)$ is a complete b-metric space. Let $T : X \to X$ be a self mapping given by*

$$Tx = \begin{cases} 4.5 & \text{if } x \in [0, 2.5[\\ 4 & \text{if } x \in [2.5, 4.5] \end{cases}.$$

Let $x, y \in X$ and denote

$$m(x,y) = -d(Tx,Ty) + \frac{d(x,Ty)+d(y,Tx)}{4d(x,Tx)+4d(y,Ty)+1}\max\{d(x,Tx),d(y,Ty)\}.$$

if $x \in [0,2.5[$ and $y \in [2.5,4.5]$, we draw the curve of the function m over this domain (Fig. 1).

We remark that it is positive, which proves the validity of the inequality (6) for all $x \in [0,2.5[$ and $y \in [2.5,4.5]$. The other cases is trivial.
Therefore, by choosing $a = b = e = 1$ and $c = f = 4$ all conditions of Theorem 2 are satisfied. Hence T has a unique fixed point $\dot{x}$ in X (here $\dot{x} = 4$).

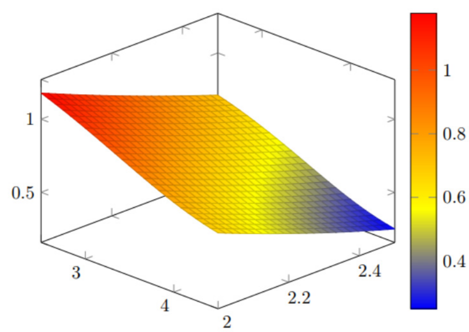

Fig. 1. Curve of the function m

If we take $s = 1$ in Theorem 2, we get the following corollary.

Corollary 2. *Let (X,d) be a complete metric space and let T be a self mapping in X. If there exist five positive real number $a, b, c, f,\ e \in \mathbb{R}^+$ such that $a \leqslant \min\{c,f\}$ or $b \leqslant \min\{c,f\}$ and for all $x, y \in X$*

$$d(Tx,Ty) \leqslant \frac{ad(x,Ty)+bd(y,Tx)}{cd(x,Tx)+fd(y,Ty)+e}\max\{d(x,Tx),d(y,Ty)\}. \tag{15}$$

Then T has a unique fixed point $\dot{x} \in X$.
Every Picard sequence converge to $\dot{x}$.

The following example illustrates and supports Corollary 2 and Theorem 2.

Example 4. *Let $X = \{0,1,2\}$ associated with a metric d such that $d(0,1) = 0.6$, $d(0,2) = 1$ and $d(1,2) = 0.4$. Also $d(x,y) = d(y,x)$ for all $x, y \in X$ and $d(x,x) = 0$ for all $x \in X$.*

Let T be a self mapping in X such that $T(0) = 2$, $T(1) = 1$ and $T(2) = 1$.

It is easy to conclude that (X,d) is a complete metric space and the inequality (15) was verified for all $x, y \in X$ with constant $a = b = c = f = 3$ and $e = \dfrac{1}{4}$. According to Corollary 2, we conclude that T has a unique fixed point. In additional, every Picard sequence converge to $\dot{x}$.

Remark 6. *It should be noted that A.C. Aouine and A. Aliouche. Theorem [2] is not applicable in this example while the generalized Corollary 2 is applicable as shown in the example above, which proves the robustness of our results.*

Third, we can generalize the previous theorems as follow:

Theorem 3. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$ or $s^2 b \leqslant \frac{1}{2}\min\{c, f\}$ and for all $x, y \in X$,*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e}\max\{d(x, Ty), d(y, Tx)\}. \tag{16}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then, $d(\dot{x}, \dot{y}) \geqslant \frac{e}{a+b}$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (16), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_{n+1}) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e}[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})] \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})]. \end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.
Since $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, then $0 \leqslant \theta_n < \frac{1}{2s}$ for all $n \in \mathbb{N}$.
On the other hand we have

$$d(x_n, x_{n+1}) \leqslant \theta_n[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})],$$

then

$$d(x_n, x_{n+1}) \leqslant \frac{\theta_n s}{1 - \theta_n s} d(x_{n-1}, x_n).$$

We denote that $\lambda_n = \dfrac{\theta_n s}{1 - \theta_n s}$ for all $n \in \mathbb{N}$.
Since $0 \leqslant \theta_n < \frac{1}{2s}$ for all $n \in \mathbb{N}$, then $0 \leqslant \lambda_n < 1$.
Then $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, then $d(x_{n+1}, x_{n+2}) < d(x_{n-1}, x_n)$.
Furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0. \end{aligned}$$

Then the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is decreasing, then

$$\begin{aligned} d(x_n,x_{n+1}) &\leqslant \lambda_n d(x_{n-1},x_n) \leqslant \\ &\leqslant \lambda_n\lambda_{n-1} d(x_{n-2},x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \lambda_n\lambda_{n-1}\cdots\lambda_1 d(x_0,x_1) \leqslant \\ &\leqslant \lambda_1^n d(x_0,x_1). \end{aligned}$$

Now for all n, $m\in\mathbb{N}$ such that $m>n$, we have

$$\begin{aligned} d(x_n,x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i,x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1}\lambda_1^i d(x_0,x_1) \leqslant \\ &\leqslant \frac{(s\lambda_1)^n-(s\lambda_1)^m}{1-s\lambda_1}\times\frac{1}{s^{n-1}} d(x_0,x_1). \end{aligned}$$

By passing to the limits $n,m\to+\infty$ on a both side of previous inequality, we get

$$\lim_{n,m\to+\infty} d(x_n,x_m)=0. \tag{17}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Case 2. If $s^2b\leqslant\dfrac{1}{2}\min\{c,f\}$, by putting $x=x_n$ and $y=x_{n-1}$ in inequality (16), we find

$$d(x_n,x_{n+1})\leqslant\frac{bd(x_{n-1},x_{n+1})}{cd(x_n,x_{n+1})+fd(x_{n-1},x_n)+e}d(x_{n-1},x_{n+1}).$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X,d,s) is complete, there exit $\dot{x}\in X$ such that

$$\lim_{n\to+\infty} x_n=\dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
Suppose that $d(\dot{x},T\dot{x})>0$.

Case 1. If $s^2a\leqslant\dfrac{1}{2}\min\{c,f\}$, by putting $x=x_n$, $y=\dot{x}$ in inequality (16), we find

$$d(T\dot{x},x_{n+1})\leqslant\frac{ad(x_n,T\dot{x})+bd(\dot{x},x_{n+1})}{cd(x_n,x_{n+1})+fd(\dot{x},T\dot{x})+e}\max\{d(\dot{x},x_{n+1}),d(x_n,T\dot{x})\}. \tag{18}$$

On the other hand, we have

$$d(\dot{x},T\dot{x}) \leqslant sd(\dot{x},x_{n+1})+sd(x_{n+1},T\dot{x})\leqslant \tag{19}$$

$$\leqslant sd(\dot{x},x_{n+1})+\frac{sad(x_n,T\dot{x})+sbd(\dot{x},x_{n+1})}{cd(x_n,x_{n+1})+fd(\dot{x},T\dot{x})+e}\max\{d(\dot{x},x_{n+1}),d(x_n,T\dot{x})\}. \tag{20}$$

By taking limit superior on both sides of (20), we have

$$d(\dot{x},T\dot{x})\leqslant\frac{sa}{fd(\dot{x},T\dot{x})+e}\Big(\limsup_{n\to+\infty} d(x_n,T\dot{x})\Big)^2. \tag{21}$$

According to Lemma 1, we get

$$d(\dot{x}, T\dot{x}) \leqslant \frac{sa}{fd(\dot{x}, T\dot{x}) + e} d(\dot{x}, T\dot{x})^2, \tag{22}$$

then

$$1 \leqslant \frac{sa}{fd(\dot{x}, T\dot{x}) + e} d(\dot{x}, T\dot{x}). \tag{23}$$

Since, $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, then $s^2 a \leqslant f$, then $s^2 ad(\dot{x}, T\dot{x}) < fd(\dot{x}, T\dot{x}) + e$ which contradict inequality (23). Then $d(\dot{x}, T\dot{x}) = 0$ that mean $T\dot{x} = \dot{x}$.

Case 2. If $s^2 b \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = \dot{x}$ and $y = x_n$ in inequality (16), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(\dot{x}, x_{n+1}) + bd(x_n, T\dot{x})}{cd(\dot{x}, T\dot{x}) + fd(x_n, x_{n+1}) + e} \max\{d(\dot{x}, x_{n+1}), d(x_n, T\dot{x})\}.$$

Similarly, as Case 1 we can deduce that $T\dot{x} = \dot{x}$.

Step 3: Suppose that T have two distinct fixed points $\dot{x}, \dot{y}$ in X and we find the distance between them.
By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (16), we find

$$d(\dot{x}, \dot{y}) \leqslant \frac{ad(\dot{x}, \dot{y}) + bd(\dot{y}, \dot{x})}{cd(\dot{x}, \dot{x}) + fd(\dot{y}, \dot{y}) + e} d(\dot{x}, \dot{y}) \leqslant \frac{(a+b)d(\dot{x}, \dot{y})}{e} d(\dot{x}, \dot{y}).$$

Then, $d(\dot{x}, \dot{y}) \geqslant \frac{e}{a+b}$. □

3. Discussion

- The Corollary 1 generalize the result of Khojasteh et al [1] and the Corollary 2 generalize the result of Aouine and Aliouche [2].
- We note that the choice of constants related to inequalities (2), (5), (6), (15) and (16) directly affects the dynamic result of Theorems 1, 2, and Corollaries 1, 2 and 3 respectively.
- Note that the ratio $\frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e}$ in the inequalities (2), (5), (6), (15) and (16) might be greater or less than 1, thus theorems is an special case of Banach contraction principle. Example 1 illustrates this point precisely.
- If *rangT* is a closed sub set of X, the inequalities (2), (5), (6), (15) and (16) can be restricted to *rangT*, and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.
- The above results can be generalized into several generalized metric spaces as $q_1 - q_2$ b-metric space, partial metric space, ...

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Некоторые новые результаты с фиксированной точкой в b-метрическом пространстве с рациональным обобщенным условием сжатия

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Аннотация. В этой статье мы улучшаем и обобщаем некоторые результаты теории неподвижных точек на b-метрическое пространство. Где мы подтверждаем существование неподвижной точки для самоотображения T, удовлетворяющего некоторым рациональным сжимающим условиям. Более того, мы устанавливаем уникальность фиксированной точки в некоторых случаях и даем динамическую информацию, связывающую неподвижные точки между собой в других случаях. Приведены наглядные примеры, демонстрирующие справедливость гипотез.

Ключевые слова: метрическое пространство, b-метрическое пространство, последовательность Пикара, фиксированная точка, отображение рационального сжатия.

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Strongly Algebraically Closed MV-algebras

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Abstract. The aim of this paper is to fully characterize strongly algebraic closed MV-algebras, extending a result of Lacava. Moreover we provide some computation relating orbit algebras, Wajsberg algebras and Łukasiewicz semirings.

Keywords: MV-algebra, strongly algebraically closed, orbit algebra.

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Introduction

The present paper continues the research started in [15] and [23]. In [23], J. Schmid proved that a distributive lattice is an algebraically closed lattice if and only if it is a Boolean lattice. Also, he shows that any strongly algebraically closed lattice is a complete Boolean lattice. Later, it is proved by the author in [20] that if a complete Boolean lattice is q'-compact, then it is an strongly algebraically closed lattice. We recall from [17–19] that an algebra A is an strongly algebraically closed in a class of algebras, if every set of equations (finite or infinite) with coefficients from A, which is solvable in some algebras of the class of algebras containing A, already has a solution in A. Similarly, the purpose of this paper is to study strongly algebraically closed MV-algebras.

MV-algebras were introduced by C. C. Chang in 1958 to give an algebraic proof of the completeness of Łukasiewicz logic reducing the problem to require the semisimplicity of the Lindenbaum–Tarski algebra. Boolean algebras stand to Boolean logic as MV-algebras stand to Łukasiewicz infinite-valued logic (see [6]).

This paper continues the examination of the structure of MV-algebras. Algebraically closed MV-algebras are studied by Lacava in [15] and [16], where an MV-algebra A is called algebraically closed if every polynomial with coefficients in A having a root in some extension of A has already a root in A. Similarly, we provide a new axiomatization of strongly algebraically closed MV-algebras and prove that an MV-algebra A is an strongly algebraically closed MV-algebra if and only if it is regular, divisible, and equationally compact. We also describe orbit algebras with other algebraic structures as Wajsberg algebras and Łukasiewicz semirings. Recall that Wajsberg algebras are special algebraic structures that naturally arise from Łukasiewicz logic and Łukasiewicz near semirings were introduced by S. Bonzio, I. Chajda, and A. Ledda in [1].

1. Algebraically closed MV-algebras

A structure $(A, \oplus, \ominus, \neg, 0, 1)$ is an MV-algebra iff A satisfies the following equations for all x, y, $z \in A$:

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1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $\neg 0 = 1$;
6. $\neg 1 = 0$;
7. $x \ominus y = \neg(\neg x \oplus \neg y)$;
8. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

By definition following two new operations $\vee$ and $\wedge$ on A, the structure $(A, \vee, \wedge, 0, 1)$ will be a bounded distributive lattice:
$x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \ominus y) \ominus y$.

We recall from [11] that MV-algebras form a variety and the notion of MV-homomorphism is just the particular cases of the corresponding universal algebraic notion. In [21], an algebra A in class of MV-algebras is called an absolute retract in the class of MV-algebras if and only if every embedding $A \hookrightarrow B$ has a left inverse (i.e., there is a homomorphism h of B onto A such that $h(a) = a$ for each $a \in A$), whenever B is in the class of MV-algebras and A is a subalgebra of B, then A is a retract of B. Also, A is equationally compact if and only if every finite subset of set of equations is satisfiable in A, then the set of equations is satisfiable in A.

Recall from [7] that a maximal ideal M of an MV-algebra A is said to have a finite rank n, for some integer $n = 2$ if $A/M \cong L_n$, otherwise one says that M has infinite rank, where $L_n = [0, 1] \cap \mathbb{Z}\dfrac{1}{n-1}$. One should observe that every maximal ideal of a Boolean algebra has finite rank. An MV-algebra A is called regular if for every prime ideal N of its Boolean center, the ideal of A generated by N is a prime ideal of A [4].
By [22], an MV-algebra A is divisible if and only if for any $a \in A - \{0\}$ and integer $n \geqslant 0$ there exist a unique least element $b \in A$ such that $\underbrace{b \oplus b \ldots \oplus b}_{n\text{-}times} = a$ and $a \cdot (\underbrace{\neg b \ominus \neg b \ominus \ldots \ominus \neg b}_{(n-1)\text{-}times}) = b$.

By an equation in an algebra A we mean a formal expression

$$p(a_1, \ldots, a_m, x_1, \ldots, x_n) \approx q(a_1, \ldots, a_m, x_1, \ldots, x_n)$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$, p and q are $(m+n)$-ary terms (in the language of A), the elements $a_1, \ldots, a_m$ belong to A and they are called parameters (or coefficients), and $x_1, \ldots, x_n$ are the unknowns of this equation.

Definition 1.1. *An MV-algebra A is called algebraically closed if every polynomial with coefficients in A having a root in some extension of A has already a root in A.*

Definition 1.2. *An MV-algebra A of the class of MV-algebras is strongly algebraically closed in the class if for every extension B of A in the class and for any system of equations with parameters taken from A, if the system has a solution in B, then it also has a solution in A.*

In [15], Lacava proved an MV-algebra is an algebraically closed if and only if it is regular and divisible. Also, following Schmid [23], if we replace "any system" by "any finite system", then we obtain the concept of an algebraically closed algebra A in the class, which this two definitions are the same. Consequently, we will have that two definition above are same. Now, we provide a new axiomatization of strongly algebraically closed MV-algebras.

Lemma 1.3. *An MV-algebra A is strongly algebraically closed algebra in variety of MV-algebras in V if it is an absolute retract in V.*

Proof. By [5], let A be strongly algebraically closed in V and $B \in V$ be an extension of A. We need to show the existence of a retraction $f \colon B \to A$. We can assume that A is a proper

subalgebra of B, because the identity map of B would obviously be a retraction $B \to A$ if $A = B$. For each element b of $B \backslash A$, we take an unknown x_b and we define $x_{\neg b} \approx \neg x_b$. For each pair $(a, b) \in B \times B$ of elements such that at least one of a and b is element of $B \backslash A$, we define an equation $E_{\oplus}(a, b)$ according to the following six rules:
(1) – If a is element of A, b is element of $B \backslash A$, and $a \oplus b$ is element of A, then $E_{\oplus}(a, b)$ is $a \oplus x_b \approx a \oplus b$.
(2) – If a is element of $B \backslash A$, b is element of A, and $a \vee b$ is element of A, then $E_{\oplus}(a, b)$ is $x_a \oplus b \approx a \oplus b$.
(3) – If a and b are elements of $B \backslash A$ and $a \oplus b$ is element of A,
then $E_{\oplus}(a, b)$ is $x_a \oplus x_b \approx a \oplus b$.
(4) – If a is element of A, b and $a \oplus b$ are elements of $B \backslash A$, then $E_{\oplus}(a, b)$ is $a \oplus x_b \approx x_{a \oplus b}$.
(5) – If a and $a \oplus b$ are elements of $B \backslash A$ and b is element of A, then $E_{\oplus}(a, b)$ is $x_a \oplus b \approx x_{a \oplus b}$.
(6) – If a, b, and $a \oplus b$ are all elements of $B \backslash A$, then $E_{\oplus}(a, b)$ is $x_a \oplus x_b \approx x_{a \oplus b}$.

Let $\widehat{E}$ be the system of all equations we have defined so far. Clearly, $\widehat{E}$ has a solution in B. Indeed, we can let $x_a := a$ for all elements b of $B \backslash A$ to obtain a solution of $\widehat{E}$. Since we have assumed that A is strongly algebraically closed in V and $\widehat{E}$ also has a solution in A. This allows us to fix a solution of $\widehat{E}$ in A. That is, we can choose an element $u_b \in A$ for each element b of A such that the equations (1)–(6) turn into true equalities when the unknowns x_b, for $b \in B \setminus A$, are replaced by the elements u_b.

Next, consider the map

$$f : B \to A, \text{ defined by } c \mapsto \begin{cases} c & \text{if } c \text{ is an element of } A, \\ u_c & \text{if } c \text{ is a element of } B \backslash A. \end{cases}$$

We claim that f is a retraction. Clearly, f acts identically on A. So we need only to show that f is a MV-homomorphism or $f(0) = 0$, $f(x \oplus y) = f(x) \oplus f(y)$ and $f(\neg x) = \neg f(x)$, for every x, $y \in A$. It suffices to verify that f commutes with $\oplus$. If $a, b \in A$, then $a \oplus b$ is also in A, and we have that $f(a) \oplus f(b) = a \oplus b = f(a \oplus b)$, as required. If, say, $a, a \oplus b \in A$ and $b \in B \setminus A$, then (1) applies and we obtain that $f(a) \oplus f(b) = a \oplus u_b = a \oplus b = f(a \oplus b)$, as required. If $a, b, a \oplus b$ are all elements of A, then we can use (6) to obtain that $f(a) \oplus f(b) = u_a \oplus u_b = u_{a \oplus b} = f(a \oplus b)$, as required. The rest of the cases follow similarly from (3)–(5). Thus, we conclude that f commutes with $\oplus$. Therefor, f is a retraction and A is an absolute retract in V. □

Now, we are in the position to state the main theorem of the paper.

Theorem 1.4. *An MV-algebra A is strongly algebraically closed algebra if and only if it is algebraically closed and equationally compact.*

Proof. Suppose that A is strongly algebraically closed MV-algebra. By Lemma 2.4, A is an absolute retract. Now, we prove that A is regular, divisible, and equationally compact. Notice that Banaschewski–Nelson in [2] and Weglorz in [26] proved that the MV-algebra A is equationally compact if and only if every pure embedding $A \hookrightarrow B$ has a left inverse, see [21]. Since any absolute retract is a pure absolute retract, and here A is equationally compact. Now, to prove that A is regular and divisible it suffices to show that A is algebraically closed MV-algebra. Suppose $i : A \hookrightarrow B$ is an arbitrary extension of A and we prove that i is pure. To do this, for any system $\Sigma(\overline{x})$ of equations with parameters taken from A, if $\Sigma(\overline{b})$ is a solution in extension B of A, then we give $p : B \longrightarrow A$, where p a retraction (left inverse) of i and $\Sigma(p(\overline{x}))$ a solution in A.

Conversely, suppose A is algebraically closed and equationally compact. We know that A is regular and divisible. If $i : A \hookrightarrow B$ is an arbitrary extension of A, then i is pure. On the other hand, A is a pure absolute retract and thus i has a left inverse. Consequently, A is an absolute retract in V.

Now, we have that $A \in V$ is an absolute retract and $B \in V$ is an extension of A, and a system $\widehat{G}$ of equations with constants taken from A has a solution in B.

Let $x, y, z, \ldots$ denote the unknowns occurring in $\widehat{G}$ (possibly, infinitely many), and let $b_x, b_y, b_z, \cdots \in B$ form a solution of $\widehat{G}$. Since we have assumed that A is an absolute retract for V, we can take a retraction $f : B \to A$. We define

$$d_x := f(b_x),\ \ d_y := f(b_y),\ \ d_z := f(b_z),\ \ \ldots;$$

they are elements of A. Let

$$p(a_1, \ldots, a_k, x, y, z, \ldots) = q(a_1, \ldots, a_k, x, y, z, \ldots)$$

be one of the equations of $\widehat{G}$; here p and q are MV-algebra terms, the constants $a_1, \ldots, a_k$ are in A, and only finitely many unknowns occur in this equation. Using that f commutes with MV-algebra terms and, at $=^*$, using also that b_x, b_y, b_z, $\ldots$ form a solution of the equation in question, we obtain that

$$\begin{aligned} p(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots) &= p(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) = \\ &= f(p(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) =^* f(q(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) = \\ q(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) &= q(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots). \end{aligned}$$

This shows that $d_x, d_y, d_z, \cdots \in A$ form a solution of $\widehat{G}$ in A. Therefore, A is strongly algebraically closed in V. $\square$

We recall from [9] that the ordinary polynomials in the language of MV-algebras are called MV-polynomials and built from variables and function symbols of the language. And as usual, the value of a polynomial is calculated inductively from the value of its variables. In [13], MV-polynomials generalized to DMV-polynomials, which are built from MV-algebra symbols plus a unary function symbol δ_n for every positive integer n and DMV-polynomials have a value in every divisible MV-algebra (not in every MV-algebra, however) and $\mathbf{x}$ and $\mathbf{y}$ are finite vectors of variables. By [9], for every MV-polynomial $f(\mathbf{x},\ \mathbf{y})$ there is a single DMV-polynomial $g_f(\mathbf{y})$ such that [0, 1] verifies the following formula:

$$\varphi_f :\ \forall \mathbf{y}, (\exists \mathbf{x}, f(\mathbf{x}, \mathbf{y}) = 0 \Longleftrightarrow g_f(\mathbf{y}) = 0).$$

Now, we can state the following theorem:

Theorem 1.5. *An MV-algebra A is an algebraically closed MV-algebra if and only if*

1. *A is divisible;*
2. *for every MV-polynomial f, A verifies the formula φ_f;*
3. *A equationally compact.*

Proof. By [22] and Theorem 7 from [8], since A is strongly algebraically closed, is divisible and for every MV-polynomial f, A verifies the formula φ_f. Using Theorem 2.5, A is equationally compact.
Conversely, suppose that A is divisible and A models the formula φ_f. By [8], Theorem 7], A is an algebraically closed algebra. On the other hand, if A is equationally compact and algebraically closed, then A is an absolute retract in V. By Theorem 2.5, we conclude that A is an strongly algebraically closed MV-algebra. $\square$

2. Representation of orbit algebras

We recall that a Boolean algebra $(\mathfrak{B}, \wedge, \vee, \neg, 0, 1)$ with a countable dense subset is called separable, where a subset A of a Boolean algebra B is dense in B if and only if every element of B is a join of a subset of A. Suppose $\mathfrak{B}$ is a separable Boolean algebra and a, $b \in \mathfrak{B}$. We define $a \sim b$ if $b = g(a)$ for some $g \in G$ of group of automorphisms of $\mathfrak{B}$. Furthermore, for $g \in Aut(\mathfrak{B})$ and $a \in \mathfrak{B}$, we denote by $g\lceil a$ a the restriction of g to the interval $[0, a]$. We define $a \to b = \neg a \vee b$ for a, $b \in \mathfrak{B}$. We furthermore write $a \perp b$ if there are no non-zero $a_0 \leqslant a$ and $b_0 \leqslant b$ such that $a_0 \sim b_0$. In a Boolean algebra $\mathfrak{B}$ with a countable dense subset if G is a group of automorphisms of $\mathfrak{B}$, then we call the pair $(\mathfrak{B}, G)$ a Boolean ambiguity algebra.

We recall from [25] that $(\mathfrak{B}, G)$ is a Boolean ambiguity algebra. Assume that the following infima and suprema exists for all a, $b \in \mathfrak{B}$:

$$[a] \odot [b] = \inf\{[a' \wedge b'] \,:\, a' \sim a,\ b' \sim b\},$$
$$[a] \to [b] = \inf\{[a' \to b'] \,:\, a' \sim a,\ b' \sim b\},$$

where $[a]$ and $[b]$ are equivalence classes with respect to $\sim$. Following [25] we call the structure $(O(\mathfrak{B}, G), \leqslant, \odot, \to, 0, 1)$ the orbit algebra of $(\mathfrak{B}, G)$, where $O(\mathfrak{B}, G)$ is the quotient by equivalence relation $\sim$. Also, we have $\neg[a] = [a] \to 0$ for $a \in \mathfrak{B}$.
A Wajsberg algebra is a structure $(W, \rightharpoondown, ^*, 1)$, where $\rightharpoondown$ is a binary operation, $*$ is a unary operation and 1 is a constant such that the following identities hold:
1. $1 \rightharpoondown a = a$;
2. $(a \rightharpoondown b) \rightharpoondown ((b \rightharpoondown c) \rightharpoondown (a \rightharpoondown c)) = 1$;
3. $(a \rightharpoondown b) \rightharpoondown b = (b \rightharpoondown a) \rightharpoondown a$;
4. $(a^* \rightharpoondown b^*) \rightharpoondown (b \rightharpoondown a) = 1$,
for all $a, b, c \in W$.
This leads us to state the following theorem.

Theorem 2.1. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \oplus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, ^*, 1)$ is a strongly algebraically closed algebra if it is algebraically closed and equationally compact, where * is an unary operation and the implication $\rightharpoondown$ is defined by $x \rightharpoondown y = x \oplus y$ and $1 = 0^*$, for all x, $y \in O_{(\mathfrak{B},G)}$.*

Proof. First we prove that $(O_{(\mathfrak{B},G)}, \leqslant, \oplus, 1)$ is an ordered monoid. If we suppose that $a' \sim a$ and $b' \sim b$ such that $[a] \oplus [b] = [a' \wedge b']$, for all a, b, $c \in \mathfrak{B}$ then we will have

$$([a] \oplus [b]) \oplus [c] = \min\{[d \wedge c'] \,:\, d \sim a' \wedge b',\, c' \sim c\} = \min\{[a'' \wedge b'' \wedge c'] \,:\, a'' \sim a,\, b'' \sim b,\, c' \sim c\},$$

whence associativity of $\oplus$ follows. Obviously, $\oplus$ is in both arguments isotone. On the other hand, $[a] \oplus [b] \leqslant [c]$ if and only if $[a] \leqslant [b] \to [c]$. Now, for any a, $b \in \mathfrak{B}$, we can obtain $ab' \sim b$ such that $[a \wedge b'] = [a] \wedge [b]$ and $[a \vee b'] = [a] \vee [b]$. On the other hand, $\neg(a' \to b') \leqslant a'$ and then $[a] \to [b] = [a' \to b']$. Finally, $[a] \oplus ([a] \to [b]) = [a'] \oplus [a' \to b'] = [a \wedge (a' \to b')] = [a' \wedge b'] = [a] \wedge [b]$. Therefore, it is divisible and $\neg[a] = [\neg a]$ for any $a \in \mathfrak{B}$; so $\neg$ is involutive. Therefor, $(O_{(\mathfrak{B},G)}, \oplus, \to, 0)$ is an MV-algebra. Using Theorem 1.4, completes the poof. $\square$

By [1], we have that if $(O_{(\mathfrak{B},G)}, \oplus, \to, 0)$ is an MV-algebra, then

$$(O_{(\mathfrak{B},G)}, \rightharpoondown, *, 1)$$

is a Wajsberg algebra, where $a \rightharpoondown b = \neg a \oplus b$, for any a, $b \in O_{(\mathfrak{B},G)}$ and $1 = 0^*$. Thus we will have the following corollary:

Corollary 2.2. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \oplus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, *, 1)$ is a Wajsberg algebra.*

We recall from [14], a BE-algebra we shall mean an algebra $(X, {}^*, 1)$ of type $(2, 0)$ satisfying the following axioms:
1. $x * x = 1$;
2. $1 * x = x$;
3. $x * (y * z) = y * (x * z)$;
4. $x * 1 = 1$,
for all $x, y, z \in X$.

A BE-algebra $(X, {}^*, 1)$ is called bounded if there exists the smallest element 0 of X (i.e. $0 * x = 1$, for all $x \in X$). Recall that Imai and Iski (1966) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. A BCI-algebra is a non-empty set X endowed with a binary operation $*$ and a constant 0 satisfies the following axioms, for all $x, y, z \in X$:
1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $x * 0 = x$;
3. $x * y = 0$ and $y * x = 0$ imply that $x = y$.
Every BCI-algebra satisfying $0 * x = 0$ for all $x \in X$ is a BCK-algebra. We recall from [27] that an algebra $(X, {}^*, 1)$ of type $(2, 0)$ is called a dual BCK-algebra (or briefly, DBCK-algebra) if
1. $x * x = 1$;
2. $x * 1 = 1$;
3. $x * y = y * x \Longrightarrow x = y$;
4. $(x * y) * ((y * z) * (x * z)) = 1$;
5. $x * ((x * y) * y) = 1$ for all $x, y, z \in X$.

We now study the relations between BE-algebras and Łukasiewicz semirings.

Theorem 2.3. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, *, 1, 0)$ is a bounded commutative BE-algebra, where $x * y = \neg x \odot y$, for all $x, y \in \mathfrak{B}$ and $1 = \neg 0$.*

Proof. We claim the structure $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ is equivalent to a bounded commutative BE-algebra. By [27], an MV-algebra $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ is a bounded commutative dual BCK-algebra $(O_{(\mathfrak{B},G)}, {}^*, 1, 0)$ with the operation $*$ and the top element 1 defined as follows:
$x * y = \neg x \odot y,\ 1 = \neg 0$, for $x, y \in O_{(\mathfrak{B},G)}$. [14], any DBCK-algebra is a bounded commutative BE-algebra. □

BL-algebras were introduced by Hajek [12] as algebraic structures of basic logic, where a *BL-algebra* is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that:
(i) $(A, \vee, \wedge, 0, 1)$is a bounded lattice;
(ii) $(A, \odot, 1)$ is a commutative monoid;
(iii) the following statements hold for every $x, y, z \in A$:
(a) $z \leqslant x \rightarrow y$ iff $x \odot z \leqslant y$;
(b) $x \wedge y = x \odot (x \rightarrow y)$;
(c) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Recall that from [24] and [9] that a BL-algebra A to be an RS-BL-algebra if, for all elements $a \in A$ holds

$$\{x \in A, a \longrightarrow x = x\} = \{x \in A \,|\, x \longrightarrow a = a\}.$$

Theorem 2.4. *If $O_{(\mathfrak{B},G)}$ is a RS-BL-algebra, then $(O_{(\mathfrak{B},G)}, \neg, {}^*, 0)$ is a Wajsberg algebra.*

Proof. Since MV-algebras are such BL-algebras that $(a \rightarrow x) \rightarrow x = (x \rightarrow a) \rightarrow a$ holds for all $x, a \in O_{(\mathfrak{B},G)}$, it is an easy task to show that MV-algebras are RS-BLalgebras. To show that MV-algebras re the only BL-algebras that are RS-BL-algebras, observe first that the following holds in all RS-BL-algebras if $x^* = 0$, then $x = 1$, where $x^* = x \rightarrow 0$. Consequently, by [24] and [19] RS-BL-algebras are equivalent to MV-algebras. By Theorem 2.1, this completes the proof. □

We recall from [8] that a semiring $(R, +, 0, \cdot, 1)$ is an algebraic structure where 0 and 1 are distinct elements of R, $+$ and $\cdot$ are binary operations on R satisfying:
(i) $(R, +)$ is a commutative monoid with identity 0;
(ii) $(R, \cdot)$ is a monoid with identity 1;
(iii) Multiplication distributes over addition;
(iv) $0 \cdot r = r \cdot 0 = 0$, for every $r \in R$.

Also, by [8], a semiring $(R, +, 0, \cdot, 1)$ is called lattice-ordered semiring iff it has the structure of a lattice such that for all $a, b \in R$:
(i) $a + b = a \vee b$;
(ii) $a \cdot b \leqslant a \wedge b$.
Groupoids were introduced by Brandt in his 1926 paper [3] and semilattices can be equivalently presented as ordered sets as well as groupoids. An algebra is a structure (A, F) where A is an arbitrary non-empty set and F is a system of operations. A type of algebra is a mapping from F to $\mathbb{N}$ (natural numbers including zero) which maps any $f \in F$ to its arity. An algebra $(S, \cdot)$ of type $\langle 2 \rangle$ is called a groupoid.
In closing this section, we mention that the Łukasiewicz semirings are also closely related with the lattice ordered semirings. We recall from [1] that a near semiring is an algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that:
(i) $(R, +, 0)$ is a commutative monoid;
(ii) $(R, \cdot, 1)$ is a groupoid satisfying $x \cdot 1 = x = 1 \cdot x$ (a unital groupoid);
(iii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$;
(iv) $x \cdot 0 = 0 \cdot x = 0$;
for all $x, y, z \in R$.

In [1] a near semiring is called a semiring if $(R, \cdot, 1)$ is a monoid and satisfies left distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all $x, y, z \in R$.

A near semiring R is called idempotent if it satisfies $x + x = x$, for all $x \in R$. It is clear that in this case $(R, +)$ is a semilattice. In particular, $(R, +)$ can be considered as a join-semilattice, where the induced order is defined as $x \leqslant y$ iff $x + y = y$ and the constant 0 is the least element [1, Remark 1].

Following [1], a map α of an idempotent near semiring, with $\leqslant$ the induced order, $(R, +, \cdot, 0, 1)$ to $(R, +, \cdot, 0, 1)$ is called an *involution* on R if it satisfies the following conditions, for each $x, y \in R$:
(1) $\alpha(\alpha(x)) = x$;
(2) if $x \leqslant y$ then $\alpha(y) \leqslant \alpha(x)$.

As is defined in [1], an involutive near semiring R is said a Łukasiewicz near semiring if it satisfies the following additional identity:

$$\alpha(x \cdot \alpha(y)) \cdot \alpha(y) = \alpha(y \cdot \alpha(x)) \cdot \alpha(x).$$

A Łukasiewicz semiring A is a Łukasiewicz near semiring such that the reduct $(A, \cdot, 1)$ is a monoid.

Recall that if $(\mathfrak{B}, G)$ is a complete Boolean ambiguity algebra, then $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, 0)$ is an MV-algebra. On the other hand, the reducts $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ is an lc-semiring, where $x \vee y = x \odot (\neg x \ominus y)$, for every $x, y \in O_{(\mathfrak{B},G)}$. In following corollaries we shall use the name lc-semiring for lattice ordered commutative semiring:

Corollary 2.5. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, {}^*, 1)$ and $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ are MV-algebra and lc-semiring, where * is an unary operation and the implication $\rightharpoondown$ is defined by $x \rightharpoondown y = x \oplus y$, $x \vee y = x \odot (\neg x \ominus y)$, and $1 = 0^*$, for all $x, y \in O_{(\mathfrak{B},G)}$.*

Corollary 2.6. *Let* $(\mathfrak{B}, G)$ *be a complete Boolean ambiguity algebra and* $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, o)$ *be the orbit algebra. Then* $(O_{(\mathfrak{B},G)}, \rightharpoondown, {}^*, 1)$ *and* $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ *are strongly algebraically closed algebra if they are algebraically closed and equationally compact, where* * *is an unary operation and the implication* $\rightharpoondown$ *is defined by* $x \rightharpoondown y = x \oplus y$, $x \vee y = x \odot (\neg x \ominus y)$, *and* $1 = 0^*$, *for all* $x, y \in O_{(\mathfrak{B},G)}$.

Example 2.7. *Let* R *denote the set of real numbers and let* Q *denote the set of rational numbers. For any* $n \in \omega,\ n \geqslant 1$ *we define* $L_{n+1} = \{0, 1/n, \ldots, (n-1)/n, 1\}$. *If* a *and* b *are real numbers we define* $a \odot b = \min(a+b, 1)$, *and* $\neg a = 1-a$. *Suppose* A *is* $(Q \cap [0, 1], \odot, \neg, 0)$ *or* $(L_{n+1}, \odot, \neg, 0)$, *where they are MV-algebras. If* $\mathfrak{B}(A)$ *denotes its* R*-generated Boolean algebra and* $G(A)$ *is a subgroup of the automorphism group of* $\mathfrak{B}(A)$, *it turns out that* $(B(A), G(A))$ *forms an MV-pair. Independently, a similar study of certain type of* $(\mathfrak{B}, G)$*-pairs which yield an MV-algebra, so called ambiguity algebras.*

Conclusions

Lacava in [16] proved that an MV-algebra is algebraically closed if and only if it is regular and divisible. So, gathering up the theorems in Section 1, we obtain a representation of strongly algebraically closed MV-algebras as regular, divisible, and equationally compact. Therefore our results give further tools which can be suitable for Łukasiewicz logic and this can be the starting point to develop a sort of Algebraic Geometry based on MV-algebras.

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Сильно алгебраически замкнутые MV-алгебры

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Аннотация. Цель этой статьи — полностью охарактеризовать сильно алгебраические замкнутые MV-алгебры, обобщая результат Лакавы. Кроме того, мы приводим некоторые вычисления, связанные с алгебрами орбит, алгебрами Вайсберга и полукольцами (Лукашевич).

Ключевые слова: MV-алгебра, сильно алгебраически замкнутая, алгебра орбит.

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A Logarithmic Barrier Approach Via Majorant Function for Nonlinear Programming

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Abstract. In this paper, we are interested in solving an optimization nonlinear programming problem using a logarithmic barrier interior point method, in which the penalty term is taken as a vector $r \in \mathbb{R}^n_+$. The descent direction has been calculated using a classical Newton method, however the step size has been calculated with a new technique of majorant functions and a secant technique. The numerical simulations show us the efficiency of our approach compared to the classical line search method.

Keywords: nonlinear convex programming, logarithmic penalty method, line search, majorant function, secant technique.

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1. Introduction and preliminaries

In this paper, we are interested in the barrier logarithmic penalty method when using a new majorant function technique instead of the classical line search method to determine the step size ([1, 2, 4]).

1.1. The problem formulation

The problem to be studied in this paper is as follows:

$$\begin{cases} \min\ g(x) \\ x \in K \subseteq \mathbb{R}^n \end{cases} . \tag{P1}$$

In which: $K = \{x \in \mathbb{R}^n :\ Bx = c,\ x \geqslant 0\}$ is the set of feasible solution of $(P1)$.

1.1.1. Assumptions

A1 g is nonlinear, convex, twice continuously differentiable function on K.

A2 $B \in \mathbb{R}^{m \times n}$ is a full rank matrix, $c \in \mathbb{R}^m$ $(m < n)$.

A3 There exists $x^0 > 0$ such that $Bx^0 = c$.

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A4 The set of optimal solutions of $(P1)$ is nonempty and bounded.

For x^* be an optimal solution in the problem $(P1)$, there exists two Lagrange multipliers $u^* \in \mathbb{R}^m, v^* \in \mathbb{R}^n_+$ such as:

$$\begin{cases} \nabla g(x^*) + B^t u^* - v^* = 0 \\ Bx^* = c \\ < v^*, x^* >= 0 \end{cases} . \tag{1}$$

We can write $g^* = g^*(x^*) = \min_{x \in B^*} g(x)$.

In the following, we replace the nonlinear constrained problem $(P1)$ with a perturbed problem. What is new in our work is that the term of penalty is taken as a vector $r \in \mathbb{R}^n_+$.

1.2. The perturbed problem

In this section, we firstly define the function $\psi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicountinuous and proper function.

ψ defined as follows:

$$\psi(r,x) = \begin{cases} \sum_{i=1}^{n} r_i \ln(r_i) - \sum_{i=1}^{n} r_i \ln(x_i) & \text{if} \quad x, r > 0 \\ 0 \qquad \text{if} \quad r = 0, \ x \geqslant 0 \\ +\infty \qquad \text{if not} \end{cases} . \tag{2}$$

Now, the convex, lower semicountinuous and proper function $\phi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\phi_r(x) = \Phi(r,x) = \begin{cases} g(x) + \sum_{i=1}^{n} r_i \ln(r_i) - \sum_{i=1}^{n} r_i \ln(x_i) & \text{if} \quad Bx = c; \quad x, r \geqslant 0 \\ +\infty \quad \text{if not} \end{cases} . \tag{3}$$

Finally, the convex function m is defined by:

$$m(r) = \inf_x \{\phi_r(x); \quad x \in \mathbb{R}^n\} \tag{P2}$$

m is clearly convex since of the convexity of ϕ_r.

We notice that the two problems $(P1)$ and $(P2)$ are coincided when $\| r \| \to 0$, then $g^* = m(0)$.

Our idea is to develop a new approach, which consist to determine the step size using a majorant function technique. We begin by studying the existence and the uniqueness of the optimal solution of the perturbed problem $(P2)$ followed by the convergence study. The resolution of the perturbed problem is based on the Newton descent direction and the majorant function technique to determine the step size.

1.2.1. Existence and uniqueness of optimal solution of perturbed problem

In order to prove that $(P2)$ admits one unique optimal solution, suffice it to prove that the cone of recession of ϕ_r is reduced to zero.

Proof. According to the fourth assumption, $(P1)$ admits one unique optimal solution then the cone of recession C_g of g is reduced to zero, we have:

$$C_g = \{d \in \mathbb{R}^n : [g]_\infty(d) \leqslant 0, \, Bd = 0, \, d \geqslant 0\} = \{0\}$$

$[g]_\infty(d)$ is the asymptotic function of b, which define by:

$$[g]_\infty(d) = \lim_{t\to+\infty} \frac{b(x_0+td)-b(x_0)}{t}.$$

We have:

$$[\phi_r]_\infty = \begin{cases} g_\infty(d) & \text{if} \quad Bd=0,\ d\geqslant 0 \\ +\infty & \text{if not} \end{cases}.$$

Then we deduce that: $\{d\in\mathbb{R}^n; \quad [\phi_r]_\infty \leqslant 0\} = \{0\}$, wich means that $C_\phi = \{0\}$.

By taking into account that ϕ_r is strictly convex, we come to conclusion that the perturbed problem $(P2)$ admits one unique optimal solution which is denoted by $x(r)\in \overline{K}$, the set of strictly feasible solution of $(P2)$, in which

$$\overline{K} = \{x\in\mathbb{R}^n : Bx=c,\ x>0\}.$$

□

1.2.2. Convergence of perturbed problem

According to the necessary and sufficient optimality conditions, there exists $\lambda(r)\in\mathbb{R}^m$ (assumption 2) verify:

$$\begin{cases} \nabla g(x(r)) - rX_r^{-1} + B^t\lambda(r) = 0 \\ Bx(r)-c=0 \end{cases}. \tag{4}$$

In which X is the diagonal matrix with diagonal entries $X_{ii}=x_i$ $\forall i=\overline{1,n}$.
We impose that

$$F(x(r),\lambda(r)) = \begin{pmatrix} \nabla g(x(r)) - rX_r^{-1} + B^t\lambda(r) \\ Bx(r)-c \end{pmatrix} = 0.$$

The two functions $r\to x(r)$ and $r\to\lambda(r)$ are differentiable on $\mathbb{R}^n_+$, by using the implicit function theorem, we get

$$\begin{pmatrix} \nabla^2 g(x(r)) + RX_r^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \nabla x(r) \\ \nabla\lambda(r) \end{pmatrix} = \begin{pmatrix} X_r^{-1} \\ 0 \end{pmatrix}, \tag{5}$$

where R is the diagonal matrix with diagonal entries $R_{ii}=r_i$ $\forall i=\overline{1,n}$. And

$$\nabla x(r) = \begin{pmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_1}{\partial r_2} & \cdots & \frac{\partial x_1}{\partial r_n} \\ \frac{\partial x_2}{\partial r_1} & \frac{\partial x_2}{\partial r_2} & \cdots & \frac{\partial x_2}{\partial r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial r_1} & \frac{\partial x_n}{\partial r_2} & \cdots & \frac{\partial x_n}{\partial r_n} \end{pmatrix}, \quad \nabla\lambda(r) = \begin{pmatrix} \frac{\partial \lambda_1}{\partial r_1} & \frac{\partial \lambda_1}{\partial r_2} & \cdots & \frac{\partial \lambda_1}{\partial r_n} \\ \frac{\partial \lambda_2}{\partial r_1} & \frac{\partial \lambda_2}{\partial r_2} & \cdots & \frac{\partial \lambda_2}{\partial r_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \lambda_m}{\partial r_1} & \frac{\partial \lambda_m}{\partial r_2} & \cdots & \frac{\partial \lambda_m}{\partial r_n} \end{pmatrix}.$$

Remember that the function m wich is differentiable on $\mathbb{R}^n_+$ is define by:

$$m(r) = g(x(r)) + \sum_{i=1}^n r_i\ln(r_i) - \sum_{i=1}^n r_i \ln(x_i(r)).$$

We have

$$\nabla m(r) = (\nabla x(r))^t \left(\nabla g(x(r)) - X_r^{-1} r\right) + (e + z_1 - z_2).$$

In which $e = (1, 1, \ldots, 1)^t$, $z_1 = (\ln r_1, \ln r_2, \ldots, \ln r_n)^t$ and $z_2 = (\ln x_1, \ln x_2, \ldots, \ln x_n)^t$. According to (4) and (5), we get

$$\begin{aligned} \nabla m(r) &= -(\nabla x(r))^t B^t \lambda(r) + (e + z_1 - z_2) = \\ &= -(B\nabla x(r))^t \lambda(r) + (e + z_1 - z_2) = \\ &= e + z_1 - z_2. \end{aligned}$$

For $x(r) \in K$ and since of the convexity of m, we get:

$$\begin{aligned} m(0) &\geqslant m(r) - r^t \nabla m(r) \geqslant \\ &\geqslant g(x(r)) + \sum_{i=1}^n r_i \ln r_i - \sum_{i=1}^n r_i \ln x_i(r) - r^t(e + z_1 - z_2) \geqslant \\ &\geqslant g(x(r)) + \sum_{i=1}^n r_i \ln r_i - \sum_{i=1}^n r_i \ln x_i(r) - \sum_{i=1}^n r_i - \sum_{i=1}^n r_i \ln r_i + \sum_{i=1}^n r_i \ln x_i(r) \geqslant \\ &\geqslant g(x(r)) - \sum_{i=1}^n r_i. \end{aligned}$$

Taking into account that: $\quad g^* = m(0) = \min_x g(x(r))$.

Then, we come conclusion $g^* \leqslant g(x(r)) \leqslant g^* + \sum\limits_{i=1}^n r_i$.

For the rest, we are interesting on the trajectory of $x(r)$ when $\| r \|$ tends to zero.

a) The case in which g is only convex.

This case is a little complicated, we impose that $\|r\|_\infty \leqslant 1$, and for that we note

$$x(r) \in \{x;\ Bx = c,\ x > 0,\ g(x) \leqslant n + g^*\}.$$

This set is convex, bounded and non empty, its cone of recession is reduced to zero. It follows that each accumulation point of x_r is an optimal solution of $(P1)$ only if $\|r\| \to 0$.

b) The case in which g is strongly convex with coefficient γ strictly positif.

We have

$$\sum_{i=1}^n r_i \geqslant g(x(r)) - g(x^*) \geqslant < \nabla g(x^*), x(r) - x^* > + \frac{\gamma}{2} \| x(r) - x^* \|^2 .$$

Using (1), we obtain

$$\sum_{i=1}^n r_i \geqslant < v^*, x(r) > \geqslant 0.$$

Then

$$\| x(r) - x^* \| \leqslant \left(\frac{2}{\gamma} \sum_{i=1}^n r_i \right)^{\frac{1}{2}} .$$

We come to conclusion that the convergence of $x(r)$ to x^* is of order $\dfrac{1}{2}$.

Remark 1.1. *If the problem $(P1)$ or the perturbed problem $(P2)$ will have an optimal solution and the values of their objective functions are equal and finite, the other problem has an optimal solution.*

The general prototype of our method is as follows:

0 Starting by $(r_0, x_0) \in \mathbb{R}^n_+ \times \overline{K}$.

1 Find an approximate solution of $(P2)$ has been noted by x_{k+1} such that:

$$\phi(r_k, x_{k+1}) \leqslant \phi(r_k, x_k).$$

2 Take: $\|r_{k+1}\|_\infty \leqslant \|r_k\|_\infty$.

The iterations continue until we obtained the approximate solution.

2. Some useful inequalities

Taking into consideration the statistical serie of n real numbers $\{z_1, \ldots, z_n\}$, we define their arithmetic mean $\overline{z}$ and their standard deviation σ_z. These quantities are defined as follows:

$$\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i, \quad \sigma_z^2 = \frac{1}{n}\sum_{i=1}^{n} z_i^2 - \overline{z}^2 = \frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2.$$

For the following result see [3,6]

Proposition 2.1.

$$\overline{z} - \sigma_z\sqrt{n-1} \;\leqslant\; \min_i z_i \;\leqslant\; \overline{z} - \frac{\sigma_z}{\sqrt{n-1}}\,,$$
$$\overline{z} + \frac{\sigma_z}{\sqrt{n-1}} \;\leqslant\; \max_i z_i \;\leqslant\; \overline{z} + \sigma_z\sqrt{n-1}\,.$$

In the case where z_i are all positifs, we deduce that:

$$\ln\left(\overline{z} - \sigma_z\sqrt{n-1}\right) \;\leqslant\; \sum_{i=1}^{n}\ln(x_i) \;\leqslant\; \ln\left(\overline{z} + \sigma_z\sqrt{n-1}\right).$$

Theorem 2.1 ([2])**.** *Assume that $z_i > 0$ for all $i = \overline{1, n}$, then:*

$$A_1 \leqslant \sum_{i=1}^{n}\ln(z_i) \leqslant A_2,$$

with:

$$A_1 = (n-1)\ln\left(\overline{z} + \frac{\sigma_z}{\sqrt{n-1}}\right) + \ln\left(\overline{z} - \sigma_z\sqrt{n-1}\right),$$
$$A_2 = \ln\left(\overline{z} + \sigma_z\sqrt{n-1}\right) + (n-1)\ln\left(\overline{z} - \frac{\sigma_z}{\sqrt{n-1}}\right).$$

3. Solving the perturbed problem

Consider the following perturbed problem defined as follows:

$$m(r) = \min_x\left\{\phi_r(x) = g(x) + \sum_{i=1}^{n}\psi(r_i, x_i) \;:\; Bx = c,\; x \geqslant 0\right\}.$$

In this section we are interested in the numerical solution of the problem $(P1)$, we begin our work by calculate the descent direction and the step size in which we use a new technique of majorant functions.

3.1. The descent direction and line search function

A descent direction d can be computed by various methods, in this task we choose the Newton's method and therefore d is given by solving the following quadratic convex minimization problem:

$$\begin{cases} \min_d \left(\frac{1}{2} < \nabla^2 \phi_r(x) d, d > + < \nabla \phi_r(x), d > \right) \\ Bd = 0 \end{cases}.$$

According to the necessary and sufficient optimality conditions, there exists $\mu \in \mathbb{R}^m$ such that:

$$\begin{cases} \nabla^2 \phi_r(x) d + \nabla \phi_r(x) + B^t \mu = 0 \\ Bd = 0 \end{cases}.$$

Which is equivalent to

$$\begin{pmatrix} \nabla^2 g(x) + RX^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} d \\ \mu \end{pmatrix} = \begin{pmatrix} X^{-1} r - \nabla g(x) \\ 0 \end{pmatrix}.$$

From which we get

$$\begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} \nabla^2 g(x) + RX^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} d \\ \mu \end{pmatrix} = \begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} X^{-1} r - \nabla g(x) \\ 0 \end{pmatrix}.$$

Then

$$< \nabla^2 g(x) d, d > + < \nabla g(x), d > = < r, X^{-1} d > - < RX^{-1} d, X^{-1} d > . \tag{6}$$

This system is equivalent to

$$\begin{pmatrix} X \nabla^2 g(x) X + R & XB^t \\ BX & 0 \end{pmatrix} \begin{pmatrix} X^{-1} d \\ \mu \end{pmatrix} = \begin{pmatrix} r - X \nabla g(x) \\ 0 \end{pmatrix}. \tag{7}$$

The Newton descent direction being calculated.

3.2. Computation of the step size

Generally, the most used methods in the search line are the classical itterative methods as Armijo–Goldstein, Wolfe, Fibonnaci, . . . , but the computational cost in there becomes high when n is very large.

In this part, we are interested to avoid this difficulty. The method that we use bellow is simple and more effective than the first, it consists on the use of majorant function of the function θ. The choice of the step size $t^* > 0$ must give us a significant decrease of the convex function θ, we have:

$$\begin{aligned} \theta_0(t) &= \phi_r(x + td) - \phi_r(x) = \\ &= g(x + td) - g(x) - \sum_{i=1}^{n} r_i ln(1 + t y_i), \qquad y = X^{-1} d. \end{aligned}$$

According to Proposition 2.1, we have: $\rho \leqslant \min_i r_i \leqslant r_i \quad \forall i = \overline{1, n}$.
In which $\rho = \overline{r} - \sigma_r \sqrt{n-1}$.
Then, we obtain

$$\theta(t) = \frac{\theta_0(t)}{\rho} \leqslant \theta_1(t) = \frac{1}{\rho} \left(g(x + td) - g(x) \right) - \sum_{i=1}^{n} ln(1 + t y_i).$$

We have

$$\theta^{'}(t) = \frac{1}{\rho}\left(< \nabla g(x+td), d > - \sum_{i=1}^{n} r_i \frac{y_i}{1+ty_i}\right),$$
$$\theta^{''}(t) = \frac{1}{\rho}\left(< \nabla^2 g(x+td)d, d > + \sum_{i=1}^{n} r_i \frac{y_i^2}{(1+ty_i)^2}\right).$$

And

$$\theta_1^{'}(t) = \frac{1}{\rho} < \nabla g(x+td), d > - \sum_{i=1}^{n} \frac{y_i}{1+ty_i},$$
$$\theta_1^{''}(t) = \frac{1}{\rho} < \nabla^2 g(x+td)d, d > + \sum_{i=1}^{n} \frac{y_i^2}{(1+ty_i)^2}.$$

We deduce from (6), that $\theta^{'}(0)+\theta^{''}(0) = 0$, and we have $\theta^{''}(0) \geqslant 0$ wich give us that $\theta^{'}(0) \leqslant 0$. Now it must to prove that $\theta_1^{'}(0) \leqslant 0$, we have:

a If $y_i \geqslant 0$, it is clearly that $\theta_1^{'}(0) \leqslant 0$.

b If $y_i < 0$, we deduce from (6) that $\theta_1^{'}(0) + \theta_1^{''}(0) \leqslant 0$ and as $\theta_1^{''}(0) \geqslant 0$ we come conclusion that $\theta_1^{'}(0) \leqslant 0$.

What is prove the significant decrease of θ_1.

3.3. The first majorant function

The choice of t^* in which $\theta^{'}(t^*) = \theta^{'}(t_{opt}) = 0$ consists of some numerical complications, so generally we can't obtain t^* directly. To solve this problem, we propose to find an approximation function of θ.

This method is based on the use of a majorant function θ_2 of the function θ.

In the following, we take: $x_i = 1 + ty_i$, $\overline{x} = 1 + t\overline{y}$, and $\sigma_x = t\sigma_y$.

Applying the inequality $\sum\limits_{i=1}^{n} ln(x_i) \geqslant A_1$ (Theorem 2.2), we get that $\theta_1(t) \leqslant \theta_2(t)$ such that

$$\theta_2(t) = \frac{1}{\rho}\left(g(x+td) - g(x)\right) - (n-1)ln(1+t\alpha) - ln(1+t\beta).$$

In which

$$\alpha = \overline{y} + \frac{\sigma_y}{\sqrt{n-1}}, \quad \beta = \overline{y} - \sigma_y\sqrt{n-1}.$$

We have

$$\theta_2^{'}(t) = \frac{1}{\rho} < \nabla g(x+td), d > -(n-1)\frac{\alpha}{1+t\alpha} - \frac{\beta}{1+t\beta},$$
$$\theta_2^{''}(t) = \frac{1}{\rho} < \nabla^2 g(x+td)d, d > +(n-1)\frac{\alpha^2}{(1+t\alpha)^2} + \frac{\beta^2}{(1+t\beta)^2}.$$

The domains of θ_2 is $H_2 =]0, T[$ in which $T = \max\{t : \ 1 + t\beta > 0\}$, this domain is content in the domain of the line search function θ.

We notice that : $\theta(0) = \theta_1(0) = \theta_2(0) = 0$, $\theta_1^{'}(0) = \theta_2^{'}(0) < 0$ and $\theta_1^{''}(0) = \theta_2^{''}(0) > 0$.

We prove that the strictly convex function θ_2 is a good approximation of θ_1 in a neighbourhood of 0, hence the unique minimum t^* of θ_2 guarantee a significant decrease of the function θ_1, and we have the follows inequalities:

$$\theta(t^*) \leqslant \theta_1(t^*) \leqslant \theta_2(t^*) < 0.$$

3.4. Case when g is linear

We impose that $g(x) = c^t x,\ \ x, c \in \mathbb{R}^n$, the auxiliary function ω is given by the following form:

$$\omega(t) = n\eta t - (n-1)ln(1+t\alpha) - ln(1+t\beta)$$

in which: $\eta = \dfrac{c^t d}{n}$.

ω have the same properties as θ_2, the unique root of $\omega^{'}(t) = 0$ is the minimum of θ_2. The unique $\bar{t}$ that we have guarantee a significant decrease of the function ϕ_r along the newton descent direction d.

3.5. Case when g is only convex

In this case, the equation $\theta_2^{'}(t) = 0$ is no longer reduces to an equation of second degree, we thought to look at another function greater than θ_2, for this we use the secant technique. Given $\bar{t} \in]0, T[$ for all $t \in]0, \bar{t}]$, we have

$$\frac{g(x+td) - g(x)}{\rho} \leqslant \frac{g(x+\bar{t}d) - g(x)}{\rho\bar{t}} t.$$

Then the auxiliary function ω is define as follows:

$$\omega(t) = n\eta t - (n-1)ln(1+t\alpha) - ln(1+t\beta).$$

Such as, we take

$$\eta = \frac{g(x+\bar{t}d) - g(x)}{n\rho\bar{t}},$$

and we calculate t^* the root of the equation $\omega^{'}(t) = 0$.

1. If $\bar{t} = 1$ and $T > 1$ then $\bar{t}$ is the optimal solution.
2. If $\bar{t} \neq 1$, then

   a If $t^* \leqslant \bar{t}$, in this case we have $\theta(t^*) \leqslant \theta_1(t^*) \leqslant \theta_2(t^*) \leqslant \omega(t^*)$, which means that we assure a significant decrease of the function ϕ_r along the direction d.

   b If $t^* > \bar{t}$, we must to choose another $\bar{t} \in]t^*, T[$ and calculate t^* for the new auxiliary function and repeat this until we have that $t^* \leqslant \bar{t}$, for example we choose

$$\bar{t} = t^* + \zeta(T - t^*); \quad \zeta \in [0, 1].$$

3.6. Minimization of the auxiliary function ω

We have

$$\omega(t) = n\eta t - (n-1)ln(1+t\alpha) - ln(1+t\beta).$$

It is easy to calculate

$$\omega^{'}(t) = n\eta - (n-1)\frac{\alpha}{1+t\alpha} - \frac{\beta}{1+t\beta},$$

$$\omega^{''}(t) = (n-1)\frac{\alpha^2}{(1+t\alpha)^2} + \frac{\beta^2}{(1+t\beta)^2}.$$

Then: $\omega(0) = 0, \quad \omega^{'}(0) = n(\eta - \overline{y}), \quad \omega^{''}(0) = n(\overline{y}^2 + \sigma_x^2) = \| y \|^2.$

We impose that $\omega^{'}(0) \leqslant 0$ and $\omega^{''}(O) \geqslant 0$.
For getting t^*, we need to calculate the root of the equation $\omega^{'}(t) = 0$:
Equivalent to

$$\eta\alpha\beta t^2 + (\eta(\alpha+\beta) - \alpha\beta)t + \eta - \overline{y} = 0$$

1. if $\eta = 0$, $t^* = \dfrac{-\overline{y}}{\alpha\beta}$,
2. if $\alpha = 0$, $t^* = \dfrac{\overline{y}-\eta}{\eta\beta}$,
3. if $\beta = 0$, $t^* = \dfrac{\overline{y}-\eta}{\eta\alpha}$,
4. if $\eta\alpha\beta \neq 0$, in this case we have two roots of the equation of the second degree but there is just only root t^* which belongs to the domain of definition of ω, both roots are:

$$t_1^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\alpha} - \frac{1}{\beta} - \sqrt{\Delta}\right), \quad t_2^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\alpha} - \frac{1}{\beta} + \sqrt{\Delta}\right).$$

In which

$$\Delta = \frac{1}{\eta^2} + \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{2}{\alpha\beta} - \left(\frac{2n-4}{n\eta}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

3.7. The second majorant function

Here, we thought to find another approximation of θ_1 simpler than θ_2 and has one logarithm. Remember that:

$$\theta_1(t) = \frac{1}{\rho}\left(g(x+td) - g(x)\right) - \sum_{i=1}^{n} ln(1+ty_i);\ \rho = \overline{r} - \sigma_r\sqrt{n-1}.$$

Using the inequality:

$$\sum_{i=1}^{n} \ln(1+ty_i) \geqslant (\| y \| + n\overline{y})t + \ln(1 - t \| y \|).$$

Then, we get a second majorant function of θ noting by θ_3 such that:

$$\theta_3(t) = \frac{1}{\rho}\left(g(x+td) - g(x)\right) - (\| y \| + n\overline{y})t - \ln(1 - t \| y \|)$$

and

$$\theta_3^{'}(t) = \frac{1}{\rho} < \nabla g(x+td), d > - \| y \| - n\overline{y} + \frac{\| y \|}{1 - t \| y \|},$$
$$\theta_3^{''}(t) = \frac{1}{\rho} < \nabla g(x+td)d, d > + \frac{\| y \|^2}{(1 - t \| y \|)^2}.$$

The domains of θ_3 is $H_3 = [0, T_3[$, with $T_3 = \max\{t; 1 - t \| y \| > 0\}$.
We remark that:

- $\theta_3(0) = \theta_1(0) = 0$,
- $\theta_3^{'}(0) = \dfrac{1}{\rho} < \nabla g(x), d > - n\overline{y} = \theta_1^{'}(0) < 0$,

- $\theta_3^{''}(0) = \dfrac{1}{\rho} < \nabla^2 g(x)d, d > + \parallel y \parallel^2 = \theta_1^{''}(0) > 0.$

The strictly convex function θ_3 is a good approximation of θ_1 in a neighbourhood of 0, the unique minimum t^* of θ_3 guarantee a significant decrease of the function θ_1, and we have:

$$\theta_1(t^*) \leqslant \theta_2(t^*) \leqslant \theta_3(t^*).$$

3.7.1. Minimization of an auxiliary function

Let us define the convex function ω_2, where it's minimum is reached at t^*.

$$\omega_2(t) = n\eta t - (\parallel y \parallel + n\overline{y})t - ln(1 - t \parallel y \parallel).$$

It is easy to calculate

$$\begin{aligned} \omega_2^{'}(t) &= n\eta - \parallel y \parallel - n\overline{y} + \frac{\parallel y \parallel}{1 - t \parallel y \parallel}, \\ \omega_2^{''}(t) &= \frac{\parallel y \parallel^2}{(1 - t \parallel y \parallel)^2}. \end{aligned}$$

Then: $\omega_2(0) = 0, \quad \omega_2^{'}(0) = n(\eta - \overline{y}), \quad \omega_2^{''}(0) = \parallel y \parallel^2.$
We impose that $\omega_2^{'}(0) \leqslant 0$ and $\omega_2^{''}(O) \geqslant 0.$
For getting t^*, we need to calculate the root of the equation $\omega_2^{'}(t) = 0.$

4. Description of the algorithm

In this part, we present the algorithm which resume our study to obtain the optimal solution x^* of the problem $(P1)$.

4.1. Algorithm

1. **Input** $\epsilon > 0$, $r_s > 0$, $x_0 \in K$, X with $X(i,i) = x_0(i)$, $r \in \mathbb{R}_+^n$, $\delta \in [0,1]^n$.
2. **Iteration**

 * Calculate d and $y = X^{-1}d$

 (a) If $\parallel y \parallel > \epsilon$ do

 a1 Calculate η, α, β and solve the equation $\omega^{'}(t) = 0$ for obtain t^*.
 a2 Calculate $x = x + t^*d$, and return to $(*)$.

 (b) If $\parallel y \parallel \leqslant \epsilon$, we obtained a good approximation of $m(r)$.

 i If $\|r\| \geqslant r_s$, $r = \delta \times r$ and return to $(*)$.
 With $\delta \times r = (\delta_1 \times r_1, \dots, \delta_n \times r_n)$.
 ii If $\|r\| \leqslant r_s$, Stop. We have a good approximation of the optimal solution.

5. Numerical tests

In the tables bellow, Iter represents the number of iterations to obtain x^*, Min represents the minimum and T(s) represents the time in seconds. Method 1 corresponds to the method of majorant function introduced in this work, method 2 corresponds to the method of majorant function introduced in [1] and method 3 corresponds to the classical line search method.

5.1. Examples with variable size

Example 1. Quadratic case [4]
We consider the following quadratic problem with $n = m + 2$

$$g^* = \min\{b(x) : Bx = c, x \geqslant 0\}.$$

In which:

$$g(x) = \frac{1}{2} < x, Qx > .$$

With

$$Q[i,j] = \begin{cases} 2 & \text{if } \ i = j = 1 \quad \text{or} \quad i = j = m \\ 4 & \text{if } \ i = j \ \text{ and } \ i \neq \{1, m\} \\ 2 & \text{if } \ i = j - 1 \quad \text{or} \quad i = j + 1 \\ 0 & \text{otherwise} \end{cases} \quad , \quad B[i,j] = \begin{cases} 1 & \text{if } \ i = j \\ 2 & \text{if } \ i = j - 1 \\ 3 & \text{if } \ i = j - 2 \\ 0 & \text{otherwise} \end{cases} .$$

$g_i = 1 \ \forall i = \overline{1, n}, \ \forall j = \overline{1, m}.$
We test this example for different value of n.

Example 2. Erikson's problem [5]
Consider the following convex problem:

$$g^* = \min[g(x) : \ Bx = c, \ x \geqslant 0].$$

Where $g(x) = \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{a_i}\right)$, $a_i, b_i \in \mathbb{R}$ are fixed, and

$$B[i,j] = \begin{cases} 1 & \text{if } \ i = j \ \text{ or } \ j = i + m \\ 0 & \text{if not} \end{cases} .$$

We test this example for different values of n, a_i and b_i.

6. Tables

Table 1. Numerical simulations for Example 1

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
4	0.285	8	0.0061	0.285	9	0.007	0.285	14	0.019
50	5.37	6	0.019	5.372	7	0.021	5.374	14	0.053
100	10.924	6	0.065	10.927	7	0.08	10.93	14	0.188
500	55.3722	8	14.5	55.372	7	13.8	55.374	14	29.815

Table 2. The case where $a_i = 1$ and $b_i = 6, \forall i = \overline{1.n}$ (Example 2)

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
10	32.94	3	0.0038	32.95	3	0.004	32.95	4	0.018
50	164.79	4	0.026	164.79	4	0.025	164.79	6	0.082
500	329.57	5	0.1	329.58	5	0.076	329.58	6	0.23

Table 3. The case where $a_i = 2$ and $b_i = 5, \forall i = \overline{1.n}$ (Example 2)

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
10	0.62×10^{-8}	2	0.0021	0.66×10^{-7}	2	0.002	9.09×10^{-6}	3	0.01
50	0.1×10^{-7}	3	0.0028	0.11×10^{-8}	3	0.003	1.86×10^{-4}	4	0.07
500	0.75×10^{-7}	3	0.0045	0.76×10^{-7}	3	0.04	1.03×10^{-4}	5	0.22

Conclusion

The effective numerical simulations show that our approach is a very important alternative and gives an encouraging results compared to the classical line search method. However, it competes with the method introduced in [1] where $r \in \mathbb{R}$.

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Логарифмический барьерный подход с использованием мажорантной функции для нелинейного программирования

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Аннотация. В данной статье нас интересует решение оптимизационной задачи нелинейного программирования с использованием метода внутренних точек с логарифмическим барьером, в котором штрафной член берется в виде вектора $r \in \mathbb{R}_+^n$. Направление спуска было рассчитано с использованием классического метода Ньютона, однако размер шага был рассчитан с использованием новой техники мажорантных функций и техники секущих. Численное моделирование показывает нам эффективность нашего подхода по сравнению с классическим методом линейного поиска.

Ключевые слова: нелинейное выпуклое программирование, метод логарифмических штрафов, линейный поиск, мажорантная функция, метод секущих.

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Series of Hypergeometric Type and Discriminants

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Abstract. The monomial of solutions of a reduced system of algebraic equations are series of hypergeometric type. The Horn–Karpranov result for hypergeometric series is extended to the case of series of hypergeometric type.

Keywords: series of hypergeometric type, logarithmic Gauss map, discriminant locus, reduced system, conjugative radii of convergence.

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1. Introduction and preliminaries

Hypergeometric functions were studied in the 19th century by many famous mathematicians such as L. Euler, C. F. Gauss, E. Kummer, B. Riemann. Most of the researches were on one variable series. At the end of 19th and the first half of 20th century the hypergeometric functions were widespread considered, including several variables cases. Among them are the functions studied by G. Lauricella [11], J. Horn [8], P. Appell [3] (see also the books [4, 5]). The hypergeometric functions are still attractive recently (see [2, 6, 13, 14]). According to Horn [8] the series

$$H(x_1, \dots, x_N) = \sum_{\alpha \in \mathbb{N}^N} c_\alpha x^\alpha \tag{1}$$

is called *hypergeometric* if the relations of neighboring coefficients

$$\mathfrak{h}_i(\alpha) = \frac{c_{\alpha+e_i}}{c_\alpha}, \quad i = 1, \dots, N, \tag{2}$$

(where the set of e_i composes the standard basis in $\mathbb{Z}^N$), are rational functions in variables $\alpha = (\alpha_1, \dots, \alpha_N)$. Limit values of functions $\mathfrak{h}_i$ along fixed directions $s = (s_1, \dots, s_N) \in \mathbb{R}^N \setminus \{0\}$

$$\mathfrak{P}_i(s) := \lim_{k\to\infty} \mathfrak{h}_i(ks)$$

play an important role. We call the vector limit

$$\frac{1}{\mathfrak{P}(s)} = \left(\frac{1}{\mathfrak{P}_1(s)}, \dots, \frac{1}{\mathfrak{P}_N(s)}\right)$$

the *Horn parameterization* or *Horn uniformization* for the hpereometric series (1). These vectors define the conjugative radii of convergence for the series (1) (about the conception of these radii see [16, Sec. 7, ch. 1]).

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In this paper we study the *hypergeometric type series*. Roughly speaking, these series satisfy the following conditions: there is a sublattice $L \subset \mathbb{Z}^N$ of rank N such that the restriction of H on the shifts of L are hypergeometric. The details about the hypergeometric type series refer to the Section 3.

We are interested in the hypergeometric type series in order to investigate the solutions to universal systems of polynomial equations. In particular, we intend to apply the discriminant apparatus considered here to the calculation of the convergence domain of these series.

Consider a general system of n polynomial equations with n unknowns $y_1, \ldots, y_n$:

$$P_i := \sum_{\lambda \in A^{(i)}} a_\lambda^{(i)} y^\lambda = 0, \quad i = 1, \ldots, n, \tag{3}$$

where $A^{(i)}$ are the finite subsets of $\mathbb{Z}^n$ and $y^\lambda = y_1^{\lambda_1} \ldots y_n^{\lambda_n}$. We assume that all coefficients $a_\lambda^{(i)}$ are independent, and call (3) an *universal algebraic system*. Applying the Stepanenko's formula (see [10]) we get the hypergeometric type series presenting the monomials with positive integer exponents of the *principal solution* to the system (4).

We will explicit the relation between the Horn parameterization $\dfrac{1}{\mathfrak{P}(s)}$ for these series and the parameterization Ψ of the discriminant locus ∇ of the system (4) (see more about Ψ and ∇ in Section 2.). According to result in [1], the parameterization Ψ is the inverse of the *logarithmic Gauss map* for ∇. (The logarithmic Gauss map $\gamma : \nabla \subset \mathbb{C}^N \to \mathbb{CP}^{N-1}$ for a hypersurface ∇, defined by polynomial P, can be defined by the formula

$$(z_1, \ldots, z_n) \longmapsto (z_1 \partial_1 P(z) : \cdots : z_n \partial_n P(z)),$$

where ∂_j is the derivative $\partial/\partial z_i$ (see [9,12])).

According to Kapranov's result in [9], the Horn parameterization $\dfrac{1}{\mathfrak{P}}$ for the hypergeometric series coincides with the parameterization $\Psi = \gamma^{-1}$ of the discriminant locus ∇. The following theorem gives an extension of the Kapranov's result in [9] for the series of hypergeometric type representing monomials of solutions to the reduced system (4).

Theorem 1. *The Horn parameterization* $\dfrac{1}{\mathfrak{P}(s)}$ *for the series (6) and the parameterization* $\Psi(s)$ *of discriminant set for the system (4) coincide:*

$$\Psi = \frac{1}{\mathfrak{P}}.$$

2. Reduced systems and their discriminants

Following the paper [1] we consider the reduced system of the system (3) in the forms

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)}} x_\lambda^{(j)} y^\lambda - 1 = 0, \quad j = 1, \ldots, n, \tag{4}$$

where each m_j is a positive integer and $\Lambda^{(j)}$ does not contain $\lambda = 0$ and $\lambda = (0, \ldots, m_j, \ldots, 0)$.

Denote by ∇^0 the set of all the coefficients for which the system (3) has multiple zeros in the torus $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$, i.e. the Jacobian of P equals zero. The *discriminant locus* ∇ of the system (3) is the closure of the set ∇^0 in the space of coefficients of polynomials $P_1, \ldots, P_n$.

Denote the matrix

$$\Lambda := (\Lambda^{(1)}, \ldots, \Lambda^{(n)}) = (\lambda_1, \ldots, \lambda_N),$$

where $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k)^T \in \Lambda^{(j)}$ are column-vector of exponents in monomials of equations (4). Also let ω_m denote the $n \times n$-diagonal matrix with values $\dfrac{1}{m_j}$ on the diagonal. Consider the matrices

$$\Phi := \omega\Lambda, \quad \tilde{\Phi} := \Phi - \chi,$$

where χ is the matrix, whose i-th row is assigned by the characteristic function of the subset $\Lambda^{(i)} \subset \Lambda$, i.e. elements of this row are 1 at the position $\lambda \in \Lambda^{(i)}$ and 0 at all positions $\lambda \in \Lambda \setminus \Lambda^{(i)}$. In addition, φ_k denotes the rows of Φ, and $\tilde{\varphi}_k$ denotes the rows of $\tilde{\Phi}$. Their elements are denoted by $\varphi_{k\lambda}$ and $\tilde{\varphi}_{k\lambda}$ correspondingly. We can interpret each row φ_k as a sequence of vectors $\varphi_k^{(1)}, \ldots, \varphi_k^{(n)}$.

We will follow two copies of $\mathbb{C}^N$. The first one is $\mathbb{C}_x^\Lambda$ with the coordinators $x = (x_\lambda)$, and the second one is $\mathbb{C}_s^\Lambda$ with the coordinators $s = (s_\lambda)$ constructed as a space with homogeneous coordinators for $\mathbb{CP}^{N-1}$. Following the result of Antipova and Tsikh (see [1]), the map

$$\Psi : \mathbb{CP}_s^{N-1} \to \mathbb{C}_x^N = \mathbb{C}_{x^{(1)}}^{\Lambda^{(1)}} \times \cdots \times \mathbb{C}_{x^{(n)}}^{\Lambda^{(n)}},$$

from a projective space to the space of coefficients $x = (x_\lambda)$ of the system (4), defined by

$$x_\lambda^{(j)} = -\frac{s_\lambda^{(j)}}{\langle \tilde{\varphi}_j, s\rangle} \prod_{k=1}^{n} \left(\frac{\langle \tilde{\varphi}_k, s\rangle}{\langle \varphi_k, s\rangle} \right)^{\varphi_{k\lambda}}, \quad \lambda \in \Lambda^{(j)},\ j = 1, \ldots, n, \tag{5}$$

gives the parameterization for the discriminant locus ∇.

3. Solutions to reduced systems of algebraic equations

For the solution $y = (y_1, \ldots, y_n)$ to (4), we consider the series representing the monomial function $y^\mu = y_1^{\mu_1} \ldots y_1^{\mu_n}$

$$y^\mu = \sum_{\alpha \in \mathbb{N}^N} c_\alpha x^\alpha. \tag{6}$$

We focus on the so-called *principal solution* to system (4): they satisfy initial condition $y(0, \ldots, 0) = (1, \ldots, 1)$. When $\mu_j > 0$ the Stepanenko's result [10] claims that the coefficients c_α in (6) admit the following expression:

$$c_\alpha = (-1)^{\alpha_1 + \cdots + \alpha_N} \cdot \Gamma_\alpha \cdot \mathrm{R}_\alpha, \tag{7}$$

where

$$\begin{aligned} \Gamma_\alpha &= \frac{\prod\limits_{j=1}^{n} \Gamma\big(\frac{\mu_j + m_j}{m_j} + \langle \varphi_j, \alpha\rangle\big)}{\prod\limits_{i=1}^{N} \Gamma(\alpha_i + 1) \prod\limits_{j=1}^{n} \Gamma\big(\frac{\mu_j + m_j}{m_j} + \langle \varphi_j, \alpha\rangle - \sum\limits_{i \in \Lambda^{(j)}} \alpha_i\big)}, \\ \mathrm{R}_\alpha &= \det \left(\delta_i^{(j)} - \frac{\langle \varphi_i^{(j)}, \alpha^{(j)}\rangle}{\mu_j + \langle \varphi_j, \alpha\rangle} \right)_{(i,j) \in P_\alpha \times P_\alpha}, \end{aligned} \tag{8}$$

with $P_\alpha \subset \{1, \ldots, n\}$. We call Γ_α the *gamma-part* and R_α the *rational-part* of the coefficient c_α.

Remark that according to expressions (7) and (8) c_α admits the expression

$$c_\alpha = t^\alpha\, \mathrm{R}(\alpha) \prod_{j=1}^{M} \Gamma(\langle a_j, \alpha\rangle + b_j),$$

where $t^\alpha = t_1^{\alpha_1} \ldots t_N^{\alpha_N}$, $t_i, b_i \in \mathbb{C}$, $a_j \in \mathbb{Q}^N$, and $\mathrm{R}(\alpha)$ is a rational function. In the case when $a_j \in \mathbb{Z}^N$ this expression presents the general coefficient Ore-Sato for hypergeometric series (see [7,15]).

4. Horn parameterization for hypergeometric type series

Here we give more details for the definition of the series of hypergeometric type and construct for them the analog of the Horn parameterization. Let $e_1, \dots, e_N$ denote the standard basis of $\mathbb{Z}^N$, i.e. $e_\lambda = (0, \dots, 1, \dots, 0)$ with 1 being on λ-th position. For a given $\nu = (\nu_1, \dots, \nu_N) \in (\mathbb{N}\setminus\{0\})^N$ we consider the sublattice $L_\nu \subset \mathbb{Z}^N$ generated by $\nu_1 e_1, \dots, \nu_N e_N$. For two vectors $\nu, s \in \mathbb{Z}^N$ we define their product $\nu s := (\nu_1 s_1, \dots, \nu_N s_N)$.

Definition 1. We say that the power series

$$\sum_{\alpha \in \mathbb{N}^N} c_\alpha x^\alpha \tag{9}$$

is of *hypergeometric type* if there exists $\nu \in (\mathbb{N} \setminus \{0\})^N$ such that all subseries

$$H_l := \sum_{\alpha \in l + L_\nu \cap \mathbb{N}^N} c_\alpha x^\alpha = t^{\frac{l}{\nu}} \sum_{s \in \mathbb{N}^N} c'_s t^s, \quad l \in J,$$

are hypergeometric in variables $t_\lambda = x_\lambda^{\nu_\lambda}$, where $c'_s = c_{l+\nu s}$ and J is the sequence of all representatives for the factor $\mathbb{Z}^N / L_\nu$:

$$J = \{(l_1, \dots, l_N) \in \mathbb{Z}^n : 0 \leqslant l_i \leqslant \nu_i - 1, i = 1, \dots, N\}.$$

The subseries H_l is hypergeometric iff all the relations

$$\mathfrak{R}_\lambda(s) := \frac{c'_{s+e_\lambda}}{c'_s}, \quad \lambda = 1, \dots, N, \tag{10}$$

are rational functions of variables $s = (s_1, \dots, s_N)$.

Proposition 1. *The series (6) with the coefficient (7) is a hypergeometric type series.*

Proof. For a vector $\nu \in (\mathbb{N} \setminus \{0\})^N$ we take $\nu = (\tau, \dots, \tau)$ where τ is the least common multiple of $m_1, \dots, m_N$.

According to (8), the relations (10) become

$$\mathfrak{R}_\lambda(s) = \frac{c_{l+\nu(s+e_\lambda)}}{c_{l+\nu s}} = \frac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}} \frac{(-1)^\tau \, \mathrm{R}_{l+\tau s+\tau e_\lambda}}{\mathrm{R}_{l+\tau s}}, \quad \lambda = 1, \dots, N, \ l \in J, \tag{11}$$

where $J = \{l = (l_1, \dots, l_N) \ : \ 0 \leqslant l_1, \dots, l_N \leqslant \tau - 1\}$. The power of the exponent $(-1)^\tau$ comes from

$$\frac{(-1)^{|l+\tau(s+e_\lambda)|}}{(-1)^{|l+\nu s|}} = (-1)^{|\tau e_\lambda|} = (-1)^\tau,$$

where $|\alpha| := \alpha_1 + \dots + \alpha_N$.

Here $l + \tau s$ denotes the restriction of α on the shifted lattice $l + L_\nu$ (i.e. $\alpha =: l + \tau s$ for some $l \in J$). Thus $\Gamma_{l+\tau s}$ and $\mathrm{R}_{l+\tau s}$ are correspondingly the restrictions of the gamma-part Γ_α and the rational-part R_α of the series (6) on the such lattice. It is clear that the second ratio in (11), the ratio for R_α, is a rational function in s.

Introduce denotations

$$A_k := \varphi_k = (\varphi_{k1}, \dots, \varphi_{kN}), \quad A_{n+k} := \tilde{\varphi}_k = (\tilde{\varphi}_{k1}, \dots, \tilde{\varphi}_{kN}), \quad A_{2n+\lambda} := e_\lambda,$$

and rewrite (8) in such a way

$$\Gamma_\alpha = \frac{\prod\limits_{p=1}^{n} \Gamma(\langle A_p, \alpha \rangle + \eta_p)}{\prod\limits_{p=n+1}^{2n} \Gamma(\langle A_p, \alpha \rangle + \eta_p) \prod\limits_{p=2n+1}^{2n+N} \Gamma(\langle A_p, \alpha \rangle + 1)},$$

where η_p are some constants independing on α. To compute the ratio of gamma-parts in (11) we use the Pochhammer symbol

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} = z(z+1)\dots(z+k-1), \quad k \in \mathbb{N}\setminus\{0\}$$

and the denotation $q_p^\lambda := \langle A_p, \tau e_\lambda\rangle$. Then it leads to

$$\frac{\Gamma_{\alpha+\tau e_\lambda}}{\Gamma_\alpha} = \frac{\prod\limits_{p=1}^{n}(\langle A_p,\alpha\rangle + \eta_p - 1 + q_p^\lambda)_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}(\langle A_p,\alpha\rangle + \eta_p - 1 + q_p^\lambda)_{q_p^\lambda}\prod\limits_{p=2n+1}^{2n+N}(\langle A_p,\alpha\rangle + q_p^\lambda)_{q_p^\lambda}}.$$

With $\alpha = l + \tau s$, we get the ratio of gamma-parts restricted on the shifted lattice $l + L_\nu$:

$$\frac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}} = \frac{\prod\limits_{p=1}^{n}(\langle \tau A_p,s\rangle + \eta'_p + q_p^\lambda)_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}(\langle \tau A_p,s\rangle + \eta'_p + q_p^\lambda)_{q_p^\lambda}\prod\limits_{p=2n+1}^{2n+N}(\tau\langle e_\lambda,s\rangle + l_\lambda + q_p^\lambda)_{q_p^\lambda}}, \tag{12}$$

where constants η'_p are independent on s.

Since m_j divide τ, the delation τA_p in (12) is a vector with integer coordinators. Then its turns out that the relation $\dfrac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}}$ in (11) is a rational function of the variables $s_1, \dots, s_N$. Thus the series (6) is of hypergeometric type. $\square$

According to Horn (see [8]) the convergence radii of hypergeometric series are defined by the limits

$$\lim_{r\to\infty} \mathfrak{h}_i(rs), \quad i = 1, \dots, N,$$

where the rational functions $\mathfrak{h}_i$ are defined by (2). In the hypergeometric type case, the convergence radii of the series (9) are defined by the limits

$$\mathfrak{P}_\lambda(s_1,\dots,s_N) = \lim_{r\to\infty}\left(\mathfrak{R}_\lambda(rs)\right)^{\frac{1}{\tau}}, \quad \lambda = 1,\dots,N, \tag{13}$$

where $\mathfrak{R}_\lambda$ are rational relations (10) and τ is the least common multiple of $\nu_1,\dots,\nu_N$, $(s_1 : \dots : s_N) \in \mathbb{RP}^{N-1}$, $s_i > 0$. Indeed $(s_1,\dots,s_N)$ are homogeneous coordinates in $\mathbb{CP}^{N-1}$, and the limits $\mathfrak{P}_i$ are rational and homogeneous of degree zero. They depend only on the ratio $s = s_1 : \dots : s_N$. The vector limit

$$\frac{1}{\mathfrak{P}(s)} := \left(\frac{1}{\mathfrak{P}_1(s)}, \dots, \frac{1}{\mathfrak{P}_N(s)}\right)$$

are called by *Horn parameterization* (or *Horn uniformation*) for hypergeometric type series since Horn is the first person who considered such a limit for hypergeometric function (see [9]).

5. The proof of the Theorem 1

According to (12) we get the following formula for the limit values of relation (11) along direction $s := (s_1,\dots,s_N) \in \mathbb{R}^N \setminus \{0\}$.

Proposition 2.

$$\mathfrak{P}_\lambda(s_1,\ldots,s_N) = -\frac{\langle\tilde{\varphi}_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle\varphi_p, s\rangle}{\langle\tilde{\varphi}_p, s\rangle}\right)^{\varphi_{p\lambda}}. \tag{14}$$

Proof. From the ratio (12) and the limits (13),

$$\mathfrak{P}_\lambda(s_1,\ldots,s_N) := \lim_{r\to\infty}\left[\frac{c_{l+\tau rs+\tau e_\lambda}}{c_{l+\tau rs}}\right]^{\frac{1}{\tau}} = \lim_{k\to\infty}\left[\frac{\Gamma_{l+\tau rs+\tau e_\lambda}}{\Gamma_{l+\tau rs}}\frac{(-1)^\tau \mathrm{R}_{l+\tau rs+\tau e_\lambda}}{\mathrm{R}_{l+\tau rs}}\right]^{\frac{1}{\tau}} =$$

$$=: -\lim_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}},$$

where

$$A := \frac{\prod\limits_{p=1}^{n}(\langle rA_p, s\rangle + \eta'_p + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}(\langle rA_p, s\rangle + \eta'_p + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}\prod\limits_{p=2n+1}^{2n+N}(r\langle e_\lambda, s\rangle + \frac{l_\lambda}{\tau} + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}},$$

$$B := \frac{\tau^{q_1^\lambda+\cdots+q_n^\lambda}}{\tau^{q_{n+1}^\lambda+\cdots+q_{2n}^\lambda}\cdot\tau^{q_{n+1}^\lambda+\cdots+q_{2n+N}^\lambda}},$$

$$C := \frac{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)}, l^{(j)}+\tau e_\lambda^{(j)}\rangle + \langle\varphi_i^{(j)}, \tau rs^{(j)}\rangle}{\mu_j+\langle\varphi_j, l+\tau e_\lambda\rangle+\langle\varphi_j, \tau rs\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)}, l^{(j)}\rangle + \langle\varphi_i^{(j)}, \tau rs^{(j)}\rangle}{\mu_j+\langle\varphi_j, l\rangle+\langle\varphi_j, \tau rs\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}.$$

Recall that

$$\begin{aligned} &A_k = (\varphi_{k1},\ldots,\varphi_{kN}),\quad A_{n+k} = (\tilde{\varphi}_{k1},\ldots,\tilde{\varphi}_{kN}),\quad A_{2n+\lambda} = e_\lambda,\\ &q_p^\lambda = \langle A_p, \tau e_\lambda\rangle,\quad p\in\{1,\ldots,2n+N\}. \end{aligned} \tag{15}$$

Since $\tau e_\lambda = (0,\ldots,\tau,\ldots,0)$ with $\lambda\in\{1,\ldots,N\}$,

$$q_p^\lambda = \begin{cases} \tau\varphi_{p\lambda} & \text{with } 1\leqslant p\leqslant n,\\ \tau\tilde{\varphi}_{(p-n)\lambda} & \text{with } n+1\leqslant p\leqslant 2n,\\ \tau & \text{with } p = 2n+\lambda,\\ 0 & \text{with } p>2n \text{ and } p\neq 2n+\lambda. \end{cases} \tag{16}$$

Thus

$$q_1^\lambda+\cdots+q_n^\lambda = (\varphi_{1\lambda}+\cdots+\varphi_{n\lambda})\tau,\ q_{2n+1}^\lambda+\cdots+q_{2n+N}^\lambda = \tau, \tag{17}$$

and with the notice that $\lambda\in\Lambda^{(j)}$ for some j,

$$q_{n+1}^\lambda+\cdots+q_{2n}^\lambda = (\tilde{\varphi}_{1\lambda}+\cdots+\tilde{\varphi}_{n\lambda})\tau = (\varphi_{1\lambda}+\cdots+\varphi_{n\lambda}-1)\tau. \tag{18}$$

The sums in (17) and (18) lead to

$$B = \frac{\tau^{(\varphi_{1\lambda}+\cdots+\varphi_{n\lambda})\tau}}{\tau^{(\varphi_{1\lambda}+\cdots+\varphi_{n\lambda}-1)\tau}\cdot\tau^\tau} = 1,$$

Let r tend to the infinity we obtain the limits:

$$\lim_{r\to\infty} A = \frac{\prod\limits_{p=1}^{n}\langle A_p, s\rangle^{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}\langle A_p, s\rangle^{q_p^\lambda}\,\langle e_\lambda, s\rangle^\tau},$$

$$\lim_{r\to\infty} C = \frac{\det\left(\delta_i^{(j)} - \frac{\langle \varphi_i^{(j)}, \tau s^{(j)}\rangle}{\langle \varphi_j, \tau s\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}{\det\left(\delta_i^{(j)} - \frac{\langle \varphi_i^{(j)}, \tau s^{(j)}\rangle}{\langle \varphi_j, \tau s\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}} = 1.$$

Thus

$$\lim_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}} = \left[\frac{\prod\limits_{p=1}^{n}\langle A_p, s\rangle^{q_p^\lambda}}{\langle e_\lambda, s\rangle^\tau \prod\limits_{p=n+1}^{2n}\langle A_p, s\rangle^{q_p^\lambda}}\right]^{\frac{1}{\tau}}.$$

Substitute coordinators of the vectors A_p formulated in (15) and the value of q_p^α in (16), then the limit $\lim\limits_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}}$ equals

$$\left[\frac{\prod\limits_{p=1}^{n}\langle \varphi_p, s\rangle^{\tau\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle^\tau \prod\limits_{p=n+1}^{2n}\langle \tilde\varphi_{p-n}, s\rangle^{\tau\tilde\varphi_{(p-n)\lambda}}}\right]^{\frac{1}{\tau}}.$$

In the square brackets each factor is an exponentiation with the power τ. The radical $\dfrac{1}{\tau}$ applying on the square brackets leads to a simpler expression for the limit:

$$\frac{\prod\limits_{p=1}^{n}\langle \varphi_p, s\rangle^{\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle \prod\limits_{p=n+1}^{2n}\langle \tilde\varphi_{p-n}, s\rangle^{\tilde\varphi_{(p-n)\lambda}}}.$$

Rewrite the index for the production in the denominator of the last expression, it will become

$$\frac{\langle \tilde\varphi_j, s\rangle \prod\limits_{p=1}^{n}\langle \varphi_p, s\rangle^{\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle \prod\limits_{p=1}^{n}\langle \tilde\varphi_p, s\rangle^{\varphi_{p\lambda}}}.$$

Combining the factors with the same index under the production signs in the numerator and in the denominator of the last expression we will get the result:

$$\frac{\langle \tilde\varphi_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle \varphi_p, s\rangle}{\langle \tilde\varphi_p, s\rangle}\right)^{\varphi_{p\lambda}}.$$

Consequently we get the formula for the limit $\mathfrak{P}_\lambda$:

$$\mathfrak{P}_\lambda(s_1,\ldots,s_N) = -\frac{\langle \tilde\varphi_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle \varphi_p, s\rangle}{\langle \tilde\varphi_p, s\rangle}\right)^{\varphi_{p\lambda}}.$$

The proposition holds. □

Now we are ready to prove the Theorem 1.

Proof of theorem 1. From (5) and (14) it turns out that

$$x_\lambda^{(j)} = \frac{1}{\mathfrak{P}_\lambda}, \quad \lambda \in \Lambda^{(j)}, \; j = 1, \ldots, n.$$

Thus the parameterization $\Psi(s)$ for the discriminant locus ∇ of the system (4) composed by the coordinators $x_\lambda^{(j)}$ coincides with the limit vector of the hypergeometeric type series (6) composed by the coordinators $\dfrac{1}{\mathfrak{P}_\lambda}$. The theorem holds. $\square$

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Ряды гипергеометрического типа и дискриминанты

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Аннотация. Одночлен решений редуцированной системы алгебраических уравнений представляет собой ряд гипергеометрического типа. Мы распространяем результат Хорна-Карпранова (для гипергеометрических рядов) на случай рядов гипергеометрического типа.

Ключевые слова: ряды гипергеометрического типа, логарифмическое отображение Гаусса, дискриминантное множество, редуцированная система, сопряженные радиусы сходимости.