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Some Properties of the Automorphisms of the Classical Domain of the First Type in the Space $\mathbb{C}[m \times n]$

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Abstract. In this article we obtain an analogue of Theorem 2.2.2 from Rudin's book [6] for classical Cartan domains of the first type.

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It is well known that thanks to Riemann's uniformization theorem an arbitrary simply connected domain whose boundary with more than one point is biholomorphically equivalent to a unit circle $U = \{z \in \mathbb{C} : |z| < 1\}$. But in $\mathbb{C}^n (n > 1)$ such a property does not hold. For example, a ball and a polidisk are not mutually biholomorphic equivalent. Therefore, the class of biholomorphic domains in the space $\mathbb{C}^n$ is very important.

Definition 1 (Homogeneous Domain). *A domain $D \subset \mathbb{C}^n$ is called homogeneous if the group $Aut\,(D)$ of automorphisms of this domain is transitive, i.e. for any pair of points $z_1, z_2 \in D$ there exists an automorphism $\varphi \in Aut\,(D)$ such that $\varphi(z_1) = z_2$.*

Definition 2 (Symmetric Domain). *A homogeneous domain $D \subset \mathbb{C}^n$ is called symmetric if for any point $\varsigma \in D$ there exists an automorphism $\varphi \in Aut\,(D)$such that:*
$\varphi(\varsigma) = \varsigma$ but $\varphi(z) \neq z$ for $z \neq \varsigma$;
$\varphi \circ \varphi = e$, where $e \in Aut\,(D)$ is the identity map.

Definition 3. *A domain $D \subset \mathbb{C}^n$ is called irreducible if it is not a direct product of bounded symmetric domains of lower dimension.*

Definition 4. *A bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group and it is transitive on it.*

In homogeneous domains, the automorphism groups ([1, 2]) can be used to find integral formulas. Domains with rich automorphism groups are often realized as matrix domains ([3,4]). They turned out to be useful in solving various problems of function theory.

Complex homogeneous bounded domains are of great interest from different points of view. This is due to the fact that they form a relatively wide class of domains in $\mathbb{C}^n$, for which a number of meaningful, essentially multidimensional results have been obtained ([3, 5, 6] and etc.).

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In the works of C. Siegel, the presence of a biholomorphic mapping of classical domains by Siegel domains is shown [7]. Such biholomorphic maps are described, and applications to the problems of holomorphic maps to unbounded domains are given [8, 9]. Therefore, classical domains play an important role in multidimensional complex analysis. The goal of this work [10] is to obtain a criterion for holomorphic extendibility into a matrix ball for functions defined on a part of the Shilov boundary (skeleton) of a matrix ball, which is close in spirit to the criterion of L. A. Aizenberg, A. M. Kytmanov [11], and G. Khudayberganov [12]. In [13], optimal estimates of Bergman kernels for classical domains $\Re_I(m,n)$, $\Re_{II}(m)$, $\Re_{III}(m)$ and $\Re_{IV}(n)$ were found, respectively, through Bergman kernels in balls from spaces $\mathbb{C}^{mn}$, $\mathbb{C}^{m(m+1)/2}$, $\mathbb{C}^{m(m-1)/2}$ and $\mathbb{C}^n$. For this purpose, the statements of the Sommer-Mehring theorem on the continuation of the Bergman kernel and some properties of the Bergman kernel are used.

The theory of functions of many complex variables, or multidimensional complex analysis, currently has a fairly rigorously constructed theory [6, 14, 15]. At the same time, many questions of classical (one-dimensional) complex analysis still do not have unambiguous multidimensional analogues. This is due to the complex structure of a multidimensional complex space, overdetermination of the Cauchy-Riemann equations, the absence of a universal integral Cauchy formula, etc. In the works of E. Cartan, C. Siegel [7], Hua Lo-Ken [3], I. I. Pyatetsky-Shapiro [16] the matrix approach is widely used for presentation of the theory of multidimensional complex analysis.

E. Cartan (see [17]) in 1935 initiated a systematic study of bounded homogeneous domains, found all homogeneous domains in the spaces $\mathbb{C}^2$ and $\mathbb{C}^3$. He gave a classification of all bounded symmetric domains. These domains are divided into equivalence classes with respect to biholomorphic mappings. Each such class can be specified by any domain that belongs to it. Moreover, it is obvious that it is sufficient to consider only irreducible classes, that is, classes of domains that are not products of bounded symmetric domains of lower dimensions. In general, as E. Cartan established [17], there are six types of classes of irreducible bounded symmetric domains. Domains belonging to four of them are called classical because their automorphism groups are classical semisimple Lie groups. Two of these types are special in the sense that each of them occurs in the space $\mathbb{C}^n$ of only one dimension n, respectively for $n = 16$ and $n = 27$.

Consider the classical domains (see. [3, 17]):

$$\Re_I(m,n) = \left\{ Z \in \mathbb{C}[m,n] : I^{(m)} - Z\bar{Z}' > 0 \right\},$$

$$\Re_{II}(m) = \left\{ Z \in \mathbb{C}[m,m] : I^{(m)} - Z\bar{Z} > 0, \, \forall Z' = Z \right\},$$

$$\Re_{III}(m) = \left\{ Z \in \mathbb{C}[m,m] : I^{(m)} + Z\bar{Z} > 0, \, \forall Z' = -Z \right\},$$

$$\Re_{IV}(n) = \left\{ Z \in \mathbb{C}^n : |\langle z,z\rangle|^2 - 2|z|^2 + 1 > 0, \, |\langle z,z\rangle| < 1 \right\},$$

here $I^{(m)}$ is the identity matrix of order m, $\bar{Z}'$ is the complex conjugate matrix of the transposed matrix Z' ($H > 0$ for a Hermitian matrix H means, as usual, that H is positive definite). All these domains are homogeneous symmetric convex complete circular domains centered at O (O is the zero matrix).

If we write the elements of the matrices $Z \in \mathbb{C}[m,n]$ as a point in the space $\mathbb{C}^{mn}$ then it is in the following form

$$z = \{z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, \ldots, z_{m1}, \ldots, z_{mn}\} \in \mathbb{C}^{mn}.$$

Then we can assume that Z is an element of the space in $\mathbb{C}^{mk}$, i.e., we arrive to the following isomorphism

$$\mathbb{C}[m \times n] \cong \mathbb{C}^{mn}.$$

Therefore, the dimensions of the classical four domains above are equal to

$$mn,\ \frac{m(m+1)}{2},\ \frac{m(m-1)}{2},\ n$$

respectively.

In [18] an analogue of Bremermann's theorem on finding the Bergman kernel is obtained for the Cartesian product of classical domains.

Writing out explicity the transitive group of automorphisms of four types of classical domains and matrix balls (see, for example, [19, 20]) associated with classical domains. By direct computation one can find the Bergman and Cauchy-Szegő kernels for these domains. And then (using the properties of the Poisson kernel), we can find the Carleman formula, which restores the value of a holomorphic function in the domain itself from its values on some boundary sets of uniqueness (see [21–24]). In this case, the scheme for finding the Bergman and Cauchy–Szegő kernels from [3, 6, 25] is used.

The properties of the matrix ball $B_2(m,n)$ of the second type are studied in [26]. In [27] the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. The full volumes of these domains are necessary to find the kernels of integral formulas for these domains (Bergman, Cauchy–Szegő, Poisson kernels, etc.) and is used for the integral representation of functions holomorphic in these domains, in the mean value theorem, and in other important concepts.

For example, in [28] the regularity and algebraicity of mappings in classical domains are studied, and in [29] harmonic Bergman functions in classical domains are studied from a new point of view.

The first type of classical domain: $\Re_1(m,n)$. Each point of the domain is a matrix with m rows and n columns satisfying the following condition

$$I^{(m)} - ZZ^* > 0,$$

where $I^{(m)}$ is a unit square matrix of the m^{th} order and Z^* is a transposed conjugate matrix for Z. Automorphisms of the classical domain of the first type $\Re_1(m,n)$ have the following form [3]

$$\varphi(Z) = (AZ + B)(CZ + D)^{-1}, \tag{1}$$

where the coefficients satisfy the following conditions:

$$AA^* - BB^* = I^{(m)},\ \ AC^* = BD^*,\ \ CC^* - DD^* = -I^{(n)}. \tag{2}$$

The automorphism (1) of the classical domain can also be represented as

$$\varphi(Z) = (ZB^* + A^*)^{-1}(ZD^* + C^*), \tag{3}$$

where the coefficients (3) satisfy the following conditions:

$$B^*B - D^*D = -I^{(n)},\ \ B^*A = D^*C,\ \ A^*A - C^*C = I^{(m)}. \tag{4}$$

We can simplify the automorphism (1) as follows

$$\varphi(Z) = (AZ + B)(CZ + D)^{-1} = A\left(Z + A^{-1}B\right)\left(D^{-1}CZ + I\right)^{-1}D^{-1}. \tag{5}$$

If we put $A = Q,\ A^{-1}B = -P,\ D = R$, then we have

$$\varphi(Z) = \varphi_P(Z) = Q(Z - P)(I - P^*Z)^{-1}R^{-1}. \tag{6}$$

Then from (2) we get

$$Q(I - PP^*)Q^* = I^{(m)},\ \ \ R(I - P^*P)R^* = I^{(n)}. \tag{7}$$

We study some useful properties of automorphism $\varphi_P(Z)$ in the following theorem (this is an analog of the theorem 2.2.2 from [6]).

Theorem 1. *For an automorphism $\varphi_P(Z) = Q(Z-P)(I-P^*Z)^{-1}R^{-1}$ of the classical domain of the first type $\Re_1(m,n)$, the following properties hold:*

1^0. $\varphi_P(P) = 0$, $\varphi_P(0) = -QPR^{-1}$. *In particular, if $QP + PR = 0$ then we have* $\varphi_P(P) = 0$, $\varphi_P(0) = P$.

2^0. *Differential of the automorphism of the domain $\Re_1(m,n)$ is equal to*

$$d(\varphi_P(P)) = QdZR^*, d(\varphi_P(0)) = (Q^*)^{-1}dZR^{-1},$$

where $d(\varphi_P(P))$ $(d(\varphi_P(0)))$ is the differential of the automorphism (6) at the point $Z = P$ $(Z = 0)$.

3^0. *For all $Z, W \in \Re_I(m,n)$ we have*

$$\det(I - \langle\varphi(Z), \varphi(W)\rangle) = \frac{\det(I - \langle P, P\rangle) \cdot \det((I - \langle Z, W\rangle))}{\det(I - \langle Z, P\rangle) \cdot \det(I - \langle P, W\rangle)},$$

$$\det(I - \langle\varphi_P(Z), \varphi_P(Z)\rangle) = \frac{\det(I - \langle P, P\rangle)\det(I - \langle Z, Z\rangle)}{\det(I - \langle Z, P\rangle)\det(I - \langle P, Z\rangle)}.$$

4^0. *If we have the following equalities $QP + PR = 0$, $R = R^*$, $Q = Q^*$ then $\varphi_P(\varphi_P(Z)) = Z$ (the property of involution).*

5^0. *Finally, $\varphi_P(Z)$ is a homeomorphism.*

Proof. Let us prove turn by turn the above properties of the automorphism (6).

1^0. The values of the automorphism $\varphi_P(Z)$ at the points $Z = P$ and $Z = 0$ are

$$\varphi_P(P) = Q(P-P)(I-P^*P)^{-1}R^{-1} = 0,$$

$$\varphi_P(0) = Q(0-P)(I-P^*\cdot 0)^{-1}R^{-1} = -QPR^{-1},$$

that is

$$\varphi_P(P) = 0,\ \varphi_P(0) = -QPR^{-1}$$

the latter equalities follow directly. The condition $QP + PR = 0$ implies $QP = -PR \Rightarrow P = -QPR^{-1}$, that is,

$$\varphi_P(0) = P.$$

We show that the matrix $-QPR^{-1}$ belongs to $\Re_1(m,n)$:

$$I - \left(-QPR^{-1}\right)\cdot\left(-QPR^{-1}\right)^* = I - QPR^{-1}\left(R^{-1}\right)^*P^*Q^*.$$

Thanks to the conditions (7) we have

$$R^{-1}(R^*)^{-1} = I - P^*P.$$

$$I - \left(-QPR^{-1}\right)\cdot\left(-QPR^{-1}\right)^* = I - QPR^{-1}\left(R^{-1}\right)^*P^*Q^* = I - QP(I-P^*P)P^*Q^* =$$
$$= I - QPP^*Q^* + QPP^*PP^*Q^* = I - QPP^*(I-PP^*)Q^* = I - QPP^*Q^{-1}Q(I-PP^*)Q^* =$$
$$= I - QPP^*Q^{-1}I^{(m)} = (Q - QPP^*)Q^{-1} = Q(I-PP^*)Q^{-1} = Q(I-PP^*)Q^*(Q^*)^{-1}Q^{-1} =$$
$$= I^{(m)}(Q^*)^{-1}Q^{-1} = (Q^*)^{-1}Q^{-1} = (Q^*)^{-1}I^{(m)}Q^{-1} > 0$$

Therefore, $\left(-QPR^{-1}\right) \in \Re_1(m,n)$ [30].

2^0. Now we calculate the differential of the automorphism $\varphi_P(Z) = Q(Z-P)(I-P^*Z)^{-1}R^{-1}$. So we have

$$d(\varphi_P(Z)) = Q\left(dZ(I-P^*Z)^{-1} + (Z-P)\,d(I-P^*Z)^{-1}\right)R^{-1} \tag{8}$$

or

$$\varphi_P(Z)R(I-P^*Z) = Q(Z-P).$$

From the last equality and according to the rules of differentiation we get

$$d(\varphi_P(Z))\,R(I-P^*Z)+\varphi_P(Z)\,Rd(I-P^*Z)=Qd(Z-P),$$
$$d(\varphi_P(Z))\,R(I-P^*Z)-\varphi_P(Z)\,RP^*dZ=QdZ,$$
$$d(\varphi_P(Z))\,R(I-P^*Z)=QdZ+\varphi_P(Z)\,RP^*dZ,$$
$$d(\varphi_P(Z))\,R(I-P^*Z)=(Q+\varphi_P(Z)\,RP^*)\,dZ,$$
$$d(\varphi_P(Z))\,R(I-P^*Z)=\Big(Q+Q(Z-P)(I-P^*Z)^{-1}R^{-1}RP^*\Big)\,dZ.$$

Then we obtain

$$d(\varphi_P(Z))=Q\Big(I+(Z-P)(I-P^*Z)^{-1}P^*\Big)\,dZ(I-P^*Z)^{-1}R^{-1}. \tag{9}$$

If we calculate the differential of $\varphi_P(Z)$ at the points $Z=P$ and $Z=0$, we get the following values

$$d(\varphi_P(P))=QdZ(I-P^*P)^{-1}R^{-1},\quad d(\varphi_P(0))=Q(I-PP^*)dZR^{-1}.$$

In accordance with the conditions (7), the following equality

$$(I-P^*P)^{-1}R^{-1}=R^*,\quad Q(I-PP^*)=(Q^*)^{-1}.$$

Then for the differentials of $\varphi_P(Z)$ at the points $Z=P$ and $Z=0$ we have

$$d(\varphi_P(P))=QdZR^*,\quad d(\varphi_P(0))=(Q^*)^{-1}dZR^{-1}$$

3^0. In order to prove this property, we use the expression (3) of the automorphism of the domain $\Re_1(m,n)$. $\varphi_P(Z)=(ZB^*+A^*)^{-1}(ZD^*+C^*)$, $\varphi_P(W)=(WB^*+A^*)^{-1}(WD^*+C^*)$

$$\begin{aligned}
I-\langle\varphi(Z),\varphi(W)\rangle &= I-(ZB^*+A^*)^{-1}(ZD^*+C^*)\Big((WB^*+A^*)^{-1}(WD^*+C^*)\Big)^*=\\
&=I-(ZB^*+A^*)^{-1}(ZD^*+C^*)(DW^*+C)(BW^*+A)^{-1}=\\
&=(ZB^*+A^*)^{-1}\big((ZB^*+A^*)(BW^*+A)-(ZD^*+C^*)(DW^*+C)\big)(BW^*+A)^{-1}=\\
&=(ZB^*+A^*)^{-1}\big(ZB^*BW^*+ZB^*A+A^*BW^*+A^*A-(ZD^*DW^*+ZD^*C+\\
&\qquad +C^*DW^*+C^*C)\big)(BW^*+A)^{-1}=\\
&=(ZB^*+A^*)^{-1}(I-ZW^*)(BW^*+A)^{-1}=\\
&=(A^*)^{-1}\Big(ZB^*(A^*)^{-1}+I\Big)^{-1}(I-ZW^*)\big(A^{-1}BW^*+I\big)^{-1}A^{-1}.
\end{aligned}$$

If we put

$$A=Q,\ A^{-1}B=-P,\ D=R$$

and use

$$A^*=Q^*,\ B^*\big(A^{-1}\big)^*=-P^*,\ D^*=R^*,$$

we get the following equality

$$\begin{aligned}
I-\langle\varphi(Z),\varphi(W)\rangle &= (Q^*)^{-1}(I-ZP^*)^{-1}(I-ZW^*)(I-PW^*)^{-1}Q^{-1}=\\
&=(Q^*)^{-1}(I-\langle Z,P\rangle)^{-1}(I-\langle Z,W\rangle)(I-\langle P,W\rangle)^{-1}Q^{-1}.
\end{aligned} \tag{10}$$

By using (7), it is not difficult to see that

$$\det\left(Q\left(I-PP^{*}\right)Q^{*}\right)=\det\left(I^{(m)}\right),\quad \det\left(\left(I-PP^{*}\right)\right)=\det\left(Q^{-1}\right)\cdot\det\left(\left(Q^{*}\right)^{-1}\right).$$

We use the last two equalities to obtain the following relation

$$\det\left(I-\langle\varphi\left(Z\right),\ \ \varphi\left(W\right)\rangle\right)=\frac{\det\left(I-\langle P,P\rangle\right)\cdot\det\left(\left(I-\langle Z,W\rangle\right)\right)}{\det\left(I-\langle Z,P\rangle\right)\cdot\det\left(I-\langle P,W\rangle\right)} \tag{11}$$

If we put $W=Z$ in (11), we get

$$\det\left(I-\langle\varphi_P\left(Z\right),\ \ \varphi_P\left(Z\right)\rangle\right)=\frac{\det\left(I-\langle P,P\rangle\right)\det\left(I-\langle Z,Z\rangle\right)}{\det\left(I-\langle Z,P\rangle\right)\det\left(I-\langle P,Z\rangle\right)}.$$

4^0. We show that the automorphism $\varphi_P\left(Z\right)=Q\left(Z-P\right)\left(I-P^{*}Z\right)^{-1}R^{-1}$ is an involution of the domain $\Re_1\left(m,n\right)$. Indeed,

$$\begin{aligned}\varphi_P\left(\varphi_P\left(Z\right)\right)&=Q\left(Q\left(Z-P\right)\left(I-P^{*}Z\right)^{-1}R^{-1}-P\right)\left(I-P^{*}Q\left(Z-P\right)\left(I-P^{*}Z\right)^{-1}R^{-1}\right)^{-1}R^{-1}=\\&=Q\left(Q\left(Z-P\right)-PR\left(I-P^{*}Z\right)\right)\left(I-P^{*}Z\right)^{-1}R^{-1}\cdot R\cdot\left(I-P^{*}Z\right)\times\\&\qquad\times\left(R\left(I-P^{*}Z\right)-P^{*}Q\left(Z-P\right)\right)^{-1}R^{-1}=\\&=Q\left(QZ-QP-PR+PRP^{*}Z\right)\cdot\left(R-RP^{*}Z-P^{*}QZ+P^{*}QP\right)^{-1}R^{-1}=\\&=Q\left(\left(Q+PRP^{*}\right)Z-\left(QP+PR\right)\right)\cdot\left(\left(R+P^{*}QP\right)-\left(RP^{*}+P^{*}Q\right)Z\right)^{-1}R^{-1}.\end{aligned}$$

Since $QP+PR=0$, $R=R^{*}$, $Q=Q^{*}$ and considering the following equalities

$$Q\left(I-PP^{*}\right)Q^{*}=I^{(m)},\quad R\left(I-P^{*}P\right)R^{*}=I^{(n)},$$

we have

$$Q\left(Q+PRP^{*}\right)=I^{(m)},\quad RP^{*}+P^{*}Q=0,\quad \left(R+P^{*}QP\right)^{-1}R^{-1}=I^{(n)}.$$

Consequently, we obtain

$$\varphi_P\left(\varphi_P\left(Z\right)\right)=Q\left(\left(Q+PRP^{*}\right)Z-\left(QP+PR\right)\right)\cdot\left(\left(R+P^{*}QP\right)-\left(RP^{*}+P^{*}Q\right)Z\right)^{-1}R^{-1}=Z,$$

i.e

$$\varphi_P\left(\varphi_P\left(Z\right)\right)=Z.$$

5^0. In order to prove this property we take $Z_1,Z_2\in\Re_1$. Then

$$\varphi_P\left(Z_1\right)=Q\left(Z_1-P\right)\left(I-P^{*}Z_1\right)^{-1}R^{-1},\varphi_P\left(Z_2\right)=Q\left(Z_2-P\right)\left(I-P^{*}Z_2\right)^{-1}R^{-1},$$

$$\begin{aligned}\varphi_P\left(Z_1\right)-\varphi_P\left(Z_2\right)&=Q\left(Z_1-P\right)\left(I-P^{*}Z_1\right)^{-1}R^{-1}-Q\left(Z_2-P\right)\left(I-P^{*}Z_2\right)^{-1}R^{-1}=\\&=Q\left(\left(Z_1-P\right)\left(I-P^{*}Z_1\right)^{-1}-\left(Z_2-P\right)\left(I-P^{*}Z_2\right)^{-1}\right)R^{-1}=\end{aligned}$$

$$=Q\left(Z_1(I-P^{*}Z_1)^{-1}-Z_2(I-P^{*}Z_2)^{-1}-P\left((I-P^{*}Z_1)^{-1}-(I-P^{*}Z_2)^{-1}\right)\right)R^{-1}=$$

$$\begin{aligned}&=Q\left(Z_1(I-P^{*}Z_1)^{-1}-Z_2(I-P^{*}Z_1)^{-1}+Z_2(I-P^{*}Z_1)^{-1}-\right.\\&\qquad\left.-Z_2(I-P^{*}Z_2)^{-1}-P\left((I-P^{*}Z_1)^{-1}-(I-P^{*}Z_2)^{-1}\right)\right)R^{-1}=\end{aligned}$$

$$= Q\left((Z_1 - Z_2)(I - P^*Z_1)^{-1} + (Z_2 - P)\left((I - P^*Z_1)^{-1} - (I - P^*Z_2)^{-1}\right)\right)R^{-1} =$$

$$= Q\left((Z_1 - Z_2)(I - P^*Z_1)^{-1} + (Z_2 - P)(I - P^*Z_2)^{-1}(I - P^*Z_2 - I + P^*Z_1)(I - P^*Z_1)^{-1}\right)R^{-1} =$$

$$= Q\left((Z_1 - Z_2)(I - P^*Z_1)^{-1} + (Z_2 - P)(I - P^*Z_2)^{-1}P^*(Z_1 - Z_2)(I - P^*Z_1)^{-1}\right)R^{-1} =$$

$$= Q\left(I + (Z_2 - P)\left((P^*)^{-1} - (P^*)^{-1}P^*Z_2\right)^{-1}\right)(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1} =$$

$$= Q\left((P^*)^{-1} - Z_2 + (Z_2 - P)\right)\left((P^*)^{-1} - Z_2\right)^{-1}(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1} =$$

$$= Q\left((P^*)^{-1} - Z_2 + (Z_2 - P)\right)\left((P^*)^{-1} - Z_2\right)^{-1}(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1} =$$

$$= Q(I - PP^*)(P^*)^{-1}P^*(I - Z_2P^*)^{-1}(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1} =$$

$$= Q(I - PP^*)(I - Z_2P^*)^{-1}(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1}.$$

Thus

$$\varphi_P(Z_1) - \varphi_P(Z_2) = Q(I - PP^*)(I - Z_2P^*)^{-1}(Z_1 - Z_2)(I - P^*Z_1)^{-1}R^{-1}.$$

Since $Q(I - PP^*)(I - Z_2P^*)^{-1}$ and $(I - P^*Z_1)^{-1}R^{-1}$ are different from O(O is the zero matrix), we can easily see that $\varphi_P(Z_1) = \varphi_P(Z_2)$ if and only if $Z_1 = Z_2$. Hence $\varphi_P(Z)$ is a homeomorphism.

The proof is complete. □

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Некоторые свойства автоморфизмов классической области первого типа в пространстве $\mathbb{C}[m \times n]$

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Аннотация. В этой статье получен аналог Теоремы 2.2.2 из книги Рудина [6] для классических областей Картана первого типа.

Ключевые слова: однородная область, симметричная область, классическая область, автоморфизм.

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On the p-fold Well-posedness of Higher Order Abstract Cauchy Problem

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Abstract. In this paper, we establish sufficient conditions for the p-fold well-posedness of higher-order abstract Cauchy problem. These conditions are expressed in terms of decay of some auxiliary pencils derived from the characteristic pencil for the operational differential equation considered. In particular, this paper improves important and interesting work.

Keywords: abstract Cauchy problems, integrated semi-groups, well-posedness.

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1. Introduction

The development of functional analysis and the theory of linear operators appeared at the beginning of the 20th century and had a great influence on the study of ordinary differential equations, partial differential equations, and boundary value problems. In particular, many results concerning the general theory of operator pencils has marked a rapid development (see, for example, [5, 9, 16] and the related sources). This theory is related to the study of boundary values problems for operator-differential equations and higher-order abstract Cauchy problems. Since integrated semi-groups were introduced at the end of the 1980s, it has become possible to deal with ill-posed first-order abstract Cauchy problems (see, [4,7,14]). Many authors have made series of direct investigations on the abstract Cauchy problem of the second order and higher order. For more information, we can refer to ([6, 11, 13, 15, 17]) and the references cited therein.

The well-posedness the Cauchy problem is a very large classical problem which has been extensively studied in many contexts. For example in [17], Tijun *et al.* presented a concise criteria for C-well-posedness and analytic well-posedness of the complete second order Cauchy problem.

In [18], Vlasenko and others studied the p-fold well-posedness of the higher order abstract Cauchy problem of the following form:

$$\sum_{k=0}^{n} A_k \frac{d^k u}{dt^k} = 0, \qquad t > 0, \tag{1.1}$$

$$u^k(0) = u_k, \qquad k = 0, \dots, n-1, \tag{1.2}$$

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where A_k $(k = 0, \ldots, n)$ are closed linear operators that act from the complex Banach space $\mathcal{X}$ into the complex Banach space $\mathcal{Y}$. We denote the norms on $\mathcal{X}$ and $\mathcal{Y}$ by $\|\|_{\mathcal{X}}$ and $\|\|_{\mathcal{Y}}$ respectively.

They obtained well-posedness conditions, which characterize the continuous dependence of solutions and their derived on the initial data.

It should be noted that the higher order abstract Cauchy problems of the form (1.1), (1.2) have been studied by many authors, since they describe several models derived from natural phenomena, such as description of vibrations (see, [1, 2, 10]), viscoelastic pipeline in [12]. This means that it deserves to study the p-fold well-posedness and exponentially p-fold well-posedness of a higher order abstract Cauchy problem. The abstract Cauchy problem for the higher order has been extensively investigated by many authors (see, [13, 18, 19] and the references therein). We refer to some researche which relates directly to our work (see, [18, 19]).

In [18], Vlasenko *et al.* investigated the equation (1.1) with a characteristic polynomial and its resolvent

$$\mathcal{P}(\lambda) = \sum_{k=0}^{n} A_k \lambda^k, \quad R(\lambda) = \mathcal{P}^{-1}(\lambda), \tag{1.3}$$

the authors derived some new conditions represented in the estimates using $R(\lambda)$ to ensure the correct setting of the p-fold of the problem (1.1), (1.2).

Motivated by this work, this paper aims to establish new sufficient conditions which guarantee the p-fold well-posedness and exponentially p-fold well-posedness of the problem (1.1), (1.2). These conditions are expressed in terms of the decay of the auxiliary pencils $\mathcal{P}_j(\lambda)$ and $\mathcal{Q}_j(\lambda)$, which are respectively defined by:

$$\mathcal{P}_j(\lambda) = \sum_{k=0}^{j} A_k \lambda^k, \tag{1.4}$$

$$\mathcal{Q}_j(\lambda) = \sum_{k=j+1}^{n} A_k \lambda^{k-j-1}, \tag{1.5}$$

for $j \in \{0, \ldots, n-1\}$.

As we mentioned before, the results of this work extend and improve the previously known results. More precisely, we will consider a higher order abstract Cauchy problem and give some new sufficient conditions to ensure p-fold well-posedness and exponentially p-fold well-posedness of the problem (1.1), (1.2). In our analysis, it is not necessary to use all the operators A_k, $k = 1, \ldots, n$, which are required in some relevant preliminary work, see [18]. This new feature makes the p-fold well-posedness of the higher order abstract Cauchy more important and useful as well. The results of this article are new and they extend and improve previously known results.

The content of this paper is organized as follows:
Some necessary concepts and preliminaries are reviewed in Section 2. Section 3 is a section of intermediate results, where we show some lemmas which are necessary for our analysis. Finally, in Section 4, we give and prove our main results.

2. Preliminaries

In this section, we briefly recall some notations and definitions which are used throughout the paper.

Let $\mathcal{G}(a, \theta)$ be the sector of the plane defined by

$$\mathcal{G}(a, \theta) = \left\{ \lambda = a + re^{i\varphi}, \ |\varphi| \leqslant \theta, \ a, r > 0, \ \frac{\pi}{2} < \theta < \pi \right\}.$$

In order to discuss the statement of our problem, we need the following definitions.

Definition 2.1 (see [18]). *By solution of the problem* (1.1), (1.2)*, we mean every function u which satisfies the conditions:*

(i) $u \in C^n(\mathbb{R}_+^*, \mathcal{X}) \bigcap C^{n-1}(\mathbb{R}_+, \mathcal{X})$;

(ii) $A_k u \in C^k(\mathbb{R}_+^*, \mathcal{Y}) \bigcap C^{k-1}(\mathbb{R}_+, \mathcal{Y})$;

(iii) u satisfies condition (1.2).

Definition 2.2 (see [18]). *The abstract Cauchy problem* (1.1), (1.2) *is said to be determined if for fixed* $\{u_k\}_0^{n-1}$*, there exists only one solution.*

Definition 2.3 (see [18]). *Let* $p \in \{1, \ldots, n\}$*, the problem* (1.1), (1.2) *is said to be p-fold well-posed if every solution u verifies:*

$$\left\|u^{p-1}(t)\right\|_{\mathcal{X}} \leqslant q(t) \sum_{k=0}^{n-1} \|u_k\|_{\mathcal{X}}, \quad t \geqslant 0, \tag{2.1}$$

where q is a non-negative function from $\mathbb{R}_+$ *to* $\mathbb{R}_+$.

Definition 2.4 (see [18]). *The problem* (1.1), (1.2) *is said to be exponentially well-posed if* $q(t) = Ce^{\omega t}$*, for* $C \geqslant 0$*, and* $\omega \geqslant 0$.

Lemma 2.5 (Jordan lemma (see [3])). *Let m be positive constant and* $Q(z)$ *be a continuous function in the upper half of complex plane, such that for* $|z| \geqslant R$

$$M_R = \max_{z \in \Gamma_R} |Q(z)| \to 0, \quad R \to \infty,$$

where Γ_R *is the semicircle of* $|z| = R$ *in the upper half of the complex plane. Then*

$$\lim_{R \to \infty} \int_{\Gamma_R} e^{imz} \mathcal{Q}(z)\, dz = 0.$$

Theorem 2.6 (see [16]). *Let Banach space* $\mathcal{X}$ *and A is closed linear operators from* $\mathcal{X}$ *into* $\mathcal{X}$*, such that* $\|A\| \leqslant q < 1$*. Then* $(I + A)$ *is reversible and* $(I + A)^{-1}$ *is bounded operator.*

3. Intermediate results

In this section, we establish three fundamental lemmas and for convenience we present their proofs. The following lemmas are useful for the proof of our main results in the last section.

Lemma 3.1. *If there exist* $j \in \{0, \ldots, n-1\}$ *such that*
$H_1)$ $\mathcal{G}(a, \theta) \subset \rho(\mathcal{Q}_j(\lambda))$,
$H_2)$ $\left\|\mathcal{Q}_j^{-1}(\lambda) A_k x\right\|_{\mathcal{X}} \leqslant C |\lambda|^{j+1} e^{\sigma|\lambda|} \|x\|_{\mathcal{X}}, \ k = 1, \ldots, n,$
$H_3)$ $\left\|\mathcal{Q}_j^{-1}(\lambda) A_k x\right\|_{\mathcal{X}} \leqslant C_j |\lambda|^{q_{k_j}} \|x\|_{\mathcal{X}}, \ k = 0, \ldots, j,$ *with* $\dfrac{\ln C_j}{\ln(a \sin\theta)} < q_{k_j} < j - k + 1,$
then,

$$\mathcal{G}(a, \theta) \subset \rho(\mathcal{P}(\lambda)),$$

and

$$\left\|\mathcal{P}^{-1}(\lambda) A_k x\right\|_{\mathcal{X}} \leqslant Ce^{\sigma|\lambda|} \|x\|_{\mathcal{X}}, \ k = 1, \ldots, n. \tag{3.1}$$

Proof. If $\lambda \in \mathcal{G}(a,\theta)$, then $\lambda \in \rho(\mathcal{Q}_j(\lambda))$, and $\mathcal{Q}_j(\lambda)$ are invertible, $j = 0,\dots,n-1$. In fact, by assumptions (1.3), (1.4), and (1.5), it can be easily write:

$$\begin{aligned}\mathcal{P}(\lambda) &= \lambda^{j+1}\mathcal{Q}_j(\lambda) + \mathcal{P}_j(\lambda) = \\ &= \lambda^{j+1}\mathcal{Q}_j(\lambda)\left[I + \lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right].\end{aligned}$$

By using the Theorem 2.6, we show that the operator (1.3) is invertible.
By (H_3), we have

$$\begin{aligned}\left\|\lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right\|_{\mathcal{X}} &\leqslant \sum_{k=0}^{j}|\lambda|^{k-j-1}\left\|\mathcal{Q}_j^{-1}(\lambda)A_k\right\|_{\mathcal{X}} \leqslant \\ &\leqslant C_j\sum_{k=0}^{j}|\lambda|^{k-j-1+q_{k_j}}.\end{aligned}$$

If we let $N_j = \max(j+1-k-q_{k,j})$, then

$$\left\|\lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right\|_{\mathcal{X}} \leqslant \frac{C_j(j+1)}{|\lambda|^{N_j}}.$$

From assumption (H_3), we decuce

$$N_j > \frac{\ln C_j}{\ln(a\sin\theta)}.$$

From the above discussion, we can get

$$\left\|\lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right\|_{\mathcal{X}} < 1. \tag{3.2}$$

We deduce that the operator pencil

$$I + \lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda),$$

is invertible as well as $\mathcal{P}(\lambda)$. Consequently

$$\mathcal{P}^{-1}(\lambda) = \lambda^{-j-1}\left(I + \lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right)^{-1}\mathcal{Q}_j^{-1}(\lambda). \tag{3.3}$$

If we put

$$\mathcal{R}_j(\lambda) = \left(I + \lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda)\mathcal{P}_j(\lambda)\right)^{-1}. \tag{3.4}$$

Make substitution of (3.4) into (3.3), we have

$$\mathcal{P}^{-1}(\lambda) = \mathcal{R}_j(\lambda)\lambda^{-j-1}\mathcal{Q}_j^{-1}(\lambda). \tag{3.5}$$

Then, we conclude

$$\left\|\mathcal{P}^{-1}(\lambda)A_kx\right\|_{\mathcal{X}} \leqslant \left\|\mathcal{R}_j(\lambda)\right\|_{\mathcal{X}}|\lambda|^{-j-1}\left\|\mathcal{Q}_j^{-1}(\lambda)A_kx\right\|_{\mathcal{X}}.$$

Using assumption (H_2), we get

$$\left\|\mathcal{P}^{-1}(\lambda)A_kx\right\|_{\mathcal{X}} \leqslant C\left\|\mathcal{R}_j(\lambda)\right\|_{\mathcal{X}}|\lambda|^{-j-1}|\lambda|^{j+1}e^{\sigma|\lambda|}\|x\|_{\mathcal{X}}.$$

From (3.2) and Theorem 2.6, we conclude that $\mathcal{R}_j(\lambda)$ is bounded, i.e, $\exists G_0 > 0$, such that

$$\left\|\mathcal{R}_j(\lambda)\right\|_{\mathcal{X}} \leqslant G_0. \tag{3.6}$$

According to (3.6), we can finaly write

$$\left\|\mathcal{P}^{-1}(\lambda)A_kx\right\|_{\mathcal{X}} \leqslant G_1e^{\sigma|\lambda|}\|x\|_{\mathcal{X}}, \quad k = 1,\dots,n, \tag{3.7}$$

where $G_1 = CG_0$. □

Corollary 3.2. *If the hypotheses* (H_1), (H_2) *and* (H_3) *are fulfilled, then*

$$\left\|\mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\right\|_{\mathcal{X}} \leqslant Ce^{\sigma|\lambda|} f_k(|\lambda|)\,, k=1,\ldots,n, \tag{3.8}$$

where

$$f_k(\lambda) = \sum_{i=k+1}^{n} \lambda^{i-k-1}. \tag{3.9}$$

Proof. From (1.5), we can easly get

$$\mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda) = \mathcal{P}^{-1}(\lambda)\sum_{i=k+1}^{n} A_k\lambda^{i-k-1},$$

according to Lemma 3.1, we find

$$\begin{aligned}\left\|\mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\right\|_{\mathcal{X}} &\leqslant \sum_{i=k+1}^{n} |\lambda|^{i-k-1}\left\|\mathcal{P}^{-1}(\lambda)\,A_k\right\|_{\mathcal{X}} \leqslant \\ &\leqslant Ce^{\sigma|\lambda|}\sum_{i=k+1}^{n}|\lambda|^{i-k-1} = Ce^{\sigma|\lambda|} f_k(|\lambda|)\,,\end{aligned}$$

where f_k is defined in (3.9).

Lemma 3.3. *If the hypotheses of Lemma 3.1 are satisfied and* u *is a solution of problem* (1.1), (1.2), *then there exists* $M>0$, *such that*

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Me^{(a-a\cos\theta)t}\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{X}}\,, \tag{3.10}$$

for

$$t > -\frac{\sigma}{\cos\theta}, \quad m=0,\ldots,n-1.$$

Proof. If the hypoteses of Lemma 3.1 are satisfied, then relation (3.1) is true and by applying Lemma 1 in [19], we obtain the following representation for solution u and its derivatives:

$$u^m(t) = -\frac{1}{2\pi i}\int_{\Gamma}\lambda^m e^{\lambda t}\mathcal{P}^{-1}(\lambda)\sum_{k=0}^{n-1}\mathcal{Q}_k(\lambda)\,u_k d\lambda, \tag{3.11}$$

for $t > -\dfrac{\sigma}{\cos\theta}$, and Γ is the boundary of $\mathcal{G}(a,\theta)$.
It follows that

$$\|u^m(t)\|_{\mathcal{X}} \leqslant \frac{C}{2\pi}\int_{\Gamma}|\lambda|^m e^{tRe\lambda}\left\|\sum_{k=0}^{n-1}\mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\,u_k\right\|_{\mathcal{X}} d\lambda. \tag{3.12}$$

As a consequence of Corollary 2.2, we can able to write

$$\|u^m(t)\|_{\mathcal{X}} \leqslant \frac{C}{2\pi}\int_{\Gamma}|\lambda|^m e^{tRe\lambda}e^{\sigma|\lambda|}\sum_{k=0}^{n-1} f_k(|\lambda|)\,\|u_k\|_{\mathcal{X}}\,d\lambda, \tag{3.13}$$

for $\lambda\in\mathcal{G}(a,\theta)$, where

$$|\lambda| \leqslant a+r, \text{ and } Re\lambda = a + r\cos\theta.$$

By using parametrisation $\lambda = a + re^{i\theta}$, then the estimate (3.13) can be written as follows

$$\|u^m(t)\|_{\mathcal{X}} \leqslant \frac{C}{\pi}\int_0^{\infty}(a+r)^m e^{t(a+r\cos\theta)}e^{\sigma(a+r)}\sum_{k=0}^{n-1} f_k(r)\|u_k\|_{\mathcal{X}}\,dr,$$

therefore

$$\begin{aligned}\|u^m(t)\|_{\mathcal{X}} &\leqslant \frac{C}{\pi}e^{t(a-a\cos\theta)}\int_0^{\infty} e^{(\sigma+t\cos\theta)(a+r)}\sum_{k=0}^{n-1}(a+r)^m f_k(r+a)\|u_k\|_{\mathcal{X}}\,dr \leqslant \\ &\leqslant \frac{C}{\pi}e^{t(a-a\cos\theta)}\sum_{k=0}^{n-1}\int_0^{\infty} e^{(\sigma+t\cos\theta)(a+r)}(a+r)^m f_k(r+a)\|u_k\|_{\mathcal{X}}\,dr. \quad (3.14)\end{aligned}$$

If we let

$$\mathcal{F}_k(t) = \int_0^{\infty} e^{(\sigma+t\cos\theta)(a+r)}(a+r)^m f_k(r+a)\,dr, \quad (3.15)$$

Make substutition of (3.9) into (3.15), we have

$$\begin{aligned}\mathcal{F}_k(t) &= \sum_{i=k+1}^{n}\int_0^{\infty} e^{(\sigma+t\cos\theta)(a+r)}(a+r)^{m+i-k-1}\,dr \\ &= \sum_{i=k+1}^{n}\frac{(m+i-k-1)!}{(\sigma+t\cos\theta)^{m+i-k}}.\end{aligned}$$

On the other hand, before proceeding next step, we first introduce this assumption
H_4) there exists $\alpha > 0$, such that $|\sigma + t\cos\theta| > \alpha$.
By assumption (H_4), we obtain

$$\mathcal{F}_k(t) \leqslant \sum_{i=k+1}^{n}\frac{(m+i-k-1)!}{\alpha^{m+i-k}}. \quad (3.16)$$

Due (3.16) and (3.14), we obtain

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Ne^{\omega t}\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{X}},$$

for

$$t > \frac{-\sigma}{\cos\theta}, \quad m = 0,\dots,n-1,$$

where

$$N = \frac{C}{\pi}\max_{\substack{k=0,\dots n-1\\ m=0,\dots n-1}}\sum_{i=k+1}^{n}\frac{(m+i-k-1)!}{\alpha^{m+i-k}},$$

and

$$\omega = a - a\cos\theta < 2a.$$

Lemma 3.4. *If $\lambda \in \mathcal{G}(a,\theta)$, and Γ is the bound of $\mathcal{G}(a,\theta)$, we have*

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\lambda^{m-k-1}d\lambda = \begin{cases} 0 & k < m, \\ \dfrac{t^{k-m}}{(k-m)!} & k \geqslant m. \end{cases}$$

Proof. Let us introduice the sets $\Gamma_R,\ \Delta_R$, and Δ as follows:

$$\Gamma_R = \{\lambda \in \Gamma,\ \ |\lambda| \leqslant R\}.$$

$$\Delta_R = \{\lambda \in \mathbb{C} \setminus \mathcal{G}(a, \theta),\ \ |\lambda| = R\}.$$

$$\Delta = \Gamma_R \cup \Delta_R.$$

We have:

$$\int_\Delta e^{\lambda t} \lambda^{m-k-1} d\lambda = \int_{\Gamma_R} e^{\lambda t} \lambda^{m-k-1} d\lambda + \int_{\Delta_R} e^{\lambda t} \lambda^{m-k-1} d\lambda. \tag{3.17}$$

We set

$$\psi(\lambda) = e^{\lambda t} \lambda^{m-k-1}.$$

By using the parametrisation $\lambda = re^{i\omega}$, for $\lambda \in \Delta_R$, with $\omega \in [\pi - \theta, \pi + \theta]$, $0 < \theta < \dfrac{\pi}{2}$, we can write:

$$|\lambda \psi(\lambda)| = \left|e^{\lambda t} \lambda^{m-k}\right| = \left|e^{\lambda t}\right| \left|\lambda^{m-k}\right| = e^{tRe\lambda} R^{m-k} = e^{tR\cos\theta} R^{m-k}.$$

Thus, if $k < m$, it is clear that

$$\lim_{R \to \infty} e^{tR\cos\theta} R^{m-k} = 0,$$

and by virtue of Jordan lemma, we obtain

$$\lim_{R \to \infty} \int_{\Delta_R} e^{\lambda t} \lambda^{m-k-1} d\lambda = 0, \tag{3.18}$$

and

$$\int_\Delta e^{\lambda t} \lambda^{m-k-1} d\lambda = 0. \tag{3.19}$$

In light of (3.17), (3.18) and (3.19), we have

$$\begin{aligned}
\int\limits_\Gamma e^{\lambda t} \lambda^{m-k-1} d\lambda &= \lim_{R \to \infty} \int_{\Gamma_R} e^{\lambda t} \lambda^{m-k-1} d\lambda = \\
&= \lim_{R \to \infty} \int_\Delta e^{\lambda t} \lambda^{m-k-1} d\lambda - \lim_{R \to \infty} \int_{\Delta_R} e^{\lambda t} \lambda^{m-k-1} d\lambda = \\
&= 0 - 0 = 0.
\end{aligned}$$

On the other hand, if $k \geqslant m,\ k + 1 > m$, we can write

$$\begin{aligned}
\int_\Delta e^{\lambda t} \lambda^{m-k-1} d\lambda &= \int_\Delta \frac{e^{\lambda t}}{\lambda^{k+1-m}} d\lambda = \\
&= \frac{2\pi i}{(k-m)!} \psi^{k-m}(0) = \\
&= \frac{2\pi i}{(k-m)!} t^{k-m},
\end{aligned} \tag{3.20}$$

which leads to

$$\begin{aligned}
\int_\Gamma e^{\lambda t} \lambda^{m-k-1} d\lambda &= \lim_{R \to \infty} \int_{\Gamma_R} e^{\lambda t} \lambda^{m-k-1} d\lambda = \\
&= \lim_{R \to \infty} \int_\Delta e^{\lambda t} \lambda^{m-k-1} d\lambda - \lim_{R \to \infty} \int_{\Delta_R} e^{\lambda t} \lambda^{m-k-1} d\lambda = \\
&= \frac{2\pi i}{(k-m)!} t^{k-m}.
\end{aligned} \tag{3.21}$$

Hence, by virtue of (3.21), we obtain

$$\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\lambda^{m-k-1}d\lambda = \frac{t^{k-m}}{(k-m)!}, \tag{3.22}$$

for $k \geqslant m$.

4. Main results

In this section, we use the results obtained in the previous section to obtain new appropriate conditions ensuring that of any solution u of problem (1.1), (1.2) and its derivatives $u', \dots, u^{(p-1)}$ at any fixed point $t = t_0 > 0$ continuously dependent on all initial data (1.2).
We are now in a position to establish the following results.

Theorem 4.1. *If hypotheses* (H_1) *and* (H_3) *of Lemma 2.1. are satisfied and, further assume that*

$$\left\|\mathcal{Q}_j^{-1}(\lambda) A_k x\right\|_{\mathcal{X}} \leqslant C\,|\lambda|^{j-p+2}\,\|x\|_{\mathcal{X}}, \tag{4.1}$$

for fixed p *in* $\{1, \dots, n\}$ *and* $k = 1, \dots, n-1$ *. Then problem* (1.1), (1.2) *is n-fold well-posed.*

Proof. From (1.5), we have

$$\begin{aligned}\mathcal{Q}_j^{-1}(\lambda) A_n &= \lambda^{j-n+1}\mathcal{Q}_j^{-1}(\lambda)\left(\mathcal{Q}_j(\lambda) - \sum_{k=j+1}^{n-1} A_k\lambda^{k-j-1}\right) = \\ &= \lambda^{j-n+1}I - \sum_{k=j+1}^{n-1}\mathcal{Q}_j^{-1}(\lambda)\lambda^{k-n}A_k.\end{aligned}$$

Due the condition (4.1), we get

$$\left\|\mathcal{Q}_j^{-1}(\lambda) A_n\right\|_{\mathcal{X}} \leqslant |\lambda|^{j-n+1} + C\sum_{k=j+1}^{n-1}|\lambda|^{k+j-n-p+2}. \tag{4.2}$$

From the above inequality (4.2), it can be easy to write

$$\left\|\mathcal{Q}_j^{-1}(\lambda) A_n\right\|_{\mathcal{X}} \leqslant |\lambda|^{j-p+1}\left(|\lambda|^{p-n} + C\sum_{k=j+1}^{n-1}|\lambda|^{k-n+1}\right). \tag{4.3}$$

Since $p \leqslant n$ and $k \leqslant n-1$, there exists $G_2 > 0$ such that:

$$\left\|\mathcal{Q}_j^{-1}(\lambda) A_n\right\|_{\mathcal{X}} \leqslant G_2\,|\lambda|^{j-p+1}. \tag{4.4}$$

This together with (4.1), leads that the hypotheses of Lemma 3.1 are verified. By virtue of Lemma 3.3, if u is a solution of the problem (1.1), (1.2), for any $\sigma > 0$

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Me^{a(1+\cos\theta)t}\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{X}}, \tag{4.5}$$

for $t > -\dfrac{\sigma}{\cos\theta}$, and $m = 0, 1 \dots, n-1$. For an arbitrary positive t, we let $t_0 = \dfrac{t}{2}$, and $\sigma_0 = -t_0\cos\theta$. We have $t > \dfrac{-\sigma_0}{\cos\theta}$, so that

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Me^{at(1+\cos\theta)}\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{X}}, \quad t > 0, \tag{4.6}$$

for $m = 0, \ldots, n-1$.
On the other hand, since $\cos\theta < 0$ which implies that $1 + \cos\theta < 1$. It follows from (4.6) that:

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Me^{at} \sum_{k=0}^{n-1} \|u_k\|_{\mathcal{X}}, \quad t > 0, \tag{4.7}$$

for $m = 0, \ldots, n-1$. Thus, the problem (1.1), (1.2) is n-fold well-posed.

Theorem 4.2. *If the conditions of Theorem 3.1 are satisfied, then the problem* (1.1), (1.2) *is p-fold exponentially well-posed.*

Proof. The remaining part of the proof is similar to Theorem 3.1. There exists $M' > 0$, such that

$$\|u^m(t)\|_{\mathcal{X}} \leqslant M'e^{at} \sum_{k=0}^{n-1} \|u_k\|_{\mathcal{X}}, \tag{4.8}$$

for $t \geqslant 1$, and $m = 0, 1 \ldots, n-1$. Suppose now $t \in]0,1[$, from (1.3), (1.4) and (1.5), we can get

$$\mathcal{Q}_k(\lambda) = \frac{1}{\lambda^{k+1}} \left(\mathcal{P}(\lambda) - \mathcal{P}_k(\lambda)\right), \tag{4.9}$$

so that

$$\begin{aligned}\sum_{k=0}^{n-2} \mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\,u_k &= \sum_{k=0}^{n-2} \lambda^{-k-1}\mathcal{P}^{-1}(\lambda)\left(\mathcal{P}(\lambda) - \mathcal{P}_k(\lambda)\right)u_k = \\ &= \sum_{k=0}^{n-2} \lambda^{-k-1}u_k - \sum_{k=0}^{n-2} \lambda^{-k-1}\mathcal{P}^{-1}(\lambda)\,\mathcal{P}_k(\lambda)\,u_k. \end{aligned} \tag{4.10}$$

Hence, we can get

$$\begin{aligned} u^m(t) &= -\frac{1}{2\pi i}\int_\Gamma \lambda^m e^{\lambda t} \sum_{k=0}^{n-2} \mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\,u_k d\lambda = \\ &= -\frac{1}{2\pi i}\int_\Gamma \lambda^m e^{\lambda t} \sum_{k=0}^{n-2} \mathcal{P}^{-1}(\lambda)\,\mathcal{Q}_k(\lambda)\,u_k d\lambda - \frac{1}{2\pi i}\int_\Gamma \lambda^m e^{\lambda t}\mathcal{P}^{-1}(\lambda)\,A_n u_{n-1} d\lambda = \\ &= -\frac{1}{2\pi i}\int_\Gamma \sum_{k=0}^{n-2} \lambda^{m-k-1} e^{\lambda t} u_k d\lambda + \frac{1}{2\pi i}\int_\Gamma \sum_{k=0}^{n-2} \lambda^{m-k-1} e^{\lambda t}\mathcal{P}^{-1}(\lambda)\,\mathcal{P}_k(\lambda)\,u_k d\lambda - \\ &- \frac{1}{2\pi i}\int_\Gamma \lambda^m e^{\lambda t}\mathcal{P}^{-1}(\lambda)\,A_n u_{n-1} d\lambda. \end{aligned} \tag{4.11}$$

From (4.11) and (3.5), we get

$$\begin{aligned} u^m(t) &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda t} \sum_{k=0}^{n-2} \lambda^{m-k-1} u_k d\lambda + \\ &+ \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} \sum_{k=0}^{n-2} \lambda^{m-k-j-2}\mathcal{R}_j(\lambda)\,\mathcal{Q}_j^{-1}(\lambda)\,\mathcal{P}_k(\lambda)\,u_k d\lambda - \\ &- \frac{1}{2\pi i}\int_\Gamma \lambda^{m-j-1} e^{\lambda t}\mathcal{R}_j(\lambda)\,\mathcal{Q}_j^{-1}(\lambda)\,A_n u_{n-1} d\lambda. \end{aligned} \tag{4.12}$$

In view of Lemma 3.4, we obtain

$$\begin{aligned} u^m(t) &= -\sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} u_k + \\ &+ \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \sum_{k=0}^{n-2} \lambda^{m-k-j-2} \mathcal{R}_j(\lambda) \mathcal{Q}_j^{-1}(\lambda) \mathcal{P}_k(\lambda) u_k d\lambda - \\ &- \frac{1}{2\pi i} \int_\Gamma \lambda^{m-j-1} e^{\lambda t} \mathcal{R}_j(\lambda) \mathcal{Q}_j^{-1}(\lambda) A_n u_{n-1} d\lambda. \end{aligned} \tag{4.13}$$

Let Γ_t be the sector of the complex plane defined by

$$\Gamma_t = \left\{ \lambda = \frac{a}{t} + re^{i\varphi},\ |\arg| \leqslant \theta \right\},$$

for $t \in]0,1[$, where a and θ are defined in $\mathcal{G}(a,\theta)$.
The functions under the sign of integration in (4.13) are analytic, so we can write

$$\begin{aligned} u^m(t) &= -\sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} u_k + \\ &+ \frac{1}{2\pi i} \int_{\Gamma_t} e^{\lambda t} \sum_{k=0}^{n-2} \lambda^{m-k-j-2} \mathcal{R}_j(\lambda) \mathcal{Q}_j^{-1}(\lambda) \mathcal{P}_k(\lambda) u_k d\lambda - \\ &- \frac{1}{2\pi i} \int_{\Gamma_t} \lambda^{m-j-1} e^{\lambda t} \mathcal{R}_j(\lambda) \mathcal{Q}_j^{-1}(\lambda) A_n u_{n-1} d\lambda. \end{aligned} \tag{4.14}$$

If we set $\mathcal{Z} = \lambda t$. Due (4.14), it is easy to verify that

$$\begin{aligned} u^m(t) &= -\sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} u_k + \\ &+ \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{t} \sum_{k=0}^{n-2} \left(\frac{\mathcal{Z}}{t}\right)^{m-k-j-2} \mathcal{R}_j\left(\frac{\mathcal{Z}}{t}\right) \mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right) \mathcal{P}_k\left(\frac{\mathcal{Z}}{t}\right) u_k d\mathcal{Z} - \\ &- \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{t} \left(\frac{\mathcal{Z}}{t}\right)^{m-j-1} \mathcal{R}_j\left(\frac{\mathcal{Z}}{t}\right) \mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right) A_n u_{n-1} d\mathcal{Z}. \end{aligned}$$

This leads to the following estimate

$$\begin{aligned} \|u^m(t)\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} \|u_k\|_{\mathcal{X}} + \\ &+ \frac{G_0}{2\pi} \int_\Gamma \frac{e^{Re\mathcal{Z}}}{t} \sum_{k=0}^{n-2} \left|\frac{\mathcal{Z}}{t}\right|^{m-k-j-2} \left\| \mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right) \mathcal{P}_k\left(\frac{\mathcal{Z}}{t}\right) \right\|_{\mathcal{X}} \|u_k\|_{\mathcal{X}} d\mathcal{Z} + \\ &+ \frac{G_0}{2\pi} \int_\Gamma \frac{e^{Re\mathcal{Z}}}{t} \left|\frac{\mathcal{Z}}{t}\right|^{m-j-1} \left\| \mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right) A_n \right\|_{\mathcal{X}} \|u_{n-1}\|_{\mathcal{X}} d\mathcal{Z}. \end{aligned} \tag{4.15}$$

Now, we estimate the second and third terms on the right side of (4.15),
Firstly, by using relation (4.1), we obtain

$$\begin{aligned}\left\|\mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right)\mathcal{P}_k\left(\frac{\mathcal{Z}}{t}\right)\right\|_{\mathcal{X}} = \left\|\mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right)\sum_{s=0}^{k}A_s\left(\frac{\mathcal{Z}}{t}\right)^s\right\|_{\mathcal{X}} &\leqslant \sum_{s=0}^{k}\left|\frac{\mathcal{Z}}{t}\right|^s\left\|\mathcal{Q}_j^{-1}A_s\right\|_{\mathcal{X}} \leqslant \\ &\leqslant C\sum_{s=0}^{k}\left|\frac{\mathcal{Z}}{t}\right|^{j-p+2+s}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=0}^{n-2}\left|\frac{\mathcal{Z}}{t}\right|^{m-k-j-2}\left\|\mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right)\mathcal{P}_k\left(\frac{\mathcal{Z}}{t}\right)\right\|_{\mathcal{X}} &\leqslant C\sum_{k=0}^{n-2}\sum_{s=0}^{k}\left|\frac{\mathcal{Z}}{t}\right|^{m-k+s-p} = \\ &= C\sum_{k=0}^{n-2}\sum_{s=0}^{k}\left|\frac{t}{\mathcal{Z}}\right|^{p-m+k-s}. \qquad (4.16)\end{aligned}$$

Secondly, by the same way, we have

$$\left\|\mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right)A_n\right\|_{\mathcal{X}} \leqslant G_2\left|\frac{\mathcal{Z}}{t}\right|^{j+1-p}.$$

Therefore

$$\left|\frac{\mathcal{Z}}{t}\right|^{m-j-1}\left\|\mathcal{Q}_j^{-1}\left(\frac{\mathcal{Z}}{t}\right)A_n\right\|_{\mathcal{X}} \leqslant G_2\left|\frac{\mathcal{Z}}{t}\right|^{m-p} = G_2\left|\frac{t}{\mathcal{Z}}\right|^{p-m}. \qquad (4.17)$$

Substituting (4.16) and (4.17) into (4.15), we obtain

$$\begin{aligned}\left\|u^m(t)\right\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2}\frac{t^{k-m}}{(k-m)!}\left\|u_k\right\|_{\mathcal{X}} + \\ &+ \frac{G_3}{2\pi}\int_\Gamma \frac{e^{Re\mathcal{Z}}}{|\mathcal{Z}|}\sum_{k=0}^{n-2}\sum_{s=0}^{k}\left(\frac{t}{|\mathcal{Z}|}\right)^{p-m+k-s-1}\left\|u_k\right\|_{\mathcal{X}}d\mathcal{Z} + \\ &+ \frac{G_4}{2\pi}\int_\Gamma \frac{e^{Re\mathcal{Z}}}{t}\frac{t^{p-m}}{|\mathcal{Z}|^{p-m}}\left\|u_{n-1}\right\|_{\mathcal{X}}d\mathcal{Z},\end{aligned}$$

where $G_3 = CG_0$, and $G_4 = G_2G_0$. This leads

$$\begin{aligned}\left\|u^m(t)\right\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2}\frac{t^{k-m}}{(k-m)!}\left\|u_k\right\|_{\mathcal{X}} + \\ &+ \frac{G_3}{2\pi}\int_\Gamma \frac{e^{Re\mathcal{Z}}}{t}\sum_{k=0}^{n-2}\sum_{s=0}^{k}\left(\frac{t}{|\mathcal{Z}|}\right)^{p-m+k-s}\left\|u_k\right\|_{\mathcal{X}}d\mathcal{Z} + \\ &+ \frac{G_4}{2\pi}\int_\Gamma \frac{e^{Re\mathcal{Z}}}{t}\frac{t^{p-m}}{|\mathcal{Z}|^{p-m}}\left\|u_{n-1}\right\|_{\mathcal{X}}d\mathcal{Z}.\end{aligned}$$

By means of parametrisation $\mathcal{Z} = at + tre^{i\theta}$, we obtain

$$\begin{aligned} \|u^m(t)\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} \|u_k\|_{\mathcal{X}} + \\ &+ \frac{G_3}{2\pi} \int_0^\infty e^{at+tr\cos\theta} \sum_{k=0}^{n-2} \sum_{s=0}^{k} \left(\frac{1}{a+r}\right)^{p-m+k-s} \|u_k\|_{\mathcal{X}}\, dr + \\ &+ \frac{G_4}{2\pi} \int_0^\infty e^{at+tr\cos\theta} \left(\frac{1}{a+r}\right)^{p-m} \|u_{n-1}\|_{\mathcal{X}}\, dr. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \|u^m(t)\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} \|u_k\|_{\mathcal{X}} + \\ &+ \frac{G_3}{2\pi} \int_0^\infty e^{at+tr\cos\theta} \sum_{k=0}^{n-2} \sum_{s=0}^{k} (a+r)^{m-p+s-k} \|u_k\|_{\mathcal{X}}\, dr + \\ &+ \frac{G_4}{2\pi} \int_0^\infty e^{at+tr\cos\theta} (a+r)^{m-p} \|u_{n-1}\|_{\mathcal{X}}\, dr. \end{aligned}$$

A simple computation shows that

$$\begin{aligned} \|u^m(t)\|_{\mathcal{X}} &\leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} \|u_k\|_{\mathcal{X}} + \\ &+ \frac{G_3}{2\pi} e^{(a-a\cos\theta)t} \int_0^\infty e^{t\cos\theta(a+r)} \sum_{k=0}^{n-2} \sum_{s=0}^{k} (a+r)^{m-p+s-k} \|u_k\|_{\mathcal{X}}\, dr + \\ &+ \frac{G_4}{2\pi} e^{(a-a\cos\theta)t} \int_0^\infty e^{t\cos\theta(a+r)} (a+r)^{m-p} \|u_{n-1}\|_{\mathcal{X}}\, dr. \end{aligned} \tag{4.18}$$

Next, we estime the second and the third terms in the right side of (4.18).
For $m < p$, we have $m - p + s - k < 0$, so

$$(a+r)^{m-p+s-k} < (a)^{m-p+s-k}.$$

If we put $M_k = \max_{s=0,\dots k} a^{m-p+s-k}$, we obtain

$$\int_0^\infty e^{t\cos\theta(a+r)} \sum_{s=0}^{k} (a+r)^{m-p+s-k}\, dr \leqslant (k+1) M_k \int_0^\infty e^{t\cos\theta(a+r)} dr = \frac{-(k+1) M_k e^{ta\cos\theta}}{t\cos\theta}.$$

In a similar way, we can write

$$\int_0^\infty e^{t\cos\theta(a+r)} (a+r)^{m-p}\, dr \leqslant a^{m-p} \int_0^\infty e^{(a+r)t\cos\theta} dr = \frac{-a^{m-p} e^{ta\cos\theta}}{t\cos\theta}.$$

Finally

$$\begin{aligned} \|u^m(t)\|_{\mathcal{X}} \leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!} \|u_k\|_{\mathcal{X}} &+ \frac{G_3}{2\pi} \sum_{k=0}^{n-2} \frac{-(k+1) M_k e^{ta\cos\theta}}{t\cos\theta} \|u_k\|_{\mathcal{X}} + \\ &+ \frac{G_4}{2\pi} \left(\frac{-a^{m-p}}{t\cos\theta}\right) e^{at} \|u_{n-1}\|_{\mathcal{X}}. \end{aligned}$$

Since $t > 0$ and $\cos\theta < 0$, we choose that

$$G_5 = \frac{G_3}{2\pi} \max_{k=0,\dots n-2} \frac{-(k+1) M_k e^{\cos\theta}}{t\cos\theta},$$

and

$$G_6 = \frac{G_4}{2\pi}\left(\frac{-a^{m-p}}{t\cos\theta}\right),$$

so

$$\|u^m(t)\|_{\mathcal{X}} \leqslant \sum_{k=m}^{n-2} \frac{t^{k-m}}{(k-m)!}\|u_k\|_{\mathcal{X}} + G_5 e^{ta}\sum_{k=0}^{n-2}\|u_k\|_{\mathcal{X}} + G_6 e^{at}\|u_{n-1}\|_{\mathcal{X}}.$$

On the other hand, we have

$$\sum_{k=m}^{n-2}\frac{t^{k-m}}{(k-m)!}\|u_k\|_{\mathcal{X}} \leqslant \sum_{k=m}^{n-2}\|u_k\|_{\mathcal{X}}.$$

If we put $R = \max(1, G_5, G_6)$, we obtain

$$\|u^m(t)\|_{\mathcal{X}} \leqslant Re^{ta}\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{X}}.$$

This together with (4.8), it follows that the problem (1.1), (1.2) is p-fold exponentially well-posed. □

Remark 4.3. *The results presented in this paper improve and extend the main result proved in [18] under appropriate conditions. As far as we know, sufficient conditions for the p-fold well-posedness of higher-order abstract Cauchy problem expressed in terms of decay of some auxiliary pencils shown in (1.4) and (1.5) considered in the present paper have not been investigated yet. For this reason, in this paper we make the first attempt to fill this gap. The method employed in this paper is different from those in related literature (Vlasenko et al [18]).*

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О p-кратной корректности абстрактной задачи Коши высшего порядка

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Аннотация. В данной статье мы устанавливаем достаточные условия p-кратной корректности абстрактной задачи Коши высокого порядка. Эти условия выражаются через затухание некоторых вспомогательных пучков, полученных из характеристического пучка рассматриваемого операционного дифференциального уравнения. В частности, эта статья совершенствует важную и интересную работу.

Ключевые слова: абстрактные задачи Коши, интегрированные полугруппы, корректность.

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The Weighted Hardy Operators and Quasi-monotone Functions

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Abstract. Some Hardy-type inequalities are established by W. T. Sulaiman. The aim of this work is to extend these inequalities for weighted Hardy operators with quasi-monotone functions. Moreover some new integral weighted inequalities were obtained.

Keywords: inequalities, Hardy's operator, quasi-monotones functions.

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1. Introduction and preliminaries

The classical Hardy inequality (See [2]) has been proved for $f(x) \geqslant 0$, $p > 1$

$$\int_0^{+\infty} \left(\frac{F(x)}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1}$$

where

$$F(x) = \int_0^x f(t)dt. \tag{2}$$

The constant $\left(\dfrac{p}{p-1}\right)^p$ is sharp (the best possible).

This inequality have many applications in the theory of differential equations (Ordinary and Partial) and led to many interesting questions and connections between different areas of mathematical analysis.

The following inequalities were proved in [4].

Let $f \geqslant 0$, $g > 0$.

1. If $\dfrac{x}{g(x)}$ is a non-increasing function, $p > 1$ and $0 < a < 1$, then

$$\int_0^{\infty} \left(\frac{F(x)}{g(x)}\right)^p dx \leqslant \frac{1}{a(p-1)(1-a)^{p-1}} \int_0^{\infty} \left(\frac{tf(t)}{g(t)}\right)^p dt. \tag{3}$$

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2. If $\dfrac{x}{g(x)}$ is a non-decreasing function, $0 < p < 1$ and $a > 0$, then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geqslant \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^p dt. \tag{4}$$

3. If $p \geqslant 2$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leqslant \int_0^\infty t^{-1} f^{p-1}(t) F(t) dt. \tag{5}$$

4. If $p > 1$, $h \geqslant 0$, h convex and non-decreasing function, then

$$\int_0^\infty h^p \left(\frac{F(x)}{x}\right) dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty h^p \left(f(t)\right) dt. \tag{6}$$

The Hardy inequalities for quasi-monotone functions are discussed for example in [1] and [3].

The objective of this work is to generalize the inequalities (3)–(5) and (6) for weighted Hardy operator and its dual with quasi-monotone functions. Moreover, other integral inequalities were obtained for quasi-monotone functions.

Throughtout this paper, we will assume that functions are non-negative integrable and the integrals are supposed to exist and are finite.

2. Main results

Consider the weighted Hardy operator and its dual

$$F_w(x) = \int_0^x f_w(t)dt, \ F_w^*(x) = \int_x^\infty f_w(t)dt,$$

where $f_w(t) = f(t)w(t)$ and $g > 0$, $f \geqslant 0$ are Lebesgue measurable functions on $(0, \infty)$. Let ω and $v > 0$ be weight functions on $(0, \infty)$, $V(x) = \int\limits_0^x v(t)dt$, $V^*(x) = \int\limits_x^\infty v(t)dt$, and $G_V(x) = g(x)V(x)$, $G_{V^*}(x) = g(x)V^*(x)$.

The following definition was introduced in [1].

Definition 1. *We say that a non-negative function f is quasi-monotone on $]0, \infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$, we say that $f \in Q_\beta$ if $x^{-\beta} f(x)$ is non-increasing and $f \in Q^\beta$ if $x^{-\beta} f(x)$ is non-decreasing.*

Theorem 1. *Let $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q_\beta$, $0 < a < 1$ and $\beta < a(\dfrac{p-1}{p})$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx \leqslant \frac{1}{(a(p-1) - p\beta)(1-a)^{p-1}} \int_0^\infty \left(\frac{tf_w(t)}{G_V(t)}\right)^p dt. \tag{7}$$

Proof. Since $\dfrac{x}{g(x)} \in Q_\beta$ and $V(x)$ is non-decreasing, then $\dfrac{x}{G_V(x)} \in Q_\beta$.

Let $K = \int\limits_0^\infty \left(\dfrac{F_w(x)}{G_V(x)}\right)^p dx$, then

$$\begin{aligned} K &= \int_0^\infty G_V^{-p}(x) \left(\int_0^x f_w(t)dt\right)^p dx = \\ &= \int_0^\infty G_V^{-p}(x) \left(\int_0^x t^{a(1-\frac{1}{p})} f_w(t) t^{-a(1-\frac{1}{p})} dt\right)^p dx. \end{aligned}$$

$\dfrac{x}{G_V(x)} \in Q_\beta$, implies that $\left(\dfrac{x}{G_V(x)}\right)^p \in Q_\beta$. By Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned} K &\leqslant \int_0^\infty G_V^{-p}(x)\left(\int_0^x t^{a(p-1)} f_w^p(t)dt\right)\left(\int_0^x t^{-a}dt\right)^{p-1} dx = \\ &= \frac{1}{(1-a)^{p-1}} \int_0^\infty x^{(1-a)(p-1)} G_V^{-p}(x) \int_0^x t^{a(p-1)} f_w^p(t) dt dx = \\ &= \frac{1}{(1-a)^{p-1}} \int_0^\infty t^{a(p-1)} f_w^p(t) \int_t^\infty x^{(1-a)(p-1)} G_V^{-p}(x) dx dt \leqslant \\ &\leqslant \frac{1}{(1-a)^{p-1}} \int_0^\infty t^{a(p-1)} f_w^p(t) \left(\frac{t^{1-\beta}}{G_V(t)}\right)^p \int_t^\infty x^{p\beta - a(p-1)-1} dx dt = \\ &= \frac{1}{(a(p-1)-p\beta)\,(1-a)^{p-1}} \int_0^\infty \left(\frac{t f_w(t)}{G_V(t)}\right)^p dt. \end{aligned}$$

□

Remark 1. *If in* (7)*, we set* $w(x) = 1$*,* $V(x) = 1$ *and* $\beta = 0$*, we get the inequality* (3)*.*

Now we consider the converse inequality of (2.1).

Theorem 2. *Let* $0 < p < 1$*,* $f \geqslant 0$*,* $g > 0$*,* $\dfrac{x}{g(x)} \in Q^\beta$*,* $a > 0$*,* $\beta < a\left(\dfrac{1-p}{p}\right)$ *and* $\beta \in \mathbb{R}$*, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_{V^*}(x)}\right)^p dx \geqslant \frac{1}{(a(1-p)-p\beta)\,(1+a)^{p-1}} \int_0^\infty \left(\frac{t f_w(t)}{G_{V^*}(t)}\right)^p dt. \tag{8}$$

Proof. Since $\dfrac{x}{g(x)} \in Q^\beta$ and $V^*(x)$ is non-increasing, then $\left(\dfrac{x}{G_{V^*}(x)}\right)^p \in Q^\beta$.

Let $I = \int\limits_0^\infty \left(\dfrac{F_w(x)}{G_{V^*}(x)}\right)^p dx$, then

$$\begin{aligned} I &= \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x f_w(t)dt\right)^p dx = \\ &= \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x t^{a(\frac{1}{p}-1)} f_w(t) t^{-a(\frac{1}{p}-1)} dt\right)^p dx. \end{aligned}$$

By converse Hölder inequality and Fubini's theorem, we get

$$\begin{aligned} I &\geqslant \int_0^\infty G_{V^*}^{-p}(x)\left(\int_0^x t^{a(1-p)} f_w^p(t)dt\right)\left(\int_0^x t^{+a}dt\right)^{p-1} dx = \\ &= \frac{1}{(1+a)^{p-1}} \int_0^\infty x^{(1+a)(p-1)} G_{V^*}^{-p}(x) \int_0^x t^{a(1-p)} f_w^p(t) dt dx = \\ &= \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{a(1-p)} f_w^p(t) \int_t^\infty x^{(1+a)(p-1)} G_{V^*}^{-p}(x) dx dt \geqslant \\ &\geqslant \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{a(1-p)} f_w^p(t) \left(\frac{t^{1-\beta}}{G_{V^*}(t)}\right)^p \int_t^\infty x^{p\beta - a(1-p)-1} dx dt = \\ &= \frac{1}{((a(1-p)-p\beta)\,(1+a)^{p-1}} \int_0^\infty \left(\frac{t f_w(t)}{G_{V^*}(t)}\right)^p dt. \end{aligned}$$

□

Remark 2. *By setting* $w(x) = 1$*,* $V^*(x) = 1$ *and* $\beta = 0$*, in* (8)*, we obtain the inequality* (4)*.*

Theorem 3. *Let* $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q^{\beta}$, $a > 1$ *and* $\beta > a(\dfrac{p-1}{p})$, *then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{G_{V^*}(x)}\right)^p dx \leqslant \frac{1}{(a(p-1)-p\beta)\,(a-1)^{p-1}} \int_0^{\infty} \left(\frac{tf_w(t)}{G_{V^*}(t)}\right)^p dt. \tag{9}$$

Proof. Let $K_1 = \int\limits_0^{\infty} \left(\dfrac{F_w^*(x)}{G_{V^*}(x)}\right)^p dx$, thus

$$\begin{aligned} K_1 &= \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} f_w(t)dt\right)^p dx = \\ &= \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} t^{a(1-\frac{1}{p})} f_w(t) t^{-a(1-\frac{1}{p})} dt\right)^p dx. \end{aligned}$$

By applying Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned} K_1 &\leqslant \int_0^{\infty} G_{V^*}^{-p}(x) \left(\int_x^{\infty} t^{a(p-1)} f_w^p(t)dt\right) \left(\int_x^{\infty} t^{-a}dt\right)^{p-1} dx = \\ &= \frac{1}{(a-1)^{p-1}} \int_0^{\infty} x^{(a-1)(p-1)} G_{V^*}^{-p}(x) \int_x^{\infty} t^{a(p-1)} f_w^p(t)dtdx = \\ &= \frac{1}{(a-1)^{p-1}} \int_0^{\infty} t^{a(p-1)} f_w^p(t) \int_0^{t} x^{(1-a)(p-1)} G_{V^*}^{-p}(x)dxdt \leqslant \\ &\leqslant \frac{1}{(a-1)^{p-1}} \int_0^{\infty} t^{a(p-1)} f_w^p(t) \left(\frac{t^{1-\beta}}{G_{V^*}(t)}\right)^p \int_0^{t} x^{p\beta - a(p-1)-1}dxdt = \\ &= \frac{1}{(a(p-1)-p\beta)\,(a-1)^{p-1}} \int_0^{\infty} \left(\frac{tf_w(t)}{G_{V^*}(t)}\right)^p dt. \end{aligned}$$

□

If in (9), we set $w(x) = 1$, $V(x)^* = 1$ and $\beta = 0$, we obtain the following corollary.

Corollary 1. *Let* $p > 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)}$ *non-decreasing function,* $a > 1$, *then*

$$\int_0^{\infty} \left(\frac{F^*(x)}{g(x)}\right)^p dx \leqslant \frac{1}{(a(p-1)(a-1)^{p-1}} \int_0^{\infty} \left(\frac{tf(t)}{g(t)}\right)^p dt. \tag{10}$$

Theorem 4. *Let* $0 < p < 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)} \in Q^{\beta}$, $a < -1$, $\beta > a\left(\dfrac{1-p}{p}\right)$ *and* $\beta \in \mathbb{R}$, *then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{G_V(x)}\right)^p dx \geqslant \frac{1}{(p\beta - a(p-1))\,((-1)(1+a))^{p-1}} \int_0^{\infty} \left(\frac{tf_w(t)}{G_V(t)}\right)^p dt. \tag{11}$$

The proof is similar to that of Theorem 3.
If in (11), $w(x) = 1$, $V(x) = 1$ and $\beta = 0$, we get the following corollary.

Corollary 2. *Let* $0 < p < 1$, $f \geqslant 0$, $g > 0$, $\dfrac{x}{g(x)}$ *non-decreasing function,* $a < -1$, *then*

$$\int_0^{\infty} \left(\frac{F_w^*(x)}{g(x)}\right)^p dx \geqslant \frac{1}{(a(1-p)-p\beta)\,(1+a)^{p-1}} \int_0^{\infty} \left(\frac{tf_w(t)}{g(t)}\right)^p dt. \tag{12}$$

Remark 3. *The inequalities (10) and (12) are the analogs of inequalities (3) and (4) respectively.*

Theorem 5. *Let $p \geqslant 2$, $f \geqslant 0$, $g > 0$, $\frac{x}{g(x)} \in Q_\beta$ and $\beta < \frac{1}{p}$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx \leqslant \frac{1}{1-p\beta}\int_0^\infty G_V^{-1}(t)\left(\frac{tf_w(t)}{G_V(t)}\right)^{p-1} F_w(t)dt. \tag{13}$$

Proof. By applying Hölder's inequality with parameters $p-1$ and its conjugate $\frac{p-1}{p-2}$ and the assumption $\frac{x}{g(x)} \in Q_\beta$, it follows that

$$\begin{aligned}
\int_0^\infty \left(\frac{F_w(x)}{G_V(x)}\right)^p dx &= \int_0^\infty G_V^{-p}(x)F_w^{p-1}(x)F_w(x)dx = \\
&= \int_0^\infty G_V^{-p}(x)F_w^{p-1}(x)\int_0^x f_w(t)dtdx = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)F_w^{p-1}(x)dxdt = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)\left(\int_0^x f_w(u)du\right)^{p-1} dxdt \leqslant \\
&\leqslant \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)\int_0^x f_w^{p-1}(u)du\left(\int_0^x du\right)^{p-2} dxdt = \\
&= \int_0^\infty f_w(t)\int_t^\infty G_V^{-p}(x)x^{p-2}\left(\int_0^x f_w^{p-1}(u)du\right) dxdt \leqslant \\
&\leqslant \int_0^\infty f_w(t)\int_t^\infty f_w^{p-1}(u)\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p\int_u^\infty x^{p\beta-2}dxdudt = \\
&= \frac{1}{1-p\beta}\int_0^\infty f_w(t)\int_t^\infty f_w^{p-1}(u)\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p u^{p\beta-1}dudt = \\
&= \frac{1}{1-p\beta}\int_0^\infty f_w^{p-1}(u)u^{p\beta-1}\left(\frac{u^{1-\beta}}{G_V(u)}\right)^p\int_0^u f_w(t)dtdu = \\
&= \frac{1}{1-p\beta}\int_0^\infty G_V^{-1}(u)\left(\frac{uf_w(u)}{G_V(u)}\right)^{p-1} F_w(u)du.
\end{aligned}$$

□

Further, setting $V(x) = 1$ and $g(x) = x$ in Theorem 5, yields the following corollary.

Corollary 3. *Let $p \geqslant 2$, $f \geqslant 0$, $\beta < \frac{1}{p}$, then*

$$\int_0^\infty \left(\frac{F_w(x)}{x}\right)^p dx \leqslant \frac{1}{1-p\beta}\int_0^\infty t^{-1}f_w^{p-1}(t)F_w(t)dt. \tag{14}$$

Remark 4. *If we take $w = 1$ and $\beta = 0$ in (14), we obtain inequality (5).*

Theorem 6. *Let $p > 1$, $\beta < \frac{1}{p}(1-\frac{1}{p})$, $h > 0$ be a convex function, $\frac{x}{g(x)} \in Q_\beta$, then*

$$\int_0^\infty x^{p\beta}h^p\left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)dx \leqslant$$

$$\leqslant \left(\frac{p}{p-p^2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta}h^p\left(\frac{(t^{1-\beta}f_w(t))}{G_V(t)}\right)dt. \tag{15}$$

Proof. By using the convexity of h and Hölder's inequality, we get

$$\int_0^\infty x^{p\beta}h^p\left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)^p dx = \int_0^\infty \left(x^\beta h\left(x^{-\beta}\frac{F_w(x)}{G_V(x)}\right)\right)^p dx =$$

$$= \int_0^\infty \left(x^\beta h\left(\frac{1}{x}\frac{x^{1-\beta}}{G_V(x)}\int_0^x f_w(t)dt\right)\right)^p dx \leqslant$$

$$\leqslant \int_0^\infty \left(\frac{x^\beta}{x}\int_0^x h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\right)^p dx =$$

$$= \int_0^\infty x^{p(\beta-1)}\left(\int_0^x h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\right)^p dx =$$

$$= \int_0^\infty x^{p(\beta-1)}\left(\int_0^x t^{\frac{1}{p}(1-\frac{1}{p})}h\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)t^{-\frac{1}{p}(1-\frac{1}{p})}dt\right)^p dx \leqslant$$

$$\leqslant \int_0^\infty x^{p(\beta-1)}\int_0^x t^{1-\frac{1}{p}}h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dt\left(\int_0^x t^{-\frac{1}{p}}dt\right)^{p-1} dx =$$

$$= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty x^{p\beta+\frac{1}{p}-2}\int_0^x t^{1-\frac{1}{p}}h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)dtdx =$$

$$= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}h^p\left(\frac{t^{1-\beta}f_w(t)}{G_V(t)}\right)\int_t^\infty x^{p\beta+\frac{1}{p}-2}dxdt =$$

$$= \left(\frac{p}{p-p2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta}h^p\left(\frac{(t^{1-\beta}f_w(t))}{G_V(t)}\right)dt.$$

□

If in (15), we set $V(x)=1$ and $g(x)=x$, we get the following corollary.

Corollary 4. *Let $p>1$, $\beta<\frac{1}{p}\Big(1-\frac{1}{p}\Big)$, $f\geqslant 0$, $h>0$, h be a convex and non-decreasing function, then*

$$\int_0^\infty x^{p\beta}h^p\left(x^{-\beta}\frac{F_w(x)}{x}\right)dx \leqslant \left(\frac{p}{p-p^2\beta-1}\right)\left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{p\beta}h^p\left(t^{-\beta}f_w(t)\right)dt. \tag{16}$$

Remark 5. *If in* (16)*, we put $w=1$ and $\beta=0$, we get* (6)*.*

The following lemma was proved in [4].

Lemma 1. *Let $h\geqslant 0$ be convex, and $h(0)=0$, then $h(x)/x$ is non-decreasing.*

Theorem 7. *Let $p>1$, $h>0$ be a convex, non-decreasing function $h(0)=0$, $g>0$, $\frac{x}{g(x)}\in Q_\beta$ and $\beta<p-1$, then*

$$\int_0^\infty \frac{x^{2-p+\beta}h\left(\frac{x^{-\beta}F_w(x)}{G_V(x)}\right)}{h(x)}dx \leqslant \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}h\left(\frac{t^{-\beta}f_w(t)}{G_V(t)}\right)}{h(t)}dt. \tag{17}$$

Proof. Let $I_1=\int\limits_0^\infty \frac{x^{2-p+\beta}h\left(\frac{x^{-\beta}F_w(x)}{G_V(x)}\right)}{h(x)}dx$, then

$$I_1 = \int_0^\infty \frac{x^{2-p+\beta} h\left(\frac{1}{x}\frac{x^{1-\beta}}{G_V(x)}\int_0^x f_w(t)dt\right)}{h(x)} dx = \\ = \int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{x^{1-\beta}}{G_V(x)} f_w(t)dt\right) dx.$$

Since $\frac{x}{g(x)} \in Q_\beta$, h is non-decreasing convex and $\frac{x}{G_V(x)} \in Q_\beta$, we have

$$\int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{x^{1-\beta}}{G_V(x)} f_w(t)dt\right) dx \leqslant \int_0^\infty \frac{x^{2-p+\beta}}{h(x)} h\left(\frac{1}{x}\int_0^x \frac{t^{1-\beta} f_w(t)}{G_V(t)} dt\right) dx \leqslant \\ \leqslant \int_0^\infty \frac{x^{1-p+\beta}}{h(x)} \int_0^x h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dtdx.$$

Applying Fubini's theorem and Lemma 1, we get

$$\int_0^\infty \frac{x^{1-p+\beta}}{h(x)} \int_0^x h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dtdx = \int_0^\infty h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) \int_t^\infty \frac{x^{1-p+\beta}}{h(x)} dxdt \leqslant \\ \leqslant \int_0^\infty h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right)\left(\frac{t}{h(t)}\right) \int_t^\infty x^{\beta-p} dxdt = \\ = \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}}{h(t)} h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dt.$$

Thus

$$\int_0^\infty \frac{x^{2-p+\beta} h\left(\frac{x^{-\beta} F_w(x)}{G_V(x)}\right)}{h(x)} dx \leqslant \frac{1}{p-\beta-1}\int_0^\infty \frac{t^{2-p+\beta}}{h(t)} h\left(\frac{t^{1-\beta} f_w(t)}{G_V(t)}\right) dt.$$

□

If in (17), we set $V(x) = 1$, $v = \omega = 1$ and $\beta = 0$, we have the following corollary.

Corollary 5. *Let $p > 1$, $h > 0$ be a convex, non-decreasing function $h(0) = 0$, $g > 0$, $\frac{x}{g(x)}$ non-increasing function, then*

$$\int_0^\infty \frac{x^{2-p} h\left(\frac{x^{-\beta} F(x)}{g(x)}\right)}{h(x)} dx \leqslant \frac{1}{p-1}\int_0^\infty \frac{t^{2-p} h\left(\frac{t^{1} f_w(t)}{g(t)}\right)}{h(t)} dt. \tag{18}$$

Definition 2. *A function h is said to be submultiplicative if $h(xy) \leqslant h(x)h(y)$.*

Theorem 8. *Let $p > 1$, $f \geqslant 0$, $\beta \in \mathbb{R}$, $\beta < p$. If $h \geqslant 0$ is a convex function and submultiplicative, $h(0) = 0$ such that $\frac{x}{h(x)} \in Q_\beta$, then*

$$\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x)) dx \leqslant \frac{1}{p-\beta}\int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t)) dt. \tag{19}$$

Proof. By using the assumption of convexity and submultiplicativity of h, $\frac{x}{h(x)} \in Q_\beta$ and the Fubini theorem, we get

$$\begin{aligned}
\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x))dx &= \int_0^\infty \frac{x^{1-p}}{h^2(x)} h\left(x\frac{1}{x}\int_0^x f(t)dt\right)dx \leqslant \\
&\leqslant \int_0^\infty \frac{x^{-p}}{h(x)} \int_0^x h(f(t))dtdx = \\
&= \int_0^\infty h(f(t)) \int_t^\infty \frac{x^{-p}}{h(x)} dxdt = \\
&= \int_0^\infty h(f(t)) \int_t^\infty \frac{x^{1-\beta}}{h(x)} x^{\beta-p-1} dxdt \leqslant \\
&\leqslant \int_0^\infty h(f(t)) \frac{t^{1-\beta}}{h(t)} \int_t^\infty x^{\beta-p-1} dxdt = \\
&= \frac{1}{p-\beta} \int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t))dt.
\end{aligned}$$

□

If we set $\beta = 0$ in (19), we obtain the following corollary.

Corollary 6. *Let $p > 1$, $f \geqslant 0$. If $h \geqslant 0$ be a convex submultiplicative function and $h(0) = 0$, then*

$$\int_0^\infty \frac{x^{1-p}}{h^2(x)} h(F(x))dx \leqslant \frac{1}{p} \int_0^\infty \frac{t^{1-p}}{h(t)} h(f(t))dt. \tag{20}$$

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Весовые операторы Харди и квазимонотонные функции

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Аннотация. Некоторые неравенства типа Харди установлены У. Т. Сулейманом. Целью данной работы является распространение этих неравенств на весовые операторы Харди с квазимонотонными функциями. Кроме того, были получены новые интегральные весовые неравенства.

Ключевые слова: неравенства, оператор Харди, квазимонотонные функции.

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On the Real Roots of Systems of Transcendental Equations

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Abstract. The work is devoted to finding the number of real roots of systems of transcendental equations. It is shown that if a system has simple roots, then the number of real coordinates of the roots is the same. Therefore, the number of real roots is related with the number of real roots of the resultant of the system.

Keywords: system of transcendental equations, resultant, residue integral.

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Introduction

Finding the number of real roots of polynomials with real coefficients is a classical problem of algebra. There are quite a few related results; the Hermite method of quadratic forms [1, ch. 16, Sec. 9], [2, Appendix I], Sturm's theorem, Descartes' sign rule, the Budan–Fourier theorem (see, for example, [3, Chapter 9]). Further development of these methods for polynomials can be found in the paper by M. Krein and M. Naimark [4] (in fact, this paper was published in 1936 in Russian, but has long become a bibliographic rarity) and the monograph by Jury [5]. For entire functions, the question of localization of real positive roots was considered in the classical works of N. G. Chebotarev [6, p. 3–18, 29-56], as well as in the work [7] (we refer to the collected works of N. G. Chebotarev, since his original works are now inaccessible).

For systems of equations, the number of real roots was studied in the articles [8–10]. In the article [11] root coordinates were related to the root first coordinates.

The monographs [12,13] consider algebraic and transcendental systems of equations. Systems of transcendental equations arise, for example, when studying the equations of chemical kinetics [14]. One of the problems that arises there is the problem of the number of real positive roots of a system of equations, or the number of roots in the reaction polyhedron.

1. Resultant of a system

Consider a system of equations of the form

$$\begin{cases} f_1(z) = 0, \\ \dots \\ f_n(z) = 0, \end{cases} \tag{1}$$

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where $f_1(z), \ldots, f_n(z)$ are entire functions of complex variables $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. In what follows we will assume that the set of roots of the system (1) is discrete. Therefore it is no more than countable. Let $\mathcal{E}$ denote the set of roots with non-zero coordinates $w_{(\nu)} = \big(w_{1(\nu)}, \ldots, w_{n(\nu)}\big)$, $\nu = 1, 2, \ldots$, numbered in ascending order of modules: $\left|w_{(1)}\right| \leqslant \left|w_{(2)}\right| \leqslant \ldots \leqslant \left|w_{(\nu)}\right| \leqslant \ldots$.

Let us consider power sums of roots S_α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a non-negative multi-index (all components are non-negative and integer) and $\alpha_1 + \ldots + \alpha_n > 0$ of the form

$$S_\alpha = \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}^{\alpha_1} \cdot w_{2(\nu)}^{\alpha_2} \cdots w_{n(\nu)}^{\alpha_n}}.$$

We will assume that all series S_α are absolutely convergent for any multi-indices α.

The concept of power sums (in the negative power) for transcendental systems of equations was considered in the works [15–18]. The results of these papers were based on the calculation of power sums through the so-called residue integrals [19].

Lemma 1. *The series S_α converge absolutely for any multi-indices α if and only if the series*

$$\sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}}, \quad \ldots, \quad \sum_{\nu=1}^{\infty} \frac{1}{w_{n(\nu)}}$$

converge absolutely.

Proof see [11].

Therefore, an entire function of genus zero is defined ([20], Chapter 7)

$$R\,(z_1) = z_1^s \cdot \prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{w_{1(\eta)}}\right), \tag{2}$$

where s is the multiplicity of the zero of the system (1) at the zero point, $s \geqslant 0$.

In the formula (2), the infinite product converges absolutely and uniformly in the complex plane $\mathbb{C}$.

We will call the function $R(z_1)$ the *resultant* of the system (1) with respect to the variable z_1. The concept of a resultant for systems of transcendental equations is not generally accepted. For the case of two equations it was introduced by N. G. Chebotarev [6] (p. 18–27). In recent years, this concept has been considered in the works of [21–24]. The results of these papers were based on the calculation of power sums through the so-called residue integrals [19]. The main problem is to find coefficients of a resultant without knowing the roots themselves. In this sense, determining the resultant for a system is not constructive. There is no formula for systems of equations like there is for the Sylvester resultant for polynomials. Some approaches to finding it can be found in the monograph [12, Sec. 3.7].

2. Auxiliary results

Let us introduce the functions $P_j^{(t)}(z_1)$

$$P_j^{(t)}(z_1) = -z_1^{s-1} \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta \neq \nu} \left(1 - \frac{z_1}{w_{1(\eta)}}\right), \quad t \geqslant 0, \quad s \geqslant 1. \tag{3}$$

Lemma 2. *Functions (3) are entire functions of the variable z_1.*

Proof. Let us write $P_j^{(t)}(z_1)$ in the form:

$$P_j^{(t)}(z_1) = -z_1^{s-1} \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{z_1}{w_{1(\eta)}}\right) \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}}.$$

The infinite product $\prod\limits_{\eta=1}^{\infty}\left(1 - \dfrac{z_1}{w_{1(\eta)}}\right)$ is an entire function of genus zero.

Let us prove that the series

$$\sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}}$$

converges absolutely and uniformly in the complex plane $\mathbb{C}$.

By Lemma 1 the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{|w_{1(\nu)}|}$ converges, then

$$\lim_{\nu\to\infty} \frac{1}{|w_{1(\nu)}|} = 0,$$

and therefore

$$\lim_{\nu\to\infty}\left(1 - \frac{z_1}{w_{1(\nu)}}\right) = 1.$$

Since $1 - \dfrac{z_1}{w_{1(\nu)}}$ is close to unity, we can assume that

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}}\right| \leqslant 2 \cdot \sum_{\nu=1}^{\infty} \frac{1}{\left|w_{j(\nu)}^t\right|}.$$

Whence it follows that the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{w_{j(\nu)}^t} \cdot \dfrac{1}{1 - \frac{z_1}{w_{1(\nu)}}}$ converges absolutely and uniformly in the complex plane $\mathbb{C}$.

This proves that the functions $P_j^{(t)}(z_1)$ are entire functions of the variable z_1. □

Theorem 1. *Let the function $R(z_1)$ have simple zeros $w_{1(\nu)}$, $\nu = 1, 2, \ldots$. Then the next equality is true*

$$\left.\frac{P_j^{(t)}(z_1)}{R'(z_1)}\right|_{z_1 = w_{1(\mu)}} = \frac{1}{w_{j(\mu)}^t} \quad \text{for any} \quad \mu.$$

Proof. Let us find the derivative with respect to z_1 of the function $R(z_1)$:

$$R'(z_1) = s \cdot z_1^{s-1} \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{z_1}{w_{1(\eta)}}\right) - z_1^s \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}} \cdot \prod_{\eta\neq\nu}\left(1 - \frac{z_1}{w_{1(\eta)}}\right).$$

The first term calculated at the point $z_1 = w_{1(\mu)}$ is equal to 0, since

$$\left.\prod_{\eta=1}^{\infty}\left(1 - \frac{z_1}{w_{1(\eta)}}\right)\right|_{z_1 = w_{1(\mu)}} = 0.$$

Let us calculate the second term at the point $z_1 = w_{1(\mu)}$:

$$-w_{1(\mu)}^s \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}} \cdot \prod_{\eta\neq\nu}\left(1 - \frac{w_{1(\mu)}}{w_{1(\eta)}}\right) = -w_{1(\mu)}^s \cdot \frac{1}{w_{1(\mu)}} \cdot \prod_{\eta\neq\mu}\left(1 - \frac{w_{1(\mu)}}{w_{1(\eta)}}\right).$$

Thus,

$$R^{'}(z_1)\Big|_{z_1=w_{1(\mu)}} = -w_{1(\mu)}^{s-1}\cdot\prod_{\eta\neq\mu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right).$$

Let us find the value of $P_j^{(t)}(z_1)$ at the point $z_1 = w_{1(\mu)}$:

$$\begin{aligned} P_j^{(t)}(z_1)\Big|_{z_1=w_{1(\mu)}} &= -w_{1(\mu)}^{s-1}\cdot\sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t}\cdot\prod_{\eta\neq\nu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right) = \\ &= -w_{1(\mu)}^{s-1}\cdot\frac{1}{w_{j(\mu)}^t}\cdot\prod_{\eta\neq\nu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right). \end{aligned}$$

After substituting the found expressions into $\left.\dfrac{P_j^{(t)}(z_1)}{R'(z_1)}\right|_{z_1=w_{1(\mu)}}$ and reducing it, we obtain the statement of the theorem. □

Thus, we get that if the first coordinates of the roots from $\mathcal{E}$ are known, then to find the remaining coordinates of the roots there is no need to find resultants for other variables.

As a resultant of the system (1), we can also take a function of the form

$$Q(z_1) = z_1^s\cdot e^{g(z_1)}\cdot\prod_{\eta=1}^{\infty}\left(1-\frac{z_1}{w_{1(\eta)}}\right). \tag{4}$$

where $g(z_1)$ is some entire function, s is the multiplicity of the zero of the system (1) at zero, $s \geqslant 0$.

It has the same roots and the same multiplicity as the resultant $R(z_1)$.

Consider the system of functions

$$V_j^{(t)}(z_1) = -z^{s-1}\cdot e^{g(z_1)}\cdot\sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t}\cdot\prod_{\eta\neq\nu}\left(1-\frac{z_1}{w_{1(\eta)}}\right),\quad t\geqslant 1,\quad s\geqslant 1.$$

Corollary 1. *Let the function $Q(z_1)$ have simple zeros $w_{1(\nu)}$, $\nu = 1, 2, \ldots$. Then the next equality is true*

$$\left.\frac{V_j^{(t)}(z_1)}{Q'(z_1)}\right|_{z_1=w_{1(\mu)}} = \frac{1}{w_{j(\mu)}^t}.$$

The proof of Corollary 1 repeats the proof of Theorem 1.

3. Main results

Let us write the Taylor series expansion in the variable z_1 in the neighborhood of zero of the function $P_j^{(t)}(z_1)$ and the function $R(z_1)$:

$$P_j^{(t)}(z_1) = -z_1^{s-1}\cdot\sum_{m=0}^{\infty}a_{jm}^{(t)}\cdot z_1^m,\qquad a_{j0}^{(t)} = 1,$$

$$R(z_1) = z_1^s\cdot\sum_{m=0}^{\infty}b_m\cdot z_1^m,\qquad b_0 = 1,$$

Theorem 2. *If a system (1) with real coefficients is such that all zeros of $R(z_1)$ are simple except for the point $z_1 = 0$, then the number of real roots of the system (1) in $\mathcal{E}$ coincides with the number of real roots of the resultant $R(z_1)$.*

Proof. If the system (1) has real coefficients, then all power sums of the roots S_α are real.

Indeed, let the system (1) has a real root w, that is, $f_j(w) = 0$, $j = 1, \ldots, n$. Then $\overline{f_j(w)} = 0$, $j = 1, \ldots, n$. Since the system (1) has real coefficients, then $f_j(\overline{w}) = 0$, $j = 1, \ldots, n$. Therefore, $\overline{w}$ is also a root. That is, complex roots are paired. This means that in the power sum S_α each non-real (complex) term corresponds to a complex conjugate term. And therefore the sum of these numbers is a real number.

Let us prove that the resultant $R(z_1) = z_1^s \cdot \sum\limits_{m=0}^{\infty} b_m \cdot z_1^m$ has real coefficients, that is, that b_m are real , $m = 0, 1, 2, \ldots$.

To do this, consider the infinite product

$$\prod_{\eta=1}^{\infty}\left(1 - \frac{z_1}{w_{1(\eta)}}\right) = 1 + z_1 \cdot \sum_{j=1}^{\infty} \frac{-1}{w_{1(j)}} + z_1^2 \cdot \sum_{j_1<j_2} \frac{1}{w_{1(j_1)} \cdot w_{1(j_2)}} +$$

$$+ z_1^3 \cdot \sum_{j_1<j_2<j_3} \frac{-1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot w_{1(j_3)}} + \ldots =$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m \cdot z_1^m \cdot \sum_{j_1<j_2<\ldots<j_m} \frac{1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot \ldots \cdot w_{1(j_m)}}.$$

The coefficients for z_1^m are equal to:

$$\begin{aligned} b_0 &= 1, \\ b_m &= (-1)^m \cdot \sum_{j_1<j_2<\ldots<j_m} \frac{1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot \ldots \cdot w_{1(j_m)}}, \quad m = 1, 2, \ldots \end{aligned} \tag{5}$$

From the form (5) it obviously follows that b_m are symmetric functions of the numbers $\frac{1}{w_{1(1)}}$, $\frac{1}{w_{1(2)}}$, $\frac{1}{w_{1(3)}}$, ..., which means b_m are real.

Let us represent $P_j^{(t)}(z_1)$ in a more convenient form.

For this, consider an auxiliary system of functions

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{\lambda}{w_{1(\nu)}}} \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right), \quad s \geqslant 1.$$

Or after the reduction:

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta\neq\nu}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) = -\lambda^{s-1} \cdot \sum_{m=0}^{\infty} a_{jm}^{(t)} \cdot \lambda^m, \qquad a_{j0}^{(t)} = 1.$$

Using the geometric progression formula for sufficiently small $|\lambda|$:

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \sum_{m=0}^{\infty}\left(\frac{\lambda}{w_{1(\nu)}}\right)^m \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) =$$

$$= -\lambda^{s-1} \cdot \sum_{m=0}^{\infty} \lambda^m \cdot \left(\sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}^m \cdot w_{j(\nu)}^t}\right) \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) =$$

$$= -\lambda^{s-1} \cdot \left(\sum_{m=0}^{\infty} S_{me_1+te_j} \cdot \lambda^m \right) \cdot \left(\sum_{k=0}^{\infty} b_k \cdot \lambda^k \right) =$$

$$= -\lambda^{s-1} \cdot \sum_{l=0}^{\infty} \lambda^s \cdot \left(\sum_{m+k=l} S_{me_1+te_j} \cdot b_k \right),$$

where $S_{me_1+te_j} = \sum\limits_{\nu=1}^{\infty} \dfrac{1}{w_{1(\nu)}^m \cdot w_{j(\nu)}^t}$ are power sums for the multi-index $me_1 + te_j = (m, 0, \ldots, 0, t, 0, \ldots, 0)$, the first component of the multi-index is equal to 1, the j-th component is equal to t, and the remaining components are zeros.

Since the coefficients of the system (1) are real, we have that

$$\sum_{m+k=l}^{\infty} S_{m+k=l} \cdot b_k, \quad l = 0, 1, 2, \ldots$$

are real.

We obtained relations for calculating $a_{jl}^{(t)}$:

$$a_{jl}^{(t)} = \sum_{m+k=l} S_{me_1+te_j} \cdot b_k,$$

where $l = 0, 1, 2, \ldots$, $b_0 = 1$, $s \geqslant 1$, $t \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$. That is, the coefficients $a_{jl}^{(t)}$ are real.

Thus, if one coordinate of the root of the system (1) is real, then all other coordinates of this root are also real. This is where the statement of the theorem follows. □

Corollary 2. *If a system (1) with real coefficients is such that all zeros of $Q(z_1)$ (that is, $R(z_1)$) are simple except for the point $z_1 = 0$ and the function $g(z_1)$ from (4) has real coefficients, then the number of real roots of the system (1) in $\mathcal{E}$ coincides with the number of real roots of the function $Q(z_1)$.*

The proof of Corollary 2 repeats the proof of Theorem 2.

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О вещественных корнях систем трансцендентных уравнений

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Аннотация. Работа посвящена нахождению числа вещественных корней систем трансцендентных уравнений. Показано, что если система имеет простые корни, то число вещественных координат корней одинаково. Поэтому число вещественных корней связано с числом вещественных корней результанта системы.

Ключевые слова: система трансцендентных уравнений, результант, вычетный интеграл.

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On Periodic Bilinear Threshold GARCH models

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Abstract. Periodic Generalized Autoregressive Conditionally Heteroscedastic ($PGARCH$) models were introduced by Bollerslev et Ghysels. These models have gained considerable interest and continued to attract the attention of researchers. This paper is devoted to extensions of the standard bilinear threshold $GARCH$ ($BLTGARCH$) model to periodically time-varying coefficients ($PBLTGARCH$) one. In this class of models, the parameters are allowed to switch between different regimes. Moreover, these models are allowed to integrate asymmetric effects in the volatility. Firstly, we give necessary and sufficient conditions ensuring the existence of stationary solutions (in periodic sense). Secondly, a quasi maximum likelihood (QML) estimation approach for estimating $PBLTGARCH$ model is developed. More precisely, the strong consistency and the asymptotic normality of the estimator are studied given mild regularity conditions, requiring strict stationarity and the finiteness of moments of some order for the errors term. The finite-sample properties of $QMLE$ are illustrated by a Monte Carlo study. Finally our proposed model is applied to model the exchange rates of the Algerian Dinar against the single European currency ($Euro$).

Keywords: periodic bilinear threshold $GARCH$ models, Strictly periodically stationary, Gaussian QML estimator.

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1. Introduction and preliminaries

In recent years, many papers discussed the periodic generalized autoregressive conditionally heteroskedastic models ($PGARCH_s$) process introduced by Bollerslev and Ghysels [10]. This process has been proved to be a power tool for modeling and forecasting many non stationary time series, which makes a distinctive by a stochastic conditional variance with periodic dynamics. Generally, by $PGARCH_s$ process we mean a discrete-time strictly stationary process

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$(\varepsilon_t, t \in \mathcal{Z}), \mathcal{Z} = \{0, \pm 1, \pm 2, \dots, \}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and satisfying the factorization

$$\varepsilon_t = h_t e_t, \tag{1.1}$$

Here, the innovation process $(e_t, t \in \mathcal{Z})$ is independent and identically distributed sequence with zero mean and unit variance $(i.i.d\,(0,1))$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ and time-varying coefficients "volatility" process $(h_t, t \in \mathcal{Z})$ satisfy the recursion

$$h_t^2 = \alpha_0\,(t) + \sum_{i=1}^{q} \alpha_i\,(t)\,\varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j\,(t)\,h_{t-j}^2, \tag{1.2}$$

where $(\alpha_i\,(t)\,, 0 \leqslant i \leqslant q)$ and $(\beta_j\,(t)\,, 0 \leqslant j \leqslant p)$ are non negative periodic functions with period s with $\alpha_0\,(t) > 0$. $PGARCH_s$ model is potentially more efficient than the standard one. It becomes increasingly important and an efficient tool to model seasonal asset returns of stocks, exchange rates and other financial time series and continues to gain a growing interest of researchers (see Ghezal [1] and Lescheb [4]). This interest is due to its multiple advantages; for instance, among others, it is able to capture the stylized facts, e.g., volatility clustering, leptokurticity, dependency without correlation and tail heaviness. However, in some asymmetric financial datasets exhibiting the so-called leverage effect characterized by $Cov\left(e_{t-k}, h_t^2\right) < 0$, for some $k > 0$, the $PGARCHs$ models are unable to model such data without further extensions. This finding led Rodriguez and Ruiz [6] to study five of the most popular specifications of the time-invariant asymmetric volatility process $(h_t, t \in \mathbb{Z})$ with leverage effect, namely, the generalized quadratic $ARCH(GQARCH)$, the threshold $GARCH$ $(TGARCH)$, the $GJR-GARCH(GJR)$, the exponential $GARCH(EGARCH)$ and the asymmetric power $GARCH(APGARCH)$ models. These models are important in modelling, forecasting and capturing the asymmetry of the volatility and hence are purported to be able to capture the leverage. Beside the above mentioned models, Choi et al [7] have recently introduced the so-called bilinear threshold $GARCH$ $(BLTGARCH)$ model defined by Equation (1.1) with time-invariant coefficients volatility process, i.e.,

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \left(\alpha_i \varepsilon_{t-i}^{+2} + \beta_i \varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k \varepsilon_{t-k}^{+} + \omega_k \varepsilon_{t-k}^{-}\right) h_{t-k} + \sum_{j=1}^{p} \gamma_j h_{t-j}^2, \tag{1.3}$$

where $\varepsilon_n^+ = \max(\varepsilon_n, 0)\,, \varepsilon_n^- = \min(\varepsilon_n, 0)\,, \varepsilon_n^{+2} = \left(\varepsilon_n^+\right)^2, \varepsilon_n^{-2} = \left(\varepsilon_n^-\right)^2$ and $d = p \wedge q$. This paper is fundamentally interested with non-stationary $BLTGARCH$ models in which the parameters are periodic in t with period s. As a result, we will provide a periodic $BLTGARCH(q, d, p)$ model $(PBLTGARCH_s)$ defined by (1.1) and

$$\begin{aligned} h_t^2 = \ & \alpha_0\,(t) + \sum_{i=1}^{q} \left(\alpha_i\,(t)\,\varepsilon_{t-i}^{+2} + \beta_i\,(t)\,\varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k\,(t)\,\varepsilon_{t-k}^{+} + \omega_k\,(t)\,\varepsilon_{t-k}^{-}\right) h_{t-k} \\ & + \sum_{j=1}^{p} \gamma_j\,(t)\,h_{t-j}^2. \end{aligned} \tag{1.4}$$

In (1.4), the functions $(\alpha_i\,(t)\,, 0 \leqslant i \leqslant q)\,,$ $(\beta_i\,(t)\,, 1 \leqslant i \leqslant q)\,,$ $(b_k\,(t)\,, 1 \leqslant k \leqslant d)\,,$ $(\omega_k\,(t)\,, 1 \leqslant k \leqslant d)$ and $(\gamma_j\,(t)\,, 1 \leqslant j \leqslant p)$ are periodic with period $s \geqslant 1$. Moreover, $(\alpha_i\,(t)\,, 0 \leqslant i \leqslant q)\,,$ $(\beta_i\,(t)\,, 1 \leqslant i \leqslant q),$ $(\gamma_j\,(t)\,, 1 \leqslant j \leqslant p)$ are non negative sequences with $\alpha_0(.) > 0$, whereas the functions $(b_k\,(t)\,, 1 \leqslant k \leqslant d)$, $(\omega_k\,(t)\,, 1 \leqslant k \leqslant d)$ have values in $(-\infty, +\infty)$. So, by transforming t into $st + v$ and setting $\varepsilon_t\,(v) = \varepsilon_{st+v}$, $h_t\,(v) = h_{st+v}$ and $e_t\,(v) = e_{st+v}$, then (1.4) may be equivalently written in periodic version as

$$\begin{aligned} h_t^2\,(v) \ & = \alpha_0\,(v) + \sum_{i=1}^{q} \left(\alpha_i\,(v)\,\varepsilon_t^{+2}\,(v-i) + \beta_i\,(v)\,\varepsilon_t^{-2}\,(v-i)\right) \\ & + \sum_{k=1}^{d} \left(b_k\,(v)\,\varepsilon_t^{+}\,(v-k) + \omega_k\,(v)\,\varepsilon_t^{-}\,(v-k)\right) h_t\,(v-k) + \sum_{j=1}^{p} \gamma_j\,(v)\,h_t^2\,(v-j)\,. \end{aligned} \tag{1.5}$$

In (1.5), the notation $\varepsilon_t(\upsilon)$ refers to ε_t during the $\upsilon - th$ "season" $\upsilon \in \mathbb{S} = \{1, \ldots, s\}$ of cycle t, and, for convenience, we set $\varepsilon_t(\upsilon) = \varepsilon_{t-1}(\upsilon + s)$, $h_t(\upsilon) = h_{t-1}(\upsilon + s)$ and $e_t(\upsilon) = e_{t-1}(\upsilon + s)$ if $\upsilon < 0$. The non-peroidic notations $(\epsilon_t), (e_t)$ and (h_t) will be used interchangeably with the periodic one $(\varepsilon_t(\upsilon))$, $(e_t(\upsilon))$ and $(h_t(\upsilon))$ whenever emphasis on seasonality is not needed. It is worth noting that, since h_t^2 is the conditional variance of ϵ_t given the past information up to time $t-1$, the positivity of the functions $(\alpha_i(t), 0 \leqslant i \leqslant q), (\beta_i(t), 1 \leqslant i \leqslant q)$ and $(\gamma_j(t), 1 \leqslant j \leqslant p)$ ensures the positivity of h_t^2 in $PTGARCH_s$ model. This is not the case in $PBLTGARCH_s$ even when $b_k(.) \geqslant 0$, $\omega_k(.) \geqslant 0$ and due to the penultimate term in (1.5), so the positivity of h_t^2 can be studied case by case and hence we shall assume throughout this paper that $h_t^2 > 0$, almost surely $(a.s)$.

Some algebraic notation and definitions are used throughout this paper. $O_{(n,m)}$ denotes the matrix of order $n \times m$ whose entries are zeros, for simplicity we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$. $I_{(n)}$ is the $n \times n$ identity matrix and $\mathbb{I}_{\Delta}$ denotes the indicator function of the set Δ. If $(M(i), i \in I)$ is $n \times n$ matrices sequence, we shall denote for any integer l and j, $\prod_{i=l}^{j} M(i) = M(l)M(l+1)\ldots M(j)$ if $l \leqslant j$ and $I_{(n)}$ otherwise. For any real random variable X, we denote $X^+ = \max(X, 0)$, $X^- = \max(-X, 0)$ so $X = X^+ - X^-$ and $|X| = X^+ - X^-$. $||.||$ refers to the induced norm in the space $\mathcal{M}(n, m)$ of $n \times m-$matrices. For instance, the norm of matrix $M = (m_{i\,j})$ is defined by $\|M\| = \sum |m_{i\,j}|$.

The main contributions of this paper can be summarized as follows. In Section 2, the Markovian representation of $PTBLGARCHs$ model is given and conditions for the existence of a strict periodic stationary (SPS) solution of (1.1)–(1.5) are established. In Section 3, the strong consistency and asymptotic normality of the $QMLE$ are studied. Numerical illustrations are given in Section 4 and an empirical application to the daily series of exchange rate of the Algerian Dinar against the single European currency is provided in Section 5.

2. Probabilistic properties of PBLTGARCHs(p,q,d)

As for many time series models, it is useful to write Equations (1.1)–(1.4) in an equivalent Markovian representation in order to facilitate their study. For this purpose, introduce the $r = (p + 2q + 2d)$-vector
$\underline{\varepsilon}_t' := \left(h_t^2, \ldots, h_{t-p+1}^2, \varepsilon_t^{+2}, \varepsilon_t^{-2}, \ldots, \varepsilon_{t-q+1}^{+2}, \varepsilon_{t-q+1}^{-2}, h_t e_t^+, h_t e_t^-, ..., h_{t-d+1}e_{t-d+1}^+, h_{t-d+1}e_{t-d+1}^-\right)$
and $\underline{H}_0' := \left(1, \underline{O}_{(r-1)}'\right), \underline{H}_1' := \left(\underline{O}_{(p)}', 1, -1, \underline{O}_{(r-p-2)}'\right)$ and $\underline{\eta}_t(e_t) := \underline{\alpha}_{0,p+1}(t) e_t^{+2} + \underline{\alpha}_{0,p+2}(t) e_t^{-2} + \underline{\alpha}_{0,r-2d+1}(t) e_t^{+} + \underline{\alpha}_{0,r-2d+2}(t) e_t^{-} + \underline{\alpha}_{0,1}(t)$ in which the $j - th$ entry of $\underline{\alpha}_{0,j}(t)$ is $\alpha_0(t)$ and all other elements are 0. With these notations, we obtain the following state-space representation $\varepsilon_t^2 = \underline{H}_1'\underline{\varepsilon}_t$ and $h_t^2 = \underline{H}_0'\underline{\varepsilon}_t$

$$\underline{\varepsilon}_t = A_t(e_t)\,\underline{\varepsilon}_{t-1} + \underline{\eta}_t(e_t),\ t \in \mathbb{Z}, \tag{2.1}$$

with $A_t(e_t) := A_1(t) e_t^{+2} + A_2(t) e_t^{-2} + A_3(t) e_t^{+} + A_4(t) e_t^{-} + A_5(t)$. Here $(A_j(t), 1 \leqslant j \leqslant 5)$ are appropriate $(r \times r)$-periodic matrices easily obtained and uniquely determined by $\{\alpha_i(t),\ \beta_i(t),\ b_k(t),\ \omega_k(t),\ \gamma_j(t), 1 \leqslant i, k, j \leqslant q \vee p\}$. Now, by iterating (2.1) s times we get the following:

$$\underline{\varepsilon}_{(t+1)s} = H(\underline{e}_t)\,\underline{\varepsilon}_{ts} + \underline{\eta}(\underline{e}_t),\ t \in \mathbb{Z}, \tag{2.2}$$

where

$$\underline{e}_{t+1} = \left(e_{(t+1)s}, ..., e_{st+1}\right)', H(\underline{e}_t) = \left\{\prod_{j=0}^{s-1} A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\}, \underline{\eta}(\underline{e}_t) =$$
$$= \sum_{k=0}^{s-1}\left\{\prod_{j=0}^{s-1} A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\}\underline{\eta}_{(t+1)s-k}\left(e_{(t+1)s-k}\right).$$

Set $\underline{\varepsilon}_{ts} = \underline{\varepsilon}(t)$ (if there is no confusion). Then,(2.2) may be rewritten as

$$\underline{\varepsilon}(t) = H\left(\underline{e}_{t-1}\right)\underline{\varepsilon}(t-1) + \underline{\eta}\left(\underline{e}_{t-1}\right),\ t \in \mathbb{Z}. \tag{2.3}$$

Note here that $H\left(\underline{e}_t\right)$ is a sequence of $i.i.d.$ random matrices independent of $\underline{\varepsilon}(k), k \leqslant t$ and $\underline{\eta}\left(\underline{e}_t\right)$ is a sequence of $i.i.d.$ vectors. So, the existence of the so-called strictly periodically stationary (SPS) and periodic ergodic (PE) solutions to (1.1)–(1.5) is now equivalent to the existence of a strict stationary and ergodic solution to (2.3). Hence, equation similar to Equation (2.3) was examined by Bougerol and Picard [8] who established that the series

$$\underline{\varepsilon}(t) = \sum_{k \geqslant 1}\left\{\prod_{i=0}^{k-1} H\left(\underline{e}_{t-i-1}\right)\right\}\underline{\eta}\left(\underline{e}_{t-k-1}\right) + \underline{\eta}\left(\underline{e}_{t-1}\right), \tag{2.4}$$

constitute the unique, strictly stationary and ergodic solution of (2.3) if and only if, the top-Lyapunov exponent $\gamma(H)$ associated with the strictly stationary and ergodic sequence of random matrices $H = \left(H\left(\underline{e}_t\right), t \in \mathbb{Z}\right)$ defined by

$$\gamma(H) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1} H\left(\underline{e}_{t-j-1}\right)\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\to\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1} H\left(\underline{e}_{t-j-1}\right),\right\|\right\} \tag{2.5}$$

is such that $\gamma(H) < 0$. However, the existence of $\gamma(H)$ is guaranteed by the fact that $E\{\log^{+}\|H(\underline{e}_t)\|\} \leqslant E\{\|H(\underline{e}_t)\|\} < \infty$, where $\log^{+}(x) = \max(\log x, 0)$ and the right-hand member in (2.5) can be justified using Kingman's [5] subadditive ergodic theorem. We summarize the above discussion in the following theorem due to Bougerol and Picard [8].

Theorem 2.1. *If $\gamma(H)$ corresponding to $PBLTGARCHs(q,d,p)$ models is strictly negative, then*
1. Equation (2.3) admits a unique, strictly stationary, causal and ergodic solution given by the series (2.4).
2. Equation (1.5) and, hence (1.1), admits a unique, SPS, causal and PE solution given by $h_t^2 = \underline{H}_0'\underline{\varepsilon}_t$ or $\varepsilon_t = e_t\left\{\underline{H}_1'\underline{\varepsilon}_t\right\}^{\frac{1}{2}}$ where $\underline{\varepsilon}_t$ is given by the series (2.4).

Proof. The proof follows essentially the same arguments as Bougerol and Picard [8]. □

Corollary 2.1. *If $\gamma(H) < 0$ and $E\{|e_0|^{2\delta}\} < \infty$ for some $\delta > 0$, then there is $\delta^* \in]0,1]$ such that $E(h_t^{\delta^*}) < \infty$ and $E({\varepsilon_t}^{\delta^*}) < \infty$.*

Remark 2.1. *Aknouche and Guerbyenne [2] have studied the conditions ensuring the existence and uniqueness of a SPS and PE solution of (1.1) and (1.5) using directly the (2.1) by showing that*

$$\inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{ts-1} A_{ts-j}\left(e_{ts-j}\right)\right\|\right\}\right\} \tag{2.6}$$

is a sufficient condition for that (2.1) to have a unique, causal, SPS and PE solution given by

$$\underline{\varepsilon}_t = \sum_{k \geqslant 1}\left\{\prod_{i=0}^{k-1} A_{t-i}\left(e_{t-i}\right)\right\}\underline{\eta}_{t-k}\left(e_{t-k}\right) + \underline{\eta}_t\left(e_t\right). \tag{2.7}$$

Remark 2.2. *It is worth noting that the condition $\gamma_L^{(s)}(H) < 0$ provides a certain global stability of model (2.1). However, when $\gamma_L^{(s)}(H) < 0$, the model (2.1) is said to be unstable and hence doesn't have a SPS solution. As an example, consider* *the* $PBLAARCH_s(1.1)$ *model defined by* $\varepsilon_t(\upsilon) = h_t(\upsilon)\, e_t(\upsilon)$ *and* $h_t^2(\upsilon) = \alpha_0(\upsilon) + \alpha_1(\upsilon)\left|e_t^2(\upsilon-1)\right| h_t^2(\upsilon-1) + b_1(\upsilon)\left|e_t(\upsilon-1)\right| h_t(\upsilon-1)$. *It is not difficult to show that* $\gamma_L^{(s)}(H) = E\left(\log\left(\prod_{\upsilon=0}^{s-1}\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)\left|e_0\right|\right|\right)\right)\right) \geqslant 0$. *Hence, the existence of some (not all) "stable regimes" (i.e.,* $E\left\{\log\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)\left|e_0\right|\right|\right)\right\} < 0$*) does not guarantee the existence of a* SPS *solution. More generally, we have the following convergence of the volatility to infinity for* $PBLAARCH_s(1,1)$ *process encompassing (2.2).*

Example 2.1. *In PBLTGARCHs(1; 1; 1) models, the necessary and sufficient condition ensuring the existence of strictly periodically stationary solution is that :*

$$\sum_{\upsilon=1}^{s} E\left\{\log\left\{\left|\alpha_1(\upsilon)\, e_0^{+2} + \beta_1(\upsilon)\, e_0^{-2} + b_1(\upsilon)\, e_0^{+} + \omega_1(\upsilon)\, e_0^{-} + \gamma(\upsilon)\right|\right\}\right\} < 0.$$

In particular, for standard $BLTARCH(1,1,1)$ *and for* $PBLTARCH_2(1,1,1)$ *with* $\alpha_1(1) = a$, $\omega_1(1) = b$, $\alpha_1(2) = 0.25a$, $\omega_1(2) = 0.25b$, $\beta_1(1) = \beta_1(2) = b_1(1) = b_1\ (2) = 0$ *and* $e_t \rightsquigarrow \mathcal{N}(0,1)$, *the stationarity zone is showed in Fig. 1.*

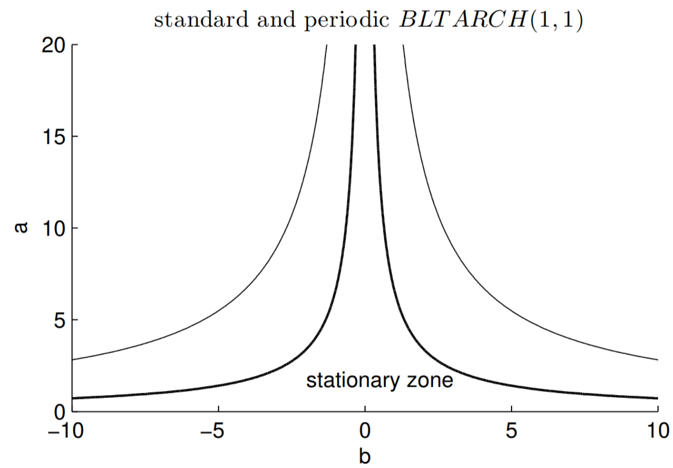

Fig. 1. Stationarity zones for standard (solid line) and periodic $BLTARCH(1,1)$ (dashed line)

It is clearly observed that the corresponding zone to the standard model is less restrictive than that corresponding to the periodic model.

2.1. Quasi-maximum likelihood estimator

In this subsection , we consider the quasi-maximum likelihood estimator ($QMLE$) for estimating the parameters of $PBLTGARCH_s$ model gathered in vector $\underline{\theta}' = \left(\underline{\theta}_1, \ldots, \underline{\theta}_{s(1+2q+2d+p)}\right) := \left(\underline{\alpha}', \underline{\beta}', \underline{b}', \underline{\omega}', \underline{\gamma}'\right) \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)}$, where $\underline{\alpha}' := \left(\underline{\alpha}_0', \underline{\alpha}_1', \ldots, \underline{\alpha}_q'\right)$, $\underline{\beta}' := \left(\underline{\beta}_1', \ldots, \underline{\beta}_q'\right)$, $\underline{b}' := \left(\underline{b}_1', \ldots, \underline{b}_d'\right)$, $\underline{\omega}' := \left(\underline{\omega}_1', \ldots, \underline{\omega}_d'\right)$, $\underline{\gamma}' := \left(\underline{\gamma}_1', \ldots, \underline{\gamma}_p'\right)$ with $\underline{\alpha}_i' := (\alpha_i(1), \ldots, \alpha_i(s))$, $\underline{\beta}_i' := (\beta_i(1), \ldots, \beta_i(s))$, $\underline{b}_k' := (b_k(1), \ldots, b_k(s))$ and $\underline{\omega}_k' := (\omega_k(1), \ldots, \omega_k(s))$, $\underline{\gamma}_j' := (\gamma_j(1), \ldots, \gamma_j(s))$ for all $0 \leqslant i \leqslant q$, $1 \leqslant k \leqslant d$ and $1 \leqslant j \leqslant p$. The true parameter value denoted by $\underline{\theta}_0 \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)}$ is unknown and, therefore, it must be estimated. For this purpose, consider a realization $\{\varepsilon_1, \ldots, \varepsilon_n; n = sN\}$ from the unique, causal, SPS and PE solution of (1.1) and (1.5) and let $h_t^2(\underline{\theta})$ be the conditional variance of ε_t given $\mathcal{F}_{t-1}$, where $\mathcal{F}_t := \sigma(\varepsilon_\tau; \tau \leqslant t)$. The Gaussian $log-$likelihood function of $\underline{\theta} \in \Theta$ conditional on some initial values $\varepsilon_0, \ldots, \varepsilon_{1-q}, h_0, \ldots, h_{1-p}$, which are generated by (1.1)–(1.5), is given up to an additive constant by $\tilde{L}_{Ns}(\underline{\theta}) = -(Ns)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} \tilde{l}_{st+v}(\underline{\theta})$ with $\tilde{l}_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{\tilde{h}_t^2(\underline{\theta})} + \log \tilde{h}_t^2(\underline{\theta})$. Here $\widetilde{h}_t^2(\underline{\theta})$ is recursively defined, for $t \geqslant 1$ by $\tilde{h}_t^2(\underline{\theta}) = \alpha_0(t) + \sum_{i=1}^{q} \left(\alpha_i(t)\varepsilon_{t-i}^{+2} + \beta_i(t)\varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k(t)\varepsilon_{t-k}^{+} + \omega_k(t)\varepsilon_{t-k}^{-}\right)\tilde{h}_{t-k}(\underline{\theta}) + \sum_{j=1}^{p} \gamma_j(t)\tilde{h}_{t-j}^2(\underline{\theta})$. A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\underline{\hat{\theta}}_{Ns}$ of $\underline{\hat{\theta}}_{Ns} = Arg \max_{\underline{\theta} \in \Theta} \tilde{L}_{Ns}(\underline{\theta}) = Arg \min_{\underline{\theta} \in \Theta} \left(-\tilde{L}_{Ns}(\underline{\theta})\right)$. In view of the strong dependency of $\tilde{h}_t^2(\underline{\theta})$ on initial values $\varepsilon_0, \ldots, \varepsilon_{1-q}, h_0, \ldots, h_{1-p}$, $\left(\tilde{l}_t(\underline{\theta})\right)_{t \geqslant 1}$ is neither a SPS nor a periodically ergodic (PE) process Therefore, it will be more convenient to work with an unobserved SPS and PE version. So, we work with an approximate version $\tilde{L}_{Ns} = -(NS)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} l_{st+v}(\underline{\theta})$ of the likelihood $\tilde{L}_{Ns}(\underline{\theta})$ with $l_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta})$.

3. Monte Carlo experiment

In this section, we describe the performance of the finite sample properties of the $QMLE$ of the unknown parameters in $BLTGARCH_s(1,1,1)$ model based on Monte Carlo experiments. To this end, we simulate $T = 500$ replications for different moderate sample sizes $n \in \{2000, 4000\}$ with standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ as innovations distributions. The vector $\underline{\theta}$ of parameters is described in the bottom of each table below and is chosen to satisfy the strictly periodically stationary condition. All empirical results were obtained via implementation of our own scripts in $Matlab$ computing language. In the tables below, the columns correspond to the average of the parameters estimates over the N simulations. In order to show the performance of $QMLE$, the roots mean square error ($RMSE$) of the each $\hat{\theta}_n(i)$, $i = 1, \ldots, s$, (results between bracket), are reported in each table. Finally, the asymptotic distributions of $\hat{\theta}_n(v)$, $v = 1, \ldots, s$ over N simulations, followed by their boxplots summary, are plotted after each appropriate table.

3.1. Periodic BLTGARCH model

The example of our Monte Carlo experiment here is devoted to estimate the periodic $BLTGARCH_s(1,1,1)$ model with $s = 2$ according to standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ as innovations distributions. The vector of parameters to be estimated is thus $\underline{\theta} = (\underline{\alpha}_0', \underline{\alpha}_1', \underline{\beta}_1', \underline{b}_1', \underline{\omega}_1', \underline{\gamma}_1')'$ where $\underline{\alpha}_0' = (\alpha_0(1), \alpha_0(2))$, $\underline{\alpha}_1' = (\alpha_1(1), \alpha_1(2))'$, etc. . . are subjected to two models Model (1) and Model (2) described as: Model(1): The parameters are chosen to ensure the locally strictly stationarity condition i.e., for each $v = 1, 2$, $E\left\{\log\left|\alpha_1(v)e_0^{+2} + \beta_1(v)e_0^{-2} + b_1(v)e_0^{+} + \omega_1(v)e_0^{-} + \gamma_1(v)\right|\right\} < 0$, so $\left(h_t^2\right)_t$ is strict periodic stationary. Model (2): The parameters are chosen such that $E\left\{\log\left|\alpha_1(1)e_0^{+2} + \beta_1(1)e_0^{-2} + b_1(1)e_0^{+} + \omega_1(1)e_0^{-} + \gamma_1(1)\right|\right\} > 0$, but $\sum_{v=1}^{2} E\left\{\log\left|\alpha_1(v)e_0^{+2} + \beta_1(v)e_0^{-2} + b_1(v)e_0^{+} + \omega_1(v)e_0^{-} + \gamma_1(v)\right|\right\} < 0$, to ensure the strict periodic stationarity condition of $\left(h_t^2\right)_t$. The results of simulation according to both models (1) and (2) are given in Tab. 1.

The asymptotic distribution of the sequence $\left(\sqrt{n}\left(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i)\right)\right)_{n \geqslant 1}$, $i = 1, \ldots, 12$ followed by their boxplot summary according to model(1) of Tab. 1 are shown in Fig. 2.

Table 1. Average and $RMSE$ of 500 simulations of $QMLE$ for $PBLTGARCH_2(1,1,1)$

n	v	$\mathcal{N}(0,1)$ 2000	$\mathcal{N}(0,1)$ 4000	$t_{(5)}$ 2000	$t_{(5)}$ 4000
$\underline{\hat{\alpha}}_0$	1	0.9888 (0.0264)	0.9953 (0.0133)	0.9561 (0.0739)	0.9921 (0.0280)
	2	0.9944 (0.0286)	0.9928 (0.0134)	0.9515 (0.0728)	0.9721 (0.0343)
$\underline{\hat{\alpha}}_1$	1	0.4999 (0.0335)	0.4947 (0.0162)	0.5048 (0.0918)	0.4971 (0.0427)
	2	0.5008 (0.0414)	0.4973 (0.0203)	0.4905 (0.0944)	0.5038 (0.0598)
$\underline{\hat{\beta}}_1$	1	0.3631 (0.0468)	0.3562 (0.0242)	0.3661 (0.1020)	0.3596 (0.0608)
	2	0.3359 (0.0324)	0.3411 (0.0154)	0.3464 (0.0725)	0.3468 (0.0352)
$\underline{\hat{b}}_1$	1	−0.2607 (0.0688)	−0.2473 (0.0333)	−0.2760 (0.1567)	−0.2482 (0.0940)
	2	−0.0027 (0.0805)	0.0058 (0.0392)	0.0084 (0.1868)	−0.0054 (0.1054)
$\underline{\hat{\omega}}_1$	1	0.3240 (0.0977)	0.3412 (0.0500)	0.3198 (0.2002)	0.3459 (0.1247)
	2	0.0126 (0.0720)	0.0087 (0.0348)	−0.0030 (0.1636)	−0.0093 (0.0814)
$\underline{\hat{\gamma}}_1$	1	0.1598 (0.0093)	0.1527 (0.0043)	0.1793 (0.0280)	0.1549 (0.0110)
	2	0.1578 (0.0090)	0.1530 (0.0044)	0.1807 (0.0284)	0.1680 (0.0124)

Model(1) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.35, 0.35, -0.25, 0.00, 0.35, 0.00, 0.15, 0.15)'$

n	v	$\mathcal{N}(0,1)$ 2000	$\mathcal{N}(0,1)$ 4000	$t_{(5)}$ 2000	$t_{(5)}$ 4000
$\underline{\hat{\alpha}}_0$	1	0.9844 (0.0554)	0.9935 (0.0242)	0.9821 (0.1133)	0.9933 (0.0586)
	2	1.0391 (0.1631)	1.0152 (0.0764)	0.9648 (0.3207)	0.9962 (0.1817)
$\underline{\hat{\alpha}}_1$	1	0.5023 (0.0231)	0.4959 (0.0119)	0.5112 (0.0662)	0.5023 (0.0350)
	2	0.4819 (0.0640)	0.4870 (0.0327)	0.5140 (0.1545)	0.5203 (0.0872)
$\underline{\hat{\beta}}_1$	1	0.2571 (0.0108)	0.2541 (0.0055)	0.2757 (0.0321)	0.2512 (0.0148)
	2	0.4179 (0.0600)	0.4324 (0.0305)	0.4290 (0.1037)	0.4434 (0.0883)
$\underline{\hat{b}}_1$	1	0.2459 (0.0324)	0.2550 (0.0166)	0.2316 (0.0893)	0.2544 (0.0496)
	2	0.1794 (0.1533)	0.1748 (0.0765)	0.1381 (0.3424)	0.1219 (0.1929)
$\underline{\hat{\omega}}_1$	1	0.1404 (0.0204)	0.1470 (0.0103)	0.1175 (0.0569)	0.1495 (0.0261)
	2	0.1905 (0.1395)	0.1739 (0.0759)	0.1477 (0.2891)	0.1490 (0.1913)
$\underline{\hat{\gamma}}_1$	1	0.1532 (0.0018)	0.1502 (0.0008)	0.1587 (0.0061)	0.1518 (0.0032)
	2	0.7415 (0.0185)	0.7419 (0.0098)	0.7734 (0.0532)	0.7584 (0.0315)

Model(2) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.25, 0.45, 0.25, 0.15, 0.15, 0.15, 0.15, 0.75)'$

Comments: A quick glance to the results of Monte Carlo experiment shows that the results of Tab. 1 provide the parameters estimates of $PBLTGARCH_s(1,1,1)$, with $s = 2$ fitted on Model (1) and Model (2) generated by standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ innovations through 500 independent simulations. First, it is clear that the results of QML associated with $t_{(5)}$ innovations have a poor performance compared with those associated to $\mathcal{N}(0,1)$. In general, it can be observed that the parameters associated to these models are quite well estimated with non significant deviations in estimated values for two innovations errors $\mathcal{N}(0,1)$ and $t_{(5)}$. It is worth noting that some values of estimates have a moderate standard deviation. In Tab. 1 where two models was simulated following a $PBLTGARCH_s(1,1,1)$ model in which the parameters of the two regimes in Model(1) are such that $E\left\{\log\left|\alpha_1(v)\,e_0^{+2}+\beta_1(v)\,e_0^{-2}+b_1(v)\,e_0^{+}+\omega_1(v)\,e_0^{-}+\gamma_1(v)\right|\right\}<0$, $v = 1,\ldots,2$, whereas, in Model(2) the second regime is explosive in the sense that $E\left\{\log\left|\alpha_1(2)\,e_0^{+2}+\beta_1(2)\,e_0^{-2}+b_1(2)\,e_0^{+}+\omega_1(2)\,e_0^{-}+\gamma_1(2)\right|\right\}>0$, but the SPS of the model is ensured. Also, one can see that the results reveal in general quite satisfactory in accordance with the asymptotic theory results.

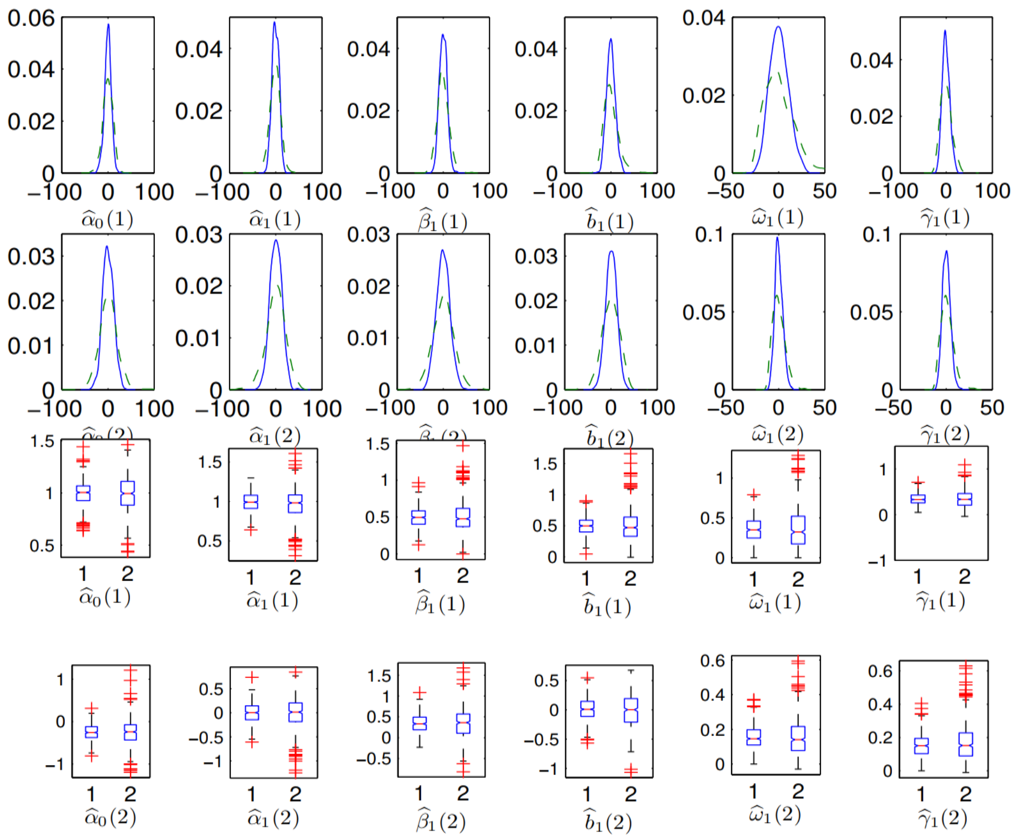

Fig. 2. Top panels: the asymptotic distribution of $\sqrt{n}(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i))$ (full line for *Normal* and dashed line for *Student*). Bottom panels: Box plot summary of $\underline{\hat{\theta}}_n(i)$, $i = 1, \ldots, 12$ (1 for *Normal* and 2 for *Student*) according to Model(1) of Tab. 1

4. Applications on exchange rates

The proposed model is investigated with real financial time series. So, we apply our model for modelling the foreign exchange rates of Algerian Dinar against the European currency (*Euro*) denoted by y_t already analyzed by Hamdi and Souam [3] via a mixture periodic *GARCH* models. We consider returns series $(r_t = 100 \times (\log(y_t/y_{t-1})))_{t\geqslant 1}$ of daily exchange rates of Algerian Dinar against Euro. The observation covers the period from January 3, 2000 to September 29, 2011. Since some weeks comprise less than five observations (due to legal holidays), we remove the entire weeks with less than five data available rather than estimating the "pseudo-missing" observations by an ad-hoc method. Thus, the final length of transformed data is 3055 observations uniformly distributed on 611 weeks. Fig. 3 displays the plots of the series (y_t) and its returns (r_t), squared return (r_t^2) and absolute return $(|r_t|)$.

By quickly examining the plots in Fig. 3, we can see that the original series are non stationary (since these do not fluctuate around a constant mean) and non-linear contrary to their returns that appear to be stationry. Moreover, there is no clear discernible behavior pattern in the returns, but some persistence is indicated in the plots of the squared and absolute returns. Additionally, some elementary statistics of the series $(y_t)_{t\geqslant 1}$ and its returns $(r_t)_{t\geqslant 1}$, squared return $(r_t^2)_{t\geqslant 1}$ and absolute return $(|r_t|)_{t\geqslant 1}$ are displeyed in Tab. 2

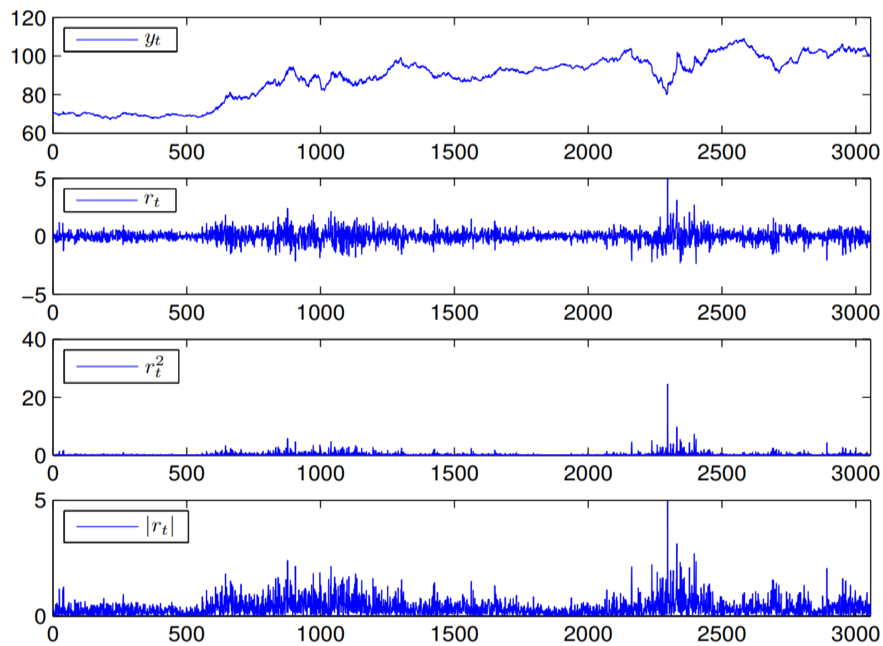

Fig. 3. The plots of the series(y_t) , squared r_t and absolute (r_t)

Table 2. Elementary statistics of the series $(y_t)_{t\geqslant 1}$, $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$

Series	Means	Std.Dev	Median	Skewness	Kurtosis
y_t	88.6118	11.5755	91.0995	−0.5181	2.1330
r_t	0.0118	0.5043	0.0123	0.3536	8.9678
r_t^2	0.2543	0.7193	0.0652	16.1027	464.3694
$\|r_t\|$	0.3575	0.3557	0.2554	2.6956	18.4307

Tab. 2 presents statistical summary of the series $(y_t)_{t\geqslant 1}$, $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ with summary measures of normality test results. The return $(r_t)_{t\geqslant 1}$ exhibits non-zero skewness and leptokurtic, while $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ exhibit significant skewness and kurtosis, indicating that their distribution is more peaked with a thicker tails than the normal distribution. Fig. 4 displays the sample autocorrelations functions (ACF) of the series $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ computed at 40 lags.

In Fig. 4, we can see that the log returns $(r_t)_{t\geqslant 1}$ show no evidence of serial correlation, but the squared and absolute returns are positively autocorrelated. Also, the decay rates of the sample autocorrelations of $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ appear to be violated compared with the correlation associated to an $ARMA$ process suggesting possibly a non linear behavior for modelling purpose.

4.1. Modeling with standard BLTGARCH model

The first attempt will be modeling the series $(r_t)_{t\geqslant 1}$ by a standard $BLTGARCH(1,1,1)$ model. The parameters estimates of volatility $(\hat{h}_t^{(s)})_{t\geqslant 1}$ to $BLTGARCH(1,1,1)$ with their $RMSE$ are given in Tab. 3.

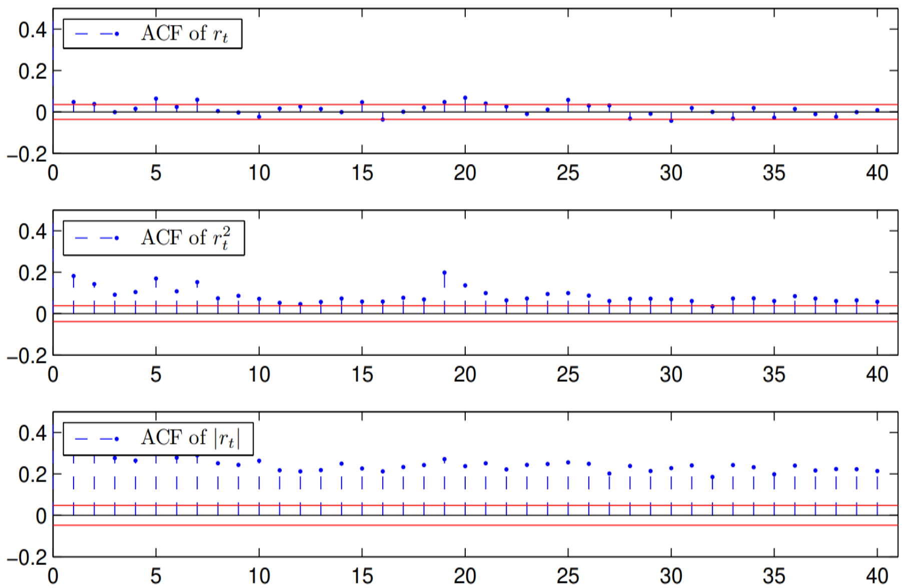

Fig. 4. The ACF of the returns and of their squred and absolute series

Table 3. Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(s)})_{t\geqslant 1}$

Parameters	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{b}_1$	$\hat{\omega}_1$	$\hat{\gamma}_1$
$\left(\hat{h}_t^{(s)}\right)_{t\geqslant 1}$	0.0007 (0.0005)	0.0304 (0.0176)	0.0591 (0.0224)	0.0276 (0.0439)	0.0283 (0.0430)	0.9540 (0.0175)

The plot of the estimated volatility $(\hat{h}_t^{(s)})_{t\geqslant 1}$ is shown later in the left side of Fig. 5.

4.2. Modeling with PBLTGARCH model

The second attempt is to look for a model able to cover the day-of -week seasonality in return (r_t) (see for instance Franses and Raap [9]). So, in order to analyze the seasonality, we fitted the following simple $PBLTGARCH_5(1,1,1)$ model for each series or equivalently. Hence, we estimate its volatility process $(h_t^2)_{t\geqslant 1}$ through five periodic effects, $r_t = h_t e_t$ and

$$h_t^2 = \alpha_0\,(t) + \left(\alpha_1\,(t)\,r_{t-1}^{+2} + \beta_1\,(t)\,r_{t-1}^{-2}\right) + \left(b_1\,(t)\,r_{t-1}^{+} + \omega_1\,(t)\,r_{t-1}^{-}\right)h_{t-k} + \gamma_1\,(t)\,h_{t-1}^2. \quad (14)$$

The parameters estimates of five-regimes (intra-day) of $(\hat{h}_t^{(p)})_{t\geqslant 1}$ and their $RMSE$ according to model (14) are reported in Tab. 4.
The plots of estimated volatilities and the squared returns associated to $(Euro)$ are showed in Fig. 5.

4.3. Comments

Tab. 3 and Tab. 4 display the $(\hat{h}_t)_{t\geqslant 1}$ estimated by Standard $BLTGARCH\,(1,1,1)$ and Periodic $BLTGARCH_5\,(1,1,1)$ models and reflect some characteristics of "spurious" $GARCH$ effects. In particular, the components of $\underline{\hat{\alpha}}_0$ are close to zeros while the components of $\underline{\hat{\gamma}}_1$ are close

Table 4. Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(p)})$

days	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{b}}_1$	$\underline{\hat{\omega}}_1$	$\underline{\hat{\gamma}}_1$
Sunday	0.0001	0.0145	0.0032	0.0165	0.0520	1.1826
	(0.0320)	(0.0329)	(0.0812)	(0.0926)	(0.1234)	(0.1894)
Monday	0.0010	0.0082	0.0419	0.0685	0.0831	1.0009
	(0.0296)	(0.0563)	(0.0588)	(0.2913)	(0.1429)	(0.1326)
Tuesday	0.0001	0.0015	0.0376	0.1162	0.0318	0.8504
	(0.0289)	(0.0651)	(0.0171)	(0.0611)	(0.0662)	(0.1156)
Wednesday	0.0025	0.0869	0.0648	0.0659	0.1768	0.7941
	(0.0142)	(0.0322)	(0.0345)	(0.1136)	(0.0951)	(0.0955)
Thursday	0.0002	0.0082	0.0645	0.0909	0.0229	0.9803
	(0.0160)	(0.0799)	(0.1260)	(0.2751)	(0.3544)	(0.2810)

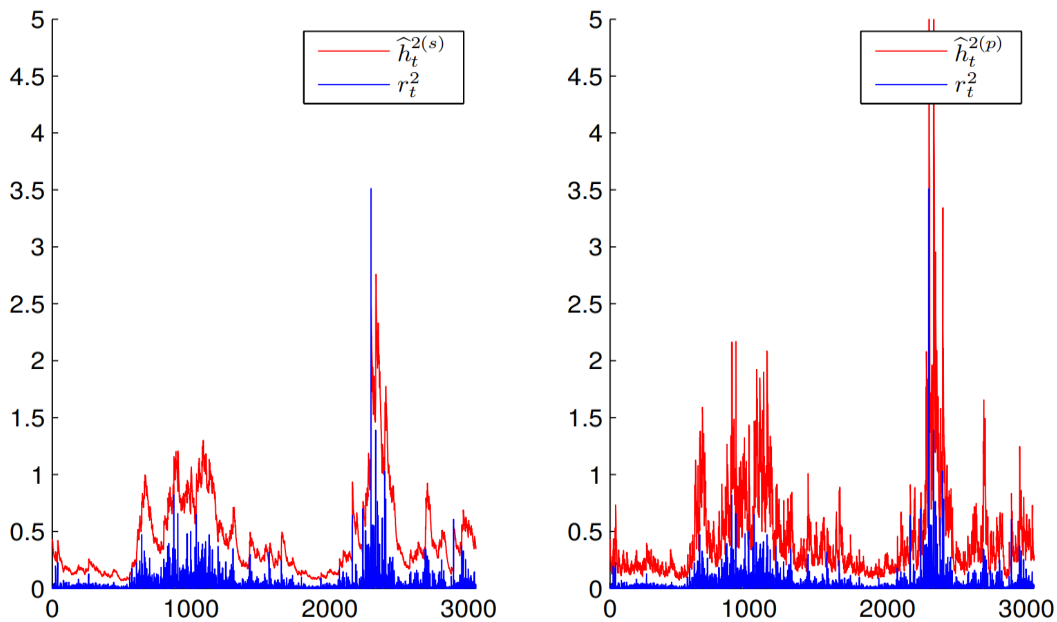

Fig. 5. Dark blue: squared returns, light red: volatilities estimates according to Standard $BLTGARCH(1,1,1)$ (left) and to Periodic $BLTGARCH_5(1,1,1)$ (right)

to ones with moderate $RMSE$. Fig. 5 represents the plots of the volatilities estimates (plots in red) according to $BLTGARCH\,(1,1,1)$ model (left) and $PBLTGARCH_5\,(1,1,1)$ model (right) and compared with the appropriate squared returns (plots in blue). It also demonstrates that a large piece of returns (positive or negative) leads to a high volatility and a small piece of returns leads to a low volatility, indicating volatility clustering. In particular, the period between 2000 and 2002 is characterized by low volatility levels compared to the period between 2009 and 2010 for both series. In addition, a high volatility cluster beginning in 2005 is observed and is mainly due to the global financial crisis. After this period of uncertainty, a cluster of low volatility is observed during 3 years. An other high volatility cluster is detected and could be related to the devaluation of the Dinar. Finally, the conditional volatility seems to be more stable after 2010. Our empirical results demonstrate that it is very difficult to distinguish between the volatilities $(\hat{h}_t^{(s)})_{t\geqslant 1}$ and $(\hat{h}_t^{(p)})_{t\geqslant 1}$ in Fig. 5, except perhaps, that the volatilities $(\hat{h}_t^{(p)})_{t\geqslant 1}$ is more fluctuated than $(\hat{h}_t^{(s)})_{t\geqslant 1}$. This finding may indicate the presence of a certain (hidden) periodicity in $(\hat{h}_t^{(p)})_{t\geqslant 1}$.

Conclusion

Beside the probabilistic structure and the conditions ensuring the existence of a SPS solution, this paper studies also the asymptotic properties of the quasi-maximum likelihood estimators of $PBLTGARCH(q, d, p)$ model. Indeed, for the first part, we have given the necessary and sufficient conditions for the existence of a strictly periodically stationary solution based on the negativity of the top-Lyapunov exponent. The paper presents for the second part, the theoretical results, which are illustrated in the third part by a Monte Carlo experiment through some usual innovations and an application to the exchange rate of the Algerian Dinar against the Euro showing its performance and its efficiency.

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О периодических билинейных пороговых моделях GARCH

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Аннотация. Периодические обобщенные авторегрессионные условно гетероскедастические модели ($PGARCH$) были представлены Bollerslev et Ghysels. Эти модели вызвали значительный интерес и продолжают привлекать внимание исследователей. Данная статья посвящена расширению стандартной билинейной пороговой модели $GARCH$ ($BLTGARCH$) до модели с периодически меняющимися во времени коэффициентами ($PBLTGARCH$). В этом классе моделей допускается переключение параметров между разными режимами. Более того, эти модели позволяют интегрировать асимметричные эффекты волатильности. Во-первых, мы приводим необходимые и достаточные условия, обеспечивающие существование стационарных решений (в периодическом смысле). Во-вторых, разработан подход оценки квазимаксимального правдоподобия (QML) для оценки модели $PBLTGARCH$. Точнее, сильная состоятельность и асимптотическая нормальность оценки изучаются при мягких условиях регулярности, требующих строгой стационарности и конечности моментов некоторого порядка для члена ошибки. Свойства $QMLE$ для конечной выборки иллюстрируются исследованием Монте-Карло. Наконец, предложенная нами модель применяется для моделирования обменных курсов алжирского динара по отношению к единой европейской валюте (*Euro*).

Ключевые слова: периодические билинейные пороговые модели $GARCH$, строго периодически стационарная, гауссовская оценка QML.

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Admissible Inference Rules of Temporal Intransitive Logic with the Operator "tomorrow"

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Abstract. We investigates non-transitive temporal logic with the "tomorrow" operator. In this logic, the operator "necessary" □ coincides with the operator “possible” ◇ (or almost coincides in reflexive case). In addition to the basic properties of the reflexive non-transitive logic $\mathcal{L}^r$ (decidability, finite approximability), admissible rules of this logic are investigated. The main result consists in proving the structural completeness of this logic and its tabular extensions.

Keywords: modal logic, frame and model Kripke, admissible and globally admissible inference rule.

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Introduction

The concept of a (structural) admissible inference rule was first introduced by Lorenzen [1] in 1955. For arbitrary logic, admissible rules of inference are those that do not change the set of provable theorems of a given logic. Any inferred rule is valid in the given logic; the reverse is not true in the general case.

Directly from the definition we can conclude that the set of all inference rules admissible in logic forms *the largest* class of inference rules with which we can expand the axiomatic system of a given logic without changing the set of provable theorems. In addition, admissible rules significantly strengthen the deductive system of a given logic. It is known that the derived inference rules can replace a certain reduce the proof linearly. Admissible rules that are not inferred by this logic can shorten the proof more significantly.

The beginning of the history of studying admissible rules can be dated back to 1975 since the appearance of H. Friedman's problem [2] on the existence of an algorithmic criterion for the admissibility of rules in the intuitionistic logic *Int*. In classical logic, the question of admissibility was resolved trivially — only deducible, provable rules are admissible. In the case of non-classical logics, the examples of Harrop, Mintz, and Post showed that there are admissible, but not provable rules of inference. In the mid-70s G. Mintz [3] obtained sufficient conditions for the deductibility of rules of a special form. A positive solution to Friedman's problem about the existence of an algorithm that recognizes the admissibility of inference rules in the intuitionistic logic *Int* was obtained by V. Rybakov in 1984 [4]. For a wide class of modal and superintuitionistic logics, the criterion for the admissibility of inference rules was later formulated in [5].

Another way of describing all admissible rules of logic goes back to the problem of A. Kuznetsov (1973) about the existence of a finite basis for admissible rules of inference of

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logic Int. Having a basis for admissible rules, all others are derived from it as consequences. The first positive result in the study of bases for admissible inference rules was obtained by A. Tsitkin [6], who found a basis for all quasi-characteristic inference rules admissible in Int.

In general, Kuznetsov's problem on the existence of a finite basis for admissible inference rules was solved negatively not only for Int (Rybakov, [7]), but also for most other basic logics. V. Rybakov (Chapter 4, [5]) showed that the logics $Int, KC, K4, S4, Grz$ and many other basic, individual logics do not have a finite basis for admissible rules from a finite number variables. Therefore, the problem of an explicit description of an easily observable basis for all admissible inference rules, at least for the main basic logics, becomes relevant.

One of the first results in this direction was obtained in 2000: in the paper [8] a recursive basis was constructed for admissible rules of intuitionistic logic Int, consisting of rules in semi-reduced form. Later, R. Iemhoff [9] obtained an explicit basis for the admissible rules of Int logic. In the article [10] V. Rybakov constructed an exact basis for all admissible rules of logic $S4$. This approach was further developed, for example, in [11, 12].

In the case of temporal (multimodal) or intransitive modal logics, relatively few results are known regarding admissible rules and their bases. The previously developed technique makes significant use of the transitivity of the reachability relation. In this work we make an attempt to fill the gap and explore the admissible rules of intransitive temporal logic $\mathcal{L}_0$ with the "tomorrow" operators and its extensions.

1. Definitions, preliminary facts

It is assumed that the reader is familiar with algebraic and Kripke semantics for modal logics, as well as some initial basic information about the rules of inference and their admissibility (although we briefly recall all the necessary facts below).

As a source on the subject as a whole, we can recommend Rybakov [5] among modern literature for a more developed technique for studying modal logics and rules of inference. In accordance with the modern interpretation, by *logic* we understand the set of all theorems that can be proven in a given axiomatic system.

In the definition, by propositional logic we mean algebraic propositional logic (see [5]), although the reader may consider λ to be modal logic, which is sufficient for our purposes. Initial information and all necessary statements used further, can be found for example in [5, ch. 2.2-2.5; 4.1].

Frame $\mathcal{F} := \langle F, R\rangle$ is a pair, where F is a non-empty set and R is a binary relation on F. The basic set and the frame itself will be further denoted by the same letter. A non-empty set $C \subseteq F$ is called a *cluster* if: 1) for any x, y from C, xRy holds; 2) for any $x \in C$ and $y\ inW$, $((xRy \& yRx) \Longrightarrow y \in C)$ is true. A cluster is called *proper* if $|C| > 1$; otherwise *singleton or degenerate*. For an element $a \in F$, let $C(a)$ denote the cluster (i.e., the set of elements mutually comparable with respect to R with a given element a) generated by the element a.

A sequence of elements $\{a_0; a_1; \ldots; a_n\}$ of an intransitive frame is called a chain of length $n+1$ if, for all $i < n$, element a_{i+1} is R-achievable from element a_i and there are no other frame elements between them.

The depth of element x of the model (frame) F is the maximum number of clusters in chains of clusters starting with the cluster $C(x)$ containing x. The set of all elements in the frame (models) F of depth no more than n will be denoted by $S_{\leqslant n}(F)$, and the set of elements of depth n will be denoted by $S_n(F)$.

Inference rule

$$\frac{\alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)}{\beta(x_1, \ldots, x_n)}$$

is called admissible in logic λ if for any formulas $\delta_1, \dots, \delta_n$ from $(\forall j \ \alpha_j(\delta_1, \dots, \delta_n) \in \lambda)$ it follows $\beta(\delta_1, \dots, \delta_n) \in \ \lambda)$.

The rule r is called a consequence of the rules $R := \{r_1; \dots; r_k\}$ in logic λ (notation $\mathcal{R} \vdash_\lambda r$), if the conclusion r is deducible from the premises r using theorems, rules $\{r_1; \dots; r_k\}$ and postulated rules of inference of λ. Inference rule $r = \{\alpha_1, \dots, \alpha_k/\beta\}$ is true on the algebra $\mathfrak{A} \in Var(\lambda)$ if and only if for any value of the variables r on $\mathfrak{A}$ as soon as $\forall \ j \ \mathfrak{A} \models_V \alpha_j$, then $\mathfrak{A} \models_V \beta$. Rule r is called a semantic consequence of the system of rules $\mathcal{R}$ in logic λ (notation $\mathcal{R} \models r$), if for any algebra $\mathfrak{A} \in Var(\lambda)$, as soon as all rules from $\mathcal{R}$ and all postulated rules of logic are true on the algebra $\mathfrak{A}$, then the rule r is also true on $\mathfrak{A}$. We say that modal logic is structurally complete if any admissible in the inference rule is deducible in .

Theorem 1.1 (Th. 1.4.11 [5]). *Let a set of inference rules $\mathcal{R} \cup \{r\}$ be given in the language algebraic logic λ. Then $\mathcal{R} \vdash r \iff \mathcal{R} \models r$. In particular, if $\mathcal{R} \not\models r$, then there is an algebra $\mathfrak{A} \in Var(\lambda)$, on which all the rules from $\mathcal{R}$ and all postulated rules of logic λ are true, but $\mathfrak{A} \not\models r$.*

For further presentation we will need n-characteristic Kripke models, with the help of which we will describe free algebras of finite ranks from the variety $Var(\lambda)$. Kripke model $\langle F, R, V\rangle$, where $V : \{p_1, p_2, \dots, p_n\} \to 2^F$, is called n-characteristic for logic λ if and only if for any formula in variables $p_1, \dots, p_n$, $\alpha \in \lambda \iff \langle F, R, V\rangle \models \alpha$.

Theorem 1.2 ([5]). *For any finitely approximable modal logic λ, the inference rule r is admissible in λ if and only if r is true on the frame $C_n(\lambda)$ for any n and for any formulaic valuation of the variables r.*

2. Логики $\mathcal{L}_0$ и $\mathcal{L}^r$

In the article [13] a temporal nontransitive logic $\mathcal{L}_0$ with the "tomorrow" operator was introduced. Let $\mathcal{L}_0 = L(F_\infty)$, where frame $F_\infty = \langle N, R\rangle$, N – set of natural numbers; and the relation R is defined as follows: $mRn \iff n = m+1$. This logic is convenient in that from each element of the $\mathcal{L}_0$-frame only one element is reachable with respect to the relation R, i.e. in this logic the operator $\Box$ coincides with the operator $\Diamond$. We also introduce the frame class F_n. Let's define $F_n = \langle\{1, \dots, n\}, R\rangle$, $n \in N, n > 0$, где $\forall i \in \{1, \dots, n-1\}(iRj \iff j = i+1) \wedge (nRn)$

We also define a nontransitive reflexive temporal logic $\mathcal{L}^r$. Let $\mathcal{L}^r = L(F^r_\infty)$, where $F^r_\infty = \langle N, R\rangle$, N is the set of natural numbers, and the relation R is defined as follows: $mRn \iff n = m+1 \vee n = m$. In this logic, from each element of the $\mathcal{L}^r$-frame, only one element different from the given one is reachable. We also introduce the class of frames $F^r_n = \langle\{1, \dots, n\}, R\rangle$, $n \in N, n > 0$, where $\forall i, j \in \{1, \dots, n-1\}(iRj \iff j = i+1 \vee i = j) \wedge (nRn)$. Let us define tabular extensions of logics $\mathcal{L}_0$ and $\mathcal{L}^r$. Let $\mathcal{L}_0$ и $\mathcal{L}^r$. Let $\mathcal{L}_n = L(F_n)$, where $F_n = \langle\{1, \dots, n\}, R\rangle$. Analogically, $\mathcal{L}^r_n = L(F^r_n)$, where $F^r_n = \langle\{1, \dots, n\}, R\rangle$.

Let α be a modal formula. The modal degree $deg(\alpha)$ of the formula α is determined as follows: $deg(p) = deg(\top) = deg(\bot) = 0$, $deg(\alpha \wedge \beta) = deg(\alpha \to \beta) = deg(\alpha \vee \beta) = \max\{deg(\alpha), deg(\beta)\}$, $deg(\neg\alpha) = deg(\alpha)$, $deg(\Box\alpha) = deg(\Diamond\alpha) = deg(\alpha) + 1$.

It is easy to prove the following statement by induction on the length of the formula:

Theorem 2.1 ([13]). *Let $deg(\alpha) = n$. Then the truth of the formula α on element x of frame F_∞ (F^r_∞) is uniquely determined by the values of all propositional variables included in the formula on elements x; $x+1$; $x+n$ of frame F_∞ (F^r_∞).*

This implies :

Theorem 2.2 ([13]). *The frame class $\{F_n | n \in N\}[\{F^r_n | n \in N\}]$ is characteristic of logic $\mathcal{L}_0$ [$\mathcal{L}^r$]. In particular, if $deg(\alpha) = n$, $n > 0$, and $F_\infty \not\models \alpha$, then $F_{n+1} \not\models \alpha$ [similarly if $deg(\alpha) = n$, $n > 0$, and $F^r_\infty \not\models \alpha$, then $F^r_{n+1} \not\models \alpha$].*

From the received statement 2.2 should also

Theorem 2.3 ([13]). *Logics $\mathcal{L}_0$ and $\mathcal{L}^r$ are finitely approximable and decidable.*

Let us construct an n-characteristic model $Ch_{\mathcal{L}_0}(n)$ by slices as follows. The first slice of the model consists of 2^n reflective elements, on which all possible valuations of propositional variables $p_1, p_2, \ldots, p_n$. To construct the second slice of this model, on each element of the first slice c_1 we hang from below $2^n - 1$ irreflexive elements with all sorts of different valuations of the propositional variables $p_1, p_2, \ldots, p_n$, different from the valuation of element c_1. We will construct the third slice of this model as follows. To each element of the second slice we assign from below 2^n irreflexive elements with all possible different valuations of the variables $p_1, p_2, \ldots, p_n$. We build all subsequent slices similarly to the third layer. Continuing the described process, as a result of construction we obtain the model $Ch_{\mathcal{L}_0}(n)$.

The n-characteristic model $Ch_{\mathcal{L}_m}(n)$ for tabular logic $\mathcal{L}_m$ is constructed in a similar way, with the only difference that the construction process continues until step (depth) m and ends at this step m. The model $Ch_{\mathcal{L}^r}(n)$ of reflexive logic is constructed in a similar way, with the only difference that at each construction step for each element c_1 we hang from below $2^n - 1$ reflexive elements with all possible different values of propositional variables $p_1, p_2, \ldots, p_n$, different from the valuation of element c_1.

Note that the frame generated by an arbitrary element of a given n-characteristic model is isomorphic to the frame F_k for some k. In the tabular case, the n-characteristic model $Ch_{\mathcal{L}_m}(n)$ is the p-morphic image of the direct union of a sufficient number of frames F_m.

Theorem 2.4. *The model $Ch_{\mathcal{L}_0}(n)$ ($Ch_{\mathcal{L}_m}(n)$, $Ch_{\mathcal{L}^r}(n)$) is n-characteristic for the logic $\mathcal{L}_0$ ($\mathcal{L}_m$, $\mathcal{L}^r$) respectively.*

Proof. In all three cases, the statement is proved in a similar way, so we will prove it only for logic $\mathcal{L}_0$. Let the formula α depend on n propositional variables. By construction, the frame of the model $\mathcal{L}_0$ is an $\mathcal{L}_0$-frame. This means that if $\alpha \in \mathcal{L}_0$, then it is true for all elements of this model.

If formula $\alpha \notin \mathcal{L}_0$, then due to the finite approximability of logic, there is a finite $\mathcal{L}_0$-frame F_m such that $F_m \not\models_V \alpha$ for some valuation V of the variables of the formula. Let's consider all possible cases:

1) All elements of F_m have different valuations of variables, i.e. an arbitrary element j and its predecessor $j-1$ are designated differently. In this case, the model is $\langle F_m, V\rangle$is an open submodel of the model $Ch_{\mathcal{L}_0}(n)$ (by construction of the latter). Therefore, $Ch_{\mathcal{L}_0}(n) \not\models_V \alpha$.

2) The reflexive element m and its R-predecessor $(m-1)$ have the same valuation for the variables of the formula α. In this case, if the elements $m,\ (m-1),\ (m-2),\ \ldots,(m-k),\ k \leqslant m$ have the same variable valuation, then we glue them slice by slice with the element m. In the resulting p-morphic image of the model, the reflexive element of the first slice and its predecessor are designated differently, and therefore is an open submodel of the n-characteristic model, i.e. $Ch_{\mathcal{L}_0}(n) \not\models_V \alpha$ is true.

For the model $Ch_{\mathcal{L}_m}(n)$ or $Ch_{\mathcal{L}^r}(n)$ the proof is similar. The statement has been proven. □

Since in all cases the various elements of the first slice of the n-characteristic model do not have a common R-predecessor, this model is a direct union of component $\mathcal{M}_i, i \leqslant 2^n$, i.e. $Ch_{\mathcal{L}}(n) = \sqcup \mathcal{M}_i,\ \mathcal{L} \in \{\mathcal{L}_0,\ \mathcal{L}_m,\ \mathcal{L}^r\}$. Each component $\mathcal{M}_i$ has the following structure: the first slice consists of a single reflective element. The second slice consists of $2^n - 1$ irreflexive (reflexive in the case of logic $Ch_{\mathcal{L}^r}(n)$) elements, the valuation of which is different from the valuation of the variables on the element of the first slice. Each element of the second and all subsequent slices has 2^n irreflexive ($2^n - 1$ reflexive in the case of logic $Ch_{\mathcal{L}^r}(n)$) immediate R-predecessor, etc. It is easy to show that in the tabular case the model $Ch_{\mathcal{L}_m}(n)$ is a p-morphic image of a finite direct union combining F_m frames. Accordingly, any $\mathcal{L}$-frame, where $\mathcal{L} \in \{\mathcal{L}_0,\ \mathcal{L}_m,\ \mathcal{L}^r\}$, is also a direct union of the components $\mathcal{M}_i$.

Lemma 2.5. *Each element of the n-characteristic model $Ch_{\mathcal{L}^r}(n)$ is formulaic.*

Proof. Let the model $Ch_{\mathcal{L}^r}(n)$ have valuation V variables $p_1, \dots, p_n$. According to the construction of this model, any two different elements $i, j \in Ch_{\mathcal{L}^r}(n) : iRj$ have different valuations of the variables. For each $x \in Ch_{\mathcal{L}^r}(n)$ of arbitrary depth m, we define the formulas:

$$\alpha(x) := \bigwedge\{p_j \mid x \models_V p_j\} \wedge \bigwedge\{\neg p_i \mid x \not\models_V p_i\};$$

$$f(x) := \alpha(x) \wedge \Diamond\alpha(x) \wedge \Diamond\alpha(x+1) \wedge \Diamond^2\alpha(x+2) \wedge \dots \wedge \Diamond^{m-1}\alpha(m) \wedge \Diamond^{m-1}\Box\alpha(m).$$

It is easy to see that $x \models_V f(x)$. Let us assume that the formula $f(x)$ is true on an element $i \in Ch_{\mathcal{L}^r}(n)$ under valuation V, other than x, and consider all possible cases of the location of this element.

1) If $x < i$ (i is located above x), then after $m - i < m - 1$ steps in relation R, stabilization occurs: $\alpha(m-i) = \alpha(m-i+1) = \dots = \alpha(m-1) = \alpha(m)$, which impossible, because according to the construction of the model, $\alpha(m-1) \neq \alpha(m)$ should be fulfilled.

2) Let now $x > i$ (i is located below x). Then, after $m - 1$ steps in relation R, stabilization should occur. Due to $i \models_V \Diamond^{m-1}\alpha(m) \wedge \Diamond^{m-1}\Box\alpha(m)$, elements reachable from element i in $m-1$ and m steps in relation R must have the same valuation. But this is not possible according to the construction of the model $Ch_{\mathcal{L}^r}(n)$.

3) If $x = i$, then after m steps with respect to R from both elements the same final element m is reachable by R. Consequently, elements x and i belong to the same component $\mathcal{M}_i$ of the n-characteristic model. Reasoning in a similar way, we find that all elements that are reachable from x and i are designated identically, i.e. according to the construction of the model, these elements $Ch_{\mathcal{L}^r}(n)$ elements coincide. □

This implies :

Lemma 2.6. *Each element of the n-characteristic model $Ch_{\mathcal{L}_m}(n)(Ch_{\mathcal{L}_m^r}(n))$ is formulaic.*

3. About structural completeness

Theorem 3.1. *Any finitely generated algebra generated by some $\mathcal{L}^r$-frame belongs to the quasivariety $\mathcal{F}_w^Q(\mathcal{L}^r)$. In particular, the variety $Var(\mathcal{L}^r)$ and the quasivariety $\mathcal{F}_w^Q(\mathcal{L}^r)$ coincide.*

Proof. Let $\mathcal{A} = G^+$ be a finitely generated $\mathcal{L}^r$-algebra. Hence, $\mathcal{A} \in HS\prod\mathcal{F}_w^Q(\mathcal{L}^r)$, i.e. this algebra is a homomorphic image of a subalgebra of the direct product of a certain number of free algebras of countable rank from the variety $Var(\mathcal{L}^r)$. Due to the local finiteness of the logic $\mathcal{L}^r$ (which is easy to verify), the algebra $\mathcal{A}$ is finite and generated by a certain finite $\mathcal{L}^r$-frame G. Consequently, this frame is an open subframe of the p-morphic image of the direct union of frames of the w-characteristic model $Ch_{\mathcal{L}^r}(w) = \sqcup\mathcal{M}_i$. Since the frame G is finite, we can take a direct union of a finite number of frames of the k-characteristic model $Ch_{\mathcal{L}^r}(k) = \sqcup\mathcal{M}_i$ for some suitable k.

As previously noted, any finite $\mathcal{L}^r$-frame is a direct union of the components $\mathcal{G}_j$. Therefore, the frame $G = \sqcup\mathcal{G}_j$ is an open subframe of the n-characteristic model frame $Ch_{\mathcal{L}^r}(n) = \sqcup\mathcal{M}_i$ for some suitable n. In particular, for all j we can assume without loss of generality that $\mathcal{G}_j \sqsubseteq \mathcal{M}_j$. Let us define a p-morphism g of a component $\mathcal{M}_j$ onto $\mathcal{G}_j$ for an arbitrary j as follows.

(1) for all elements of components $\mathcal{G}_j$ ($\mathcal{G}_j \sqsubset \mathcal{M}_j$), we define a p-morphism g as identical, i.e. $\forall\, x \in \mathcal{G}_j\ g(x) := x$. In particular, for the element $x_0 \in S_1(\mathcal{G}_j)$ we define $g(x_0) := x_0$.

(2) Let us now define g by slices on the entire component $\mathcal{M}_i \sqsubset Ch_{\mathcal{L}^r}(n)$ as follows. Let the p-morphism not yet be defined on the elements $y_1, \dots, y_k \in S_2(\mathcal{M}_j)$. Let's choose an arbitrary element $x_1 \in S_2(\mathcal{G}_j) \sqsubseteq S_2(\mathcal{M}_j)$. By (1) on such an element the p-morphism is already defined as

identical. Then we define $g(y_i) := x_1,\ 1 \leqslant i \leqslant k$. Thus, we define a p-morphism on the entire second slice $S_2(\mathcal{M}_j)$, preserving the depth of the elements.

Now let the p-morphism on the element $y \in S_3(\mathcal{M}_j)$ not yet be defined and let $yRz \& y \neq z$, where $z \in S_2(\mathcal{M}_j))$. The image $g(z) = e \in S_2(\mathcal{G}_j)$ is already defined. If e is R-maximal in $\mathcal{G}_j$, i.e. is not reachable from elements of a strictly greater depth in $\mathcal{G}_j$, then for all elements $\{t | \exists m \in N\ tR^m y\}$ we define $g(t) := e$ (i.e. the entire lower cone of the element y is p-morphically compressed into the element e).

If a given element e has immediate R-predecessors $\{e_1, \dots, e_k\}$ in the component $\mathcal{G}_j$, then element y is compressible with one of the elements $e_i, i \leqslant k$. For definiteness, we put $g(y) := e_1$. With this additional definition of p-morphism, the depth of the element is preserved.

By force of the arbitrariness of the choice of element y, we extend the p-morphism on the entire third layer of the component $S_3(\mathcal{M}_j)$. For elements of depth 4 and all subsequent layers of the component $\mathcal{M}_j$ we define a p-morphism in exactly the same way as above. Thus, as a result, the p-morphism g will be defined on the entire component $\mathcal{M}_j$ and $g(\mathcal{M}_j) = \mathcal{G}_j$.

(3) For all $\mathcal{M}_j \sqsubset Ch_{\mathcal{L}^r}(n) \setminus G$ we define $g(\mathcal{M}_j) := x_0$, where x_0 some fixed element of the first slice of an arbitrary component $\mathcal{G}_j \sqsubset G$.

Again, from the arbitrariness of the choice of j, we conclude that the required p-morphism g is defined on the entire frame of the n-characteristic model $Ch_{\mathcal{L}^r}(n) = \sqcup \mathcal{M}_i$. By Theorem 3.3.8 [5], the algebra generated by an arbitrary $\mathcal{L}^r$-frame G is a subalgebra of the free algebra $\mathcal{F}_q$ for some q, and therefore belongs to the quasivariety $\mathcal{F}_w^Q(\mathcal{L}^r)$.

A similar statement is also true for tabular irreflexive logic. It is easy to show in a similar way that for an arbitrary $\mathcal{L}_m$-frame there is a p-morphism from the frame of the n-characteristic model $Ch_{\mathcal{L}_m}(n)$ for a given frame. Taking into account the formulaicity and finiteness of this model, the following theorem is valid:

Theorem 3.2. *The algebra generated by an arbitrary finite $\mathcal{L}_m$-frame belongs to the quasivariety $\mathfrak{F}_w^Q(\mathcal{L}_m)$. In particular, the variety $Var(\mathcal{L}_m)$ and the quasivariety $\mathfrak{F}_w^Q(\mathcal{L}_m)$ coincide.*

Next, suppose that the rule $r := \alpha_1, \dots, \alpha_n/\beta$ not derivable in logic $\mathcal{L}_m$. Then, by Theorem 1.4.11 [5], this rule will be refuted on some finite $\mathcal{L}_m$-algebra A. It follows from the theorem that the rule will also be refuted on a free algebra of countable rank $\mathfrak{F}_w(\mathcal{L}_m)$. Therefore, rule r is not admissible in logic $\mathcal{L}_m$. Because any derived rule is also admissible, then the statement is proven:

Theorem 3.3. *The inference rule r is admissible in the logic $\mathcal{L}_m \iff$ this rule is derivable in the logic $\mathcal{L}_m$. In particular, the logic $\mathcal{L}_m$ is structurally complete.*

By virtue of Theorem 3.1, we can prove in exactly the same way

Theorem 3.4. *The inference rule r is admissible in the logic $\mathcal{L}_0^r \iff$ this rule is derivable in the logic $\mathcal{L}_0^r$. In particular, the logic $\mathcal{L}_0^r$ is structurally complete.*

Note that due to $\mathcal{L}_0 \subseteq \mathcal{L}_m$, for all natural numbers $n; m$ is executed $Ch_{\mathcal{L}_m}(n) \sqsubseteq Ch_{\mathcal{L}_0}(n)$. Directly from the definition of logics we conclude $\mathcal{L}_0 = \bigcap \mathcal{L}_m$. This implies:

Theorem 3.5. *If for all natural numbers m the inference rule r is admissible in the logic $\mathcal{L}_m$, then the rule r is admissible in the logic $\mathcal{L}_0$.*

Proof. Let the inference rule $r := \{\alpha_1, \dots, \alpha_n/\beta\}$ is not admissible in logic $\mathcal{L}_0$. Let us show that this rule r is not admissible in some tabular logic $\mathcal{L}_k$. In this case, the rule r is refuted at some formulaic valuation V on the frame of the n-characteristic model $Ch_{\mathcal{L}_0}(n)$ for some n: the premise of the rule is true on $Ch_{\mathcal{L}_0}(n)$, but there is an element $b \in Ch_{\mathcal{L}_0}(n)$, on which the conclusion of the rule is refuted with a given valuation:

$$Ch_{\mathcal{L}_0}(n) \models_V \alpha_i,\ 1 \leqslant i \leqslant n; \quad b \not\models_V \beta.$$

Without loss of generality, we can assume that a given element b is R-maximal among all such elements on which the conclusion of the rule is refuted and has depth k. Then frame

$$b^{\leqslant R} := \{x| \ \exists\ l \ : bRx_lRx_{l-1}\dots Rx\},$$

generated by element b is isomorphic to frame F_k, where k is the depth of element b. Therefore it is fulfilled

$$\langle F_k,\ V\rangle \models_V \alpha_i,\ 1 \leqslant i \leqslant n; \quad b \not\models_V \beta.$$

Let us show that the rule r will be inadmissible in the logic $\mathcal{L}_k = L(F_k)$. Indeed, as noted earlier, the frame of the k-characteristic model $Ch_{\mathcal{L}_k}(n)$ is the p-morphic image of the direct union of a sufficient (finite!) number of isomorphic copies of the frame F_k. Consequently, transferring the valuation V from the model $\langle F_k,\ V\rangle$ with the help of this p-morphism to the model $Ch_{\mathcal{L}_k}(n)$ we obtain $Ch_{\mathcal{L}_k}(n) \models_V \alpha_i,\ 1 \leqslant i \leqslant n;\quad \exists\, b \in Ch_{\mathcal{L}_k}(n)\ :\ b \not\models_V \beta$. Therefore, the inference rule $r := \{\alpha_1,\ \dots,\ \alpha_n/\beta\}$ is not admissible in the logic $\mathcal{L}_k$. □

Theorem 3.6 ([14])**.** *Logic $\mathcal{L}_0$ is not structurally complete.*

Proof. Let's define the inference rule

$$\mathcal{R} = \frac{(p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)}{\bot}.$$

Recall that the first skice of the n-characteristic model contains only reflexive elements. On these elements, for any value of the variable p, the premise of the rule R is not satisfied, which entails its admissibility in logic $\mathcal{L}_0$.

Let us now assume that this rule is derivable in $\mathcal{L}_0$. Then, by the deduction theorem, the following is true for modal logics:

$$\exists n_1, \dots, n_k \ \vdash_{\mathcal{L}_0} \Box^{n_1}((p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)) \wedge \dots \wedge \Box^{n_k}((p \wedge \Box\neg p) \vee (\neg p \wedge \Box p)) \to \bot.$$

At the same time, with the valuation $V(p) = \{2k \,|\, k \in N\}$, the premise of the rule is true on frame $\mathcal{F}_\infty$, and the conclusion is false, which entails the falsity of this formula, and therefore the rule R on frame $\mathcal{F}_\infty$. Therefore, rule R is not derivable in logic $\mathcal{L}_0$. □

Thus, the question arises about the existence of a finite basis for the admissible rules of logic $\mathcal{L}_0$ and its description, if it exists. Note that if an arbitrary $\mathcal{L}_0$-frame does not contain infinitely increasing chains of elements and R-maximal elements of finite depth (i.e. elements of finite depth that are not reachable with respect to R from other elements), then similarly to how it was done earlier, we can prove that the associated algebra belongs to the quasivariety $\mathfrak{F}^Q_w(\mathcal{L}_0)$. Consequently, as soon as on some $\mathcal{L}_0$-frame a rule admissible in the logic $\mathcal{L}_0$ is refuted, then a given frame contains an infinitely increasing chain of elements, or a chain of irreflexive elements of finite length. It is clear that in the first case on this frame Rule R is refuted.

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Допустимые правила временной нетранзитивной логики с оператором "завтра"

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Аннотация. В статье исследуется нетранзитивная временная логика с оператором "завтра". В этой логике оператор "необходимо" $\Box$ совпадает с оператором "возможно" $\Diamond$ (или почти совпадает в рефлексивном случае). Помимо базовых свойств рефлексивной нетранзитивной логики $\mathcal{L}^r$ (разрешимость, финитная аппроксимируемость) исследуются допустимые правила этой логики. Основной результат состоит в доказательстве структурной полноты данной логики и ее табличных расширений.

Ключевые слова: модальная логика, фрейм и модель Крипке, допустимое правило вывода, глобально допустимые правила вывода.

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Equilibrium Problem for a Kirchhoff-Love Plate Contacting by the Side Edge and the Bottom Boundary

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Abstract. A new model of a Kirchhoff–Love plate which is in contact with a rigid obstacle of a certain given configuration is proposed in the paper. The plate is in contact either on the side edge or on the bottom surface. A corresponding variational problem is formulated as a minimization problem for an energy functional over a non-convex set of admissible displacements subject to a non-penetration condition. The inequality type non-penetration condition is given as a system of inequalities that describe two cases of possible contacts of the plate and the rigid obstacle. Namely, these two cases correspond to different types of contacts by the plate side edge and by the plate bottom. The solvability of the problem is established. In particular case, when contact zone is known equivalent differential statement is obtained under the assumption of additional regularity for the solution of the variational problem.

Keywords: contact problem, non-penetration condition, non-convex set, variational problem.

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Introduction

Contact problems for solids with inequality type constraints have attracted attention of scientists since 1933s [1–3]. Problems of this kind are associated with the use of boundary conditions that describe non-penetration constraints on the contact surfaces or curves. For this Signorini problem it is assumed that some properties of displacements for points where a solid is in contact with a rigid obstacle [4, 5] or with another deformable body [6–9] are know in advance. It was established with the use of the fictitious domain method that a certain class of contact

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problems are connected with crack problems subject to non-penetration conditions on crack faces [10–12]. Point-wise contact problems were considered [13,14] where minimization problems over non-convex sets were studied.

In contrast to previous studies (see [4, 11]), a certain initial configuration of Kirchhoff–Love plate and an obstacle with a given geometrical shape are considered. The plate is in contact interaction with a rigid obstacle by its side edge or by its given front surface which is located below with respect to the selected coordinate system. In this case two types of restrictions are imposed. Namely, the first type is described by inequality for deflection functions (vertical displacements). The second type is described by inequalities for deflection functions and horizontal displacements. The main problem is formulated as a minimization of an energy functional over a non-convex set of admissible displacements. The solvability of the non-linear equilibrium problem is established. In particular case, when types of contact zones are known in advance an equivalent differential statement is obtained under the assumption of additional regularity for the solution of the variational problem.

1. The variational problem

Let $\Omega \subset \mathbf{R}^2$ be a bounded with a smooth boundary Γ which consists of two continuous curves $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\mathrm{mes}(\Gamma_0) > 0$. For convenience, it is supposed that

$$\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 = \psi(x_2), \quad x_2 \in [a, b]\},$$

where ψ is a given function, $a < b$, $a, b \in \mathbb{R}$. Let us denote the unit normal vector to Γ by $\nu = (\nu_1, \nu_2)$. For simplicity, suppose that plate has uniform thickness $2h$. Let us assign three-dimensional Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega\} \times \{0\} \subset \mathbf{R}^3$ corresponding to the middle plane of the plate.

Let us denote the displacement vector of the mid-surface points ($x \in \Omega$) by $\chi = \chi(x) = (W, w)$, displacements in the plane $\{x_1, x_2\}$ by $W = (w_1, w_2)$ and displacements along the axis z (deflections) by w. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$ and $\sigma_{ij} = \sigma_{ij}(W)$, respectively [5]:

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \quad \sigma_{ij}(W) = a_{ijkl}\varepsilon_{kl}(W), \quad i, j = 1, 2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor that is assumed to be symmetric and positive definite:

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega),$$

$$a_{ijkl}\xi_{ij}\xi_{kl} \geqslant c_0|\xi|^2 \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const > 0.$$

A summation convention over repeated indices is assumed. Bending moments are [5]

$$m_{ij}(w) = -d_{ijkl}w_{,kl}\,, \quad i, j = 1, 2, \quad \left(w_{,kl} = \frac{\partial^2 w}{\partial x_k \partial x_l}\right)$$

where tensor $\{d_{ijkl}\}$ has the same symmetry, boundedness, and positive definiteness characteristics as tensor $\{a_{ijkl}\}$. Let $B(\cdot\,,\cdot)$ be a bilinear form defined by the equality

$$B(\chi, \bar{\chi}) = \int_\Omega \left\{\sigma_{ij}(W)\,\varepsilon_{ij}(\bar{W}) - m_{ij}(w)\bar{w}_{,ij}\,\right\}dx, \tag{1}$$

where $\chi = (W, w)$, $\bar{\chi} = (\bar{W}, \bar{w})$.

Let us introduce Sobolev spaces

$$H^{1,0}_{\Gamma_0}(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_0 \right\},$$
$$H^{2,0}_{\Gamma_0}(\Omega) = \left\{ v \in H^2(\Omega) \;\middle|\; v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\},$$
$$H(\Omega) = H^{1,0}_{\Gamma_0}(\Omega)^2 \times H^{2,0}_{\Gamma_0}(\Omega).$$

It is well known that standard expression for potential energy functional of a Kirchhoff–Love plate has the following representation

$$\Pi(\chi) = \frac{1}{2} B(\chi, \chi) - \int_\Omega F\,\chi dx, \qquad \chi = (W, w),$$

where vector $F = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describes the body forces [5]. Note that the following inequality providing coercivity of functional $\Pi(\chi)$

$$B(\chi, \chi) \geqslant c\|\chi\|^2 \quad \forall\, \chi \in H(\Omega), \quad (\|\chi\| = \|\chi\|_{H(\Omega)}) \tag{2}$$

with a constant $c > 0$ that is independent of χ holds for bilinear form $B(\cdot, \cdot)$ [5].

An obstacle is described by the following part of the cylindrical surface

$$\{(x_1, x_2, z) \,|\, (x_1, x_2) \in \Gamma_1, \quad z \in (-\infty, -h]\}.$$

It restricts displacements on the side edge of the plate. Deflections are restricted by the following part of the plane

$$\{(x_1, x_2, z) \,|\, x_1 \leqslant \psi(x_2), \quad x_2 \in [a, b], \quad z = -h\,\}.$$

It is assumed that for the initial state the elastic plate touches a rigid obstacle with a given shape by its side edge corresponding to the points of curve Γ_1 as shown in Fig. 1:

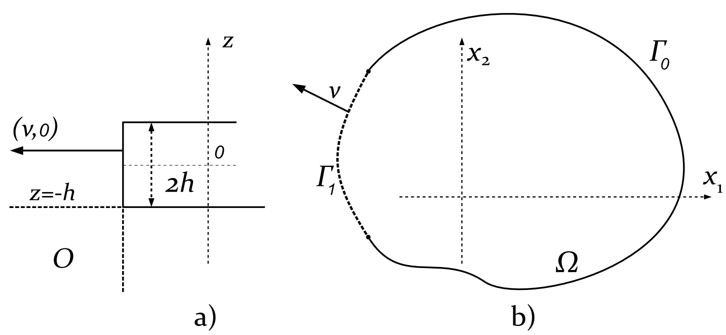

Fig. 1. a) cross section of the plate and the obstacle O; b) midsurface of the plate

At the part Γ_1 of the boundary the condition describing non-penetration of the plate points into the rigid obstacle is considered

$$w \geqslant 0 \quad \text{on} \quad \Gamma_1 \quad \text{or if this is not the case} \quad w \leqslant 0 \quad \text{and} \quad W\nu + h\frac{\partial w}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1. \tag{3}$$

It is worth to mention that according to (3) function $\chi = (W, w)$ satisfies either $w \geqslant 0$ on Γ_1 or two last inequalities (3). Now one can introduce the following set of admissible functions

$$K = \{\chi = (W, w) \in H(\Omega) \,|\, \chi \text{ satisfies (3)}\}.$$

Note that K is not convex set since for some $\alpha > 0$ one can construct functions $\tilde{\chi} = (\tilde{W}, \tilde{w}) \in K$ and $\hat{\chi} = (\hat{W}, \hat{w}) \in K$ with properties

$$\tilde{w} < -\alpha, \quad -\frac{\alpha}{2} < \tilde{W}\nu + h\frac{\partial \tilde{w}}{\partial \nu} \leqslant 0, \quad \frac{\alpha}{2} > \hat{w} > 0, \quad \hat{W}\nu + h\frac{\partial \hat{w}}{\partial \nu} > \alpha \quad \text{on} \quad \gamma \subset \Gamma_1, \quad \text{mes}(\gamma) > 0.$$

Obviously, function $\chi_s = \frac{1}{2}(\tilde{\chi} + \hat{\chi})$, $\chi_s = (W_s, w_s)$ does not belong to K because of

$$w_s \leqslant 0 \quad \text{and} \quad W_s\nu + h\frac{\partial w_s}{\partial \nu} > 0 \quad \text{on} \quad \gamma.$$

Let us formulate a variational statement of an equilibrium problem. It is required to find a function $\xi = (U, u) \in K$ such that

$$\Pi(\xi) = \inf_{\chi \in K} \Pi(\chi). \tag{4}$$

Theorem 1.1. *Problem* (4) *has a solution.*

Proof. The existence of a solution of the problem is established in accordance with the Weierstrass theorem [15]. It is well known that energy functional has properties of coercivity and weak lower semicontinuity on $H(\Omega)$ [16]. First, it is proved that set K is weakly closed. Let an arbitrary sequence $\{\chi_n\} \subset K$ be given with the property $\chi_n \to \chi$ in space $H(\Omega)$. By virtue of embedding theorems, this implies that there is a subsequence $\{\chi_n\}$, still denoted in the same way, that converges almost everywhere on Γ to χ. Let us prove that limit function χ also belongs to K. Indeed, there are the following relations for $\chi_n = (W_n, w_n)$

$$w_n \geqslant 0 \quad \text{or} \quad w_n \leqslant 0 \quad \text{and} \quad W_n\nu + h\frac{\partial w_n}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1 \backslash B$$

That are satified for each point of $\Gamma_1 \backslash B$, $\text{mes}(B) = 0$ and for all $n \in \mathbb{N}$. Therefore, for every fixed $x \in \Gamma_1 \backslash B$ one can obtain that

$$w_n(x) \geqslant 0 \quad \text{or} \quad w_n(x) \leqslant 0 \quad \text{and} \quad W_n(x)\nu + h\frac{\partial w_n(x)}{\partial \nu} \leqslant 0.$$

There must exists either a subsequence $\{\chi_{n_k}\} \subset \chi_n$ for which

$$w_{n_k}(x) \geqslant 0 \tag{5}$$

or a subsequence $\{\chi_{n_m}\}$ with the following property

$$w_{n_m}(x) \leqslant 0 \quad \text{and} \quad W_{n_m}(x)\nu + h\frac{\partial w_{n_m}(x)}{\partial \nu} \leqslant 0. \tag{6}$$

In both cases one can take the limit in corresponding inequalities, namely, for $\{\chi_{n_k}\}$ in (5), and for $\{\chi_{n_m}\}$ in (6). As a result, the following relations are obtained for limiting function χ

$$w(x) \geqslant 0$$

for the case of subsequence $\{\chi_{n_k}\}$ and

$$w(x) \leqslant 0 \quad \text{and} \quad W(x)\nu + h\frac{\partial w(x)}{\partial \nu} \leqslant 0.$$

for the case of subsequence $\{\chi_{n_m}\}$. Note that if both subsequences $\{\chi_{n_m}\}$ and $\{\chi_{n_k}\}$ with the mentioned properties exist then it means that

$$w(x) = 0 \quad \text{and} \quad W(x)\nu + h\frac{\partial w(x)}{\partial \nu} \leqslant 0.$$

Since point $x \in \Gamma_1 \backslash B$ is arbitrary, condition (3) is fulfilled for limiting function χ. Therefore, set K is weakly closed in $H(\Omega)$. Finally, for problem (4) conditions of the Weierstrass theorem for both functional $\Pi(\chi)$ and set of admissible functions K are satisfied. Then problem (4) has at least one solution. The theorem is proved.

2. Differential statement for the case of known contact zones

In this section the case when types of contact zones are known is considered. Let us assume that curve Γ_1 consists of disjoint curves Γ_1^e and Γ_1^b. Namely, it is supposed that inequalities

$$w \leqslant 0 \quad \text{and} \quad W\nu + h\frac{\partial w}{\partial \nu} \leqslant 0 \quad \text{on} \quad \Gamma_1^e, \tag{7}$$

describing a contact of the plate side edge are fulfilled on Γ_1^e. There is the following condition on the rest part Γ_1^b of curve Γ_1

$$w \geqslant 0 \quad \text{on} \quad \Gamma_1^b \tag{8}$$

which corresponds to a contact of the plate bottom with the rigid obstacle. A new set of admissible functions is introduced as follows

$$K_2 = \{\chi = (W, w) \in H(\Omega) \,|\, \chi \text{ satisfies } (7), (8)\}.$$

One can see that set K_2 is convex and closed. The convexity of set K_2 allow us to represent the following minimization problem

$$\Pi(\xi) = \inf_{\chi \in K_2} \Pi(\chi) \tag{9}$$

as variational inequality [5]

$$\xi \in K_2, \quad B(\xi, \chi - \xi) \geqslant \int_\Omega F(\chi - \xi)dx \quad \forall\, \chi \in K_2. \tag{10}$$

Suppose that solution $\xi = (U, u) \in K$ is sufficiently smooth. Next, let us apply the following Green's formulas for functions $\chi = (W, w) \in K$ [5]

$$\int_\Omega \sigma_{ij}(U)\, \varepsilon_{ij}(W)dx = -\int_\Omega \sigma_{ij,j}(U)w_i\, dx + \int_\Gamma \Big(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau\Big)\, d\Gamma, \tag{11}$$

$$\int_\Omega m_{ij}(u)w_{,ij}\, dx = \int_\Omega m_{ij,ij}(u)\, w\, dx + \int_\Gamma \Big(t^\nu(u)w - m_\nu(u)\frac{\partial w}{\partial \nu}\Big)\, d\Gamma, \tag{12}$$

where

$$\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j, \quad m_\nu(u) = -m_{ij}\nu_i\nu_j,$$

$$\sigma_\tau(U) = (\sigma_\tau^1(U), \sigma_\tau^2(U)) = (\sigma_{1j}(U)\nu_j, \sigma_{2j}(U)\nu_j) - \sigma_\nu(U)\nu,$$

$$t^\nu(u) = -m_{ij,k}\tau_k\tau_j\nu_i - m_{ij,j}\nu_i, \quad \tau = (-\nu_2, \nu_1),$$

$$W\nu = w_i\nu_i, \quad W\tau = (W_\tau^1, W_\tau^2), \quad w_i = (W\nu)\nu_i + W_\tau^i, \quad i = 1, 2.$$

Along with variational statement (9), one can deal with corresponding differential statement. Namely, the following theorem holds.

Theorem 2.1. *Supposing that solution $\xi = (U, u)$ is sufficiently smooth, variational problem* (9) *is equivalent to the following boundary value problem*

$$-m_{ij,ij}(u) = f_3 \quad in \quad \Omega, \tag{13}$$

$$-\sigma_{ij,j}(U) = f_i \quad in \quad \Omega, \quad i = 1, 2, \tag{14}$$

$$t^\nu(u) \leqslant 0, \quad u \leqslant 0, \quad \sigma_\nu(U) \leqslant 0, \quad \sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0 \quad on \quad \Gamma_1^e, \tag{15}$$

$$\sigma_\tau(U) = (0, 0), \quad U\nu + h\frac{\partial u}{\partial \nu} \leqslant 0 \quad on \quad \Gamma_1^e, \tag{16}$$

$$\sigma_\tau(U) = (0, 0), \quad \sigma_\nu(U) = m_\nu(u) = 0, \quad t_\nu(u) \geqslant 0, \quad u \geqslant 0 \quad on \quad \Gamma_1^b, \tag{17}$$

$$\sigma_\nu(U)U\nu - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_1^e, \quad t^\nu(u)u = 0 \quad on \quad \Gamma_1^b, \tag{18}$$

$$U = u = \frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_0. \tag{19}$$

Proof. Substituting $\bar{\chi} = \xi \pm \tilde{\chi}$, where $\tilde{\chi} \in C_0^\infty(\Omega)^3$, as a test function into (10), one can obtain the following relation

$$\int_\Omega (\sigma_{ij}(U)\, \varepsilon_{ij}(\tilde{W}) - m_{ij}(u)\tilde{w}_{,ij}\,)dx = \int_\Omega F\tilde{\chi}dx$$

, that is, equilibrium equations

$$-m_{ij,ij}(u) = f_3 \quad \text{in} \quad \Omega, \tag{20}$$

$$-\sigma_{ij,j}(U) = f_i \quad \text{in} \quad \Omega, \quad i = 1, 2, \tag{21}$$

hold in terms of distribution.

Applying Green's formulas to (10) and using (20), (21), one can show that

$$\int_\Gamma \Bigg(\sigma_\nu(U)(W - U)\nu + \sigma_\tau(U)(W - U)\tau -$$

$$-t^\nu(u)(w - u) + m_\nu(u)\left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\Bigg) d\Gamma \geqslant 0 \quad \forall\, \chi = (W, w) \in K. \tag{22}$$

Since K is convex cone in $H(\Omega)$, one can substitute $\chi = \lambda\xi$ in (22) and deduce

$$\int_\Gamma \left(\sigma_\nu(U)U\nu + \sigma_\tau(U)\bar{U}\tau - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu}\right) d\Gamma = 0, \tag{23}$$

$$\int_\Gamma \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu}\right) d\Gamma \geqslant 0, \tag{24}$$

for all $\chi = (W, w) \in K$. Let us suppose that $\chi = (W, w) \in K$ and $\chi = 0$ on Γ_1^b. In this case one can rewrite (24) as follows

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu}\right) d\Gamma \geqslant 0. \tag{25}$$

Since $W\tau$ is not included in inequalities (7), due to arbitrariness of $W\tau$ on Γ_1^e one can conclude that

$$\sigma_\tau(U) = 0 \quad \text{on} \quad \Gamma_1^e.$$

Therefore, inequality (25) can be reduced to

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0. \tag{26}$$

By choosing functions $\chi = (W, w)$ such that $W = 0$, $\dfrac{\partial w}{\partial \nu} = 0$ on Γ_1^e, one can obtain in (26) that

$$t^\nu(u) \geqslant 0 \quad \text{on} \quad \Gamma_1^e.$$

Now one can substitute test functions with properties $w = 0$, $W\nu + h\dfrac{\partial w}{\partial \nu} = 0$ and obtain

$$\int_{\Gamma_1^e} \left(\sigma_\nu(U)W\nu - \frac{1}{h} m_\nu(u)W\nu \right) d\Gamma \geqslant 0. \tag{27}$$

Then

$$\sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0 \quad \text{on} \quad \Gamma_1^e.$$

since the value of $W\nu$ can be arbitrary. The last equality allow us to represent (27) in the form

$$\int_{\Gamma_1^e} \sigma_\nu(U)\Big(W\nu + h\frac{\partial w}{\partial \nu} \Big) d\Gamma \geqslant 0,$$

Then, it follows that

$$\sigma_\nu(U) \leqslant 0 \quad \text{on} \quad \Gamma_1^e.$$

Let us assume that $\chi \in K$, $\chi = 0$ on Γ_1^e. Then one can obtain on Γ_1^b that $w \geqslant 0$ and

$$\int_{\Gamma_1^b} \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geqslant 0. \tag{28}$$

Due to arbitrariness of W, $\dfrac{\partial w}{\partial \nu}$ on Γ_1^b there are following equalities

$$\sigma_\tau(U) = (0,0), \quad \sigma_\nu(U) = m_\nu(u) = 0 \quad \text{on} \quad \Gamma_1^b.$$

Now, it remains to deduce from the reduced inequality

$$-\int_{\Gamma_1^b} t^\nu(u)w \, d\Gamma \geqslant 0 \tag{29}$$

the following inequality

$$t^\nu(u) \leqslant 0 \quad \text{on} \quad \Gamma_1^b.$$

Let us consider relation (23). Let us take into account that $\xi = (U, u) \in K$,

$$\sigma_\nu(U) \leqslant 0, \quad \sigma_\nu(U) - \frac{1}{h} m_\nu(u) = 0, \quad t^\nu(u) \geqslant 0 \quad \text{on} \quad \Gamma_1^e,$$

and

$$\sigma_\tau(U) = (0,0), \quad \sigma_\nu(U) = m_\nu(u) = 0, \quad t^\nu(u) \leqslant 0 \quad \text{on} \quad \Gamma_1^b.$$

Then corresponding integrand of (23) is non-negative a.e. on Γ. Therefore

$$\sigma_\nu(U)U\nu - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1^e, \quad t^\nu(u)u = 0 \quad \text{on} \quad \Gamma_1^b.$$

Conversely, in order to obtain variational inequality (10) from (13)–(19) relation (13) is multiplied by $(u-w)$ and relations (14) are multiplied by $(u_i - w_i)$, $i = 1, 2$, where $W = (w_1, w_2)$ and w such that $\chi = (W, w) \in K$. Then after integrating over Ω and summing, one can obtain

$$-\int_\Omega (\sigma_{ij,j}(U)(U-W) + m_{ij,ij}(u)(w-u))dx = \int_\Omega F(\chi - \xi)dx.$$

Now let us use the Green formulas and obtain

$$\begin{aligned}
&\int_\Omega \Big(\sigma_{ij}(U)\,\varepsilon_{ij}(W-U)dx - m_{ij}(u)(w-u)_{,ij}\Big)dx - \\
&-\int_\Gamma \Big(\sigma_\nu(U)(W\nu - U\nu) + \sigma_\tau(U)(W\tau - U\tau)\Big)\, d\Gamma + \\
&+\int_\Gamma \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu})\Big)\, d\Gamma = \int_\Omega F(\chi-\xi)dx.
\end{aligned} \tag{30}$$

Taking into account that $\sigma_\tau(U) = 0$ on Γ_1, $\xi = \chi = 0$ on Γ_0, the sum of integrals over Γ in the left side of (30) can be represented as follows

$$I = \int_{\Gamma_1} \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\, d\Gamma. \tag{31}$$

Then bearing in mind the equalities $\sigma_\tau(U) = (0,0)$, $\sigma_\nu(U) = m_\nu(u) = 0$ on Γ_1^b, relation (31) can be represented as the following sum

$$\begin{aligned}
I = &\int_{\Gamma_1^b} t^\nu(u)(w-u)d\Gamma + \\
&+\int_{\Gamma_1^e} \Big(t^\nu(u)(w-u) - m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\, d\Gamma.
\end{aligned} \tag{32}$$

Finally, relation (32) can be transformed into

$$\begin{aligned}
I = &\int_{\Gamma_1^b} t^\nu(u)(w-u)d\Gamma + \int_{\Gamma_1^e} \Big(t^\nu(u)w - \sigma_\nu(U)\big(W\nu + h\frac{\partial w}{\partial \nu})\Big)\, d\Gamma - \\
&-\int_{\Gamma_1^e} \Big(t^\nu(u)u - \sigma_\nu(U)U\nu - m_\nu(u)\frac{\partial u}{\partial \nu}\Big)\, d\Gamma.
\end{aligned}$$

Taking into account relations (15)–(18) and $\chi \in K$, one can see that each term in the last sum is non-positive. It remains to note that since $I \leqslant 0$ equality (30) yields variational inequality (10). □

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Задача о равновесии пластины Кирхгофа-Лява, контактирующей боковой кромкой и лицевой поверхностью

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Аннотация. Предложена новая модель пластины Кирхгофа–Лява, которая может соприкасаться либо по боковой грани, либо по одной из лицевых поверхностей с жестким препятствием определенной заданной конфигурации. Соответствующая вариационная задача формулируется в виде задачи минимизации функционала энергии над невыпуклым множеством допустимых перемещений с условием непроникания. Условие непроникания представлено в виде системы неравенств, описывающей два случая возможного контакта пластины и жесткого препятствия. А именно эти два случая соответствуют разным типам контактов: со стороны боковой кромки пластины и со стороны ее известной лицевой поверхности. Установлена разрешимость задачи. В частном случае, когда зоны контакта заранее известны, найдена эквивалентная дифференциальная постановка в предположении дополнительной регулярности решения вариационной задачи.

Ключевые слова: контактная задача, условие непроникания, невыпуклое множество, вариационная задача.

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On the Collection Formulas for Positive Words

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Abstract. For any formal commutator R of a free group F, we constructively prove the existence of a logical formula $\mathcal{E}_R$ with the following properties. First, if we apply the collection process to a positive word W of the group F, then the structure of $\mathcal{E}_R$ is determined by R, and the logical values of $\mathcal{E}_R$ are determined by W and the arrangement of the collected commutators. Second, if the commutator R was collected during the collection process, then its exponent is equal to the number of elements of the set $D(R)$ that satisfy $\mathcal{E}_R$, where $D(R)$ is determined by R. We provide examples of $\mathcal{E}_R$ for some commutators R and, as a consequence, calculate their exponents for different positive words of F. In particular, an explicit collection formula is obtained for the word $(a_1 \dots a_n)^m$, $n, m \geqslant 1$, in a group with the Abelian commutator subgroup. Also, we consider the dependence of the exponent of a commutator on the arrangement of the commutators collected during the collection process.

Keywords: commutator, collection process, free group.

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Introduction

We continue our research [1] on the collection process, the concept of which was introduced by P. Hall [2]. Let W be a positive word of the free group $F = F(a_1, \dots, a_n)$, $n \geqslant 2$, i.e. W does not contain inverses of $a_1, \dots, a_n$. By rearranging step by step consecutive occurrences of elements in W with use of commutators: $QR = RQ[Q, R]$, $Q, R \in F$, the collection process transforms W into the following form:

$$W = q_1^{e_1} \dots q_j^{e_j} T_j, \qquad j \geqslant 1, \tag{1}$$

where $q_1, \dots, q_j$ are commutators in $a_1, \dots, a_m$ arranged in order of increasing weights, T_j consists of commutators of weights not less than $w(q_j)$ (the weight of q_j), the exponents $e_1, \dots, e_j$ are positive integers. Further we will not impose restrictions on the arrangement of $q_1, \dots, q_j$.

Research is developing in two directions. The first one is connected with divisibility properties of the exponents $e_1, \dots, e_j$ for some words W. In [2, Theorems 3.1 and 3.2] the application of the collection process to the word $W = (a_1 a_2)^m$, $m \geqslant 1$, leads to the formula

$$(a_1 a_2)^m = q_1^{e_1} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \geqslant 2, \tag{2}$$

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where $\Gamma_s(F)$ is the s-th term of the lower central series of F, which is defined as follows: $\Gamma_1(F) = F$, $\Gamma_k(F) = [\Gamma_{k-1}(F), F]$, $k \geqslant 2$, and the exponents of the commutators are expressed in the following form:

$$e_i(m) = \sum_{k=1}^{w(q_i)} c_k \binom{m}{k}, \tag{3}$$

where non-negative integers c_k do not depend on m. This result is significant for the theory of p-groups, since the expression $e_i(p^\alpha)$ is divisible by the prime power p^α if $w(q_i) < p$. In [3, Theorem 12.3.1] the same result is obtained for the word $(a_1 \dots a_n)^m$, $n \geqslant 1$. In [4, Theorems 5.13A and 5.13B] a similar formula with divisibility properties of the exponents of the commutators is proved for W^m, where W is an arbitrary word (not necessarily positive), $m \geqslant 1$. The work [5, Lemma 4] devoted to nilpotent products of cyclic groups and also the works [6, 7] consider the word $W = a_1^{m_1} a_2^{m_2}$ (with some restrictions on $m_1, m_2 \geqslant 1$) for which some divisibility properties of the exponents of the commutators are obtained. The author's work [1] proposes an approach to studying the exponents e_j in (1) and gives generalizations of the above results using this approach.

The second direction is connected with an explicit form of the exponents e_j in the P. Hall's collection formula (2) and, as a consequence, with explicit collection formulas (2) in groups with some restrictions (solvable length, nilpotency class of the group, etc). For example, the explicit formula

$$(a_1 a_2)^m = a_1^m a_2^m [a_2, a_1]^{\binom{m}{2}}$$

is well known for a group G, where $a_1, a_2 \in G$, $[a_2, a_1] \in Z(G)$. Formula (2) and the exponent $\binom{m}{i+1}$ of the commutator $[a_2, {}_i a_1]$, $i \geqslant 1$, have been used to prove the $(p-1)$-th Engel congruence $[a_2, {}_{p-1} a_1] = 1 \pmod{\Gamma_{p+1}(G)}$ for a group G of prime exponent p, which was the key to investigation of the restricted Burnside problem for groups of prime exponent p [3, p. 327]. With use of the exponents for more complex commutators, the 14-th Engel congruence has been proved for groups of exponent 8 in [8, 9]. Also, explicit collection formulas (2) for groups with some restrictions are considered in the works [10–13]. The explicit formula (2) for a group where any commutator with more than two occurrences of a_2 is equal to 1 has been used to prove the non-regularity of the Sylow p-subgroup of the general linear group $GL_n(\mathbb{Z}_{p^m})$ for $n \geqslant (p+2)/3$ and $m \geqslant 3$, when $(p+2)/3$ is an integer [14]. This has lead to partial solution to Wehrfritz's problem [15, Question 8.3]. The exponents for several series of the commutators in (2) have been found in an explicit form in the author's work [16].

In this paper, for any formal commutator R of the group $F(a_1, \dots, a_n)$, we constructively prove the existence of a logical formula $\mathcal{E}_R$ using which one can calculate the exponent of R using information about the initial word W in the collection process and the arrangement of the collected commutators $q_1, \dots, q_j$ (Theorem 1). The formula $\mathcal{E}_R$ has the following properties. First, its structure is determined by R, and its logical values are determined by the word W and the arrangement of the collected commutators. Second, if R was collected during the collection process, then its exponent is equal to the number of elements of the set $D(R)$ that satisfy the formula $\mathcal{E}_R$, where $D(R)$ is determined by R. We provide examples of $\mathcal{E}_R$ for some commutators R (Lemmas 1, 2) and, as a consequence, calculate their exponents for different positive words (Theorem 2). In particular, an explicit collection formula is obtained for the word $(a_1 \dots a_n)^m$, $n, m \geqslant 1$, in a group with the Abelian commutator subgroup (Theorem 3). Also, we consider the dependence of the exponent of a commutator on the arrangement of the collected commutators $q_1, \dots, q_j$ (Corollary 2).

1. Basic notation

In this paper we use the concepts formally defined in Sections 2 and 3 of the article [1]. The basic properties of the collection process and examples are also given there. In this section we will briefly describe some important concepts.

The *collection process* is a construction of the sequence of words:

$$W_0 \equiv T_0, \quad W_1 \equiv q_1^{e_1} T_1, \quad W_2 \equiv q_1^{e_1} q_2^{e_2} T_2, \quad \ldots, \quad W_j = q_1^{e_1} \ldots q_j^{e_j} T_j, \quad \ldots \tag{4}$$

by the following rules. The initial word W_0 is a positive word of the free group $F = F(a_1, \ldots, a_n)$, $n \geqslant 2$. All occurrences of the letters $a_1, \ldots, a_n$ (commutators of weight 1) have *labels* (integer sequences) assigned to them, and different occurrences of the same letter have pairwise different labels of the same length. Let q_j be an arbitrary commutator from the *uncollected part* T_{j-1}. The word W_j, $j \geqslant 1$, is obtained from W_{j-1} by moving step by step all the occurrences of q_j to the beginning of the word T_{j-1} with use of commutators:

$$Q(\Lambda_u)R(\Lambda_v) = R(\Lambda_v)Q(\Lambda_u)[Q, R](\Lambda_u\Lambda_v),$$

where $\Lambda_u\Lambda_v$ is the concatenation of the labels Λ_u and Λ_v.

Denote by $D(a_k)$, $k \in \overline{1, n}$, an arbitrary fixed set of integer sequences of the same length that contains all labels of the occurrences of a_k in W_0. Assume that a commutator R arose during the collection process (4), and the parenthesis-free notation of R is $(a_{i_1}, \ldots, a_{i_{w(R)}})$. Then any occurrence of R has a label that belongs to the Cartesian product $D(R) = D(a_{i_1}) \times \cdots \times D(a_{i_{w(R)}})$.

Suppose that some uncollected part in (4) contains an occurrence of the commutator R. The *existence condition* of the commutator R is the predicate E_R^Λ, $\Lambda \in D(R)$, that is equal to 1 iff there exists a word in (4) such that its uncollected part contains the occurrence $R(\Lambda)$.

Suppose that some uncollected part in (4) contains occurrences of the commutators R and Q. The *precedence condition* for the commutators R and Q is the predicate $P_{Q,R}^{\Lambda_1\Lambda_2}$, $\Lambda_1\Lambda_2 \in D(Q) \times D(R)$, that is equal to 1 iff there exists a word in (4) such that, in its uncollected part, $Q(\Lambda_1)$ precedes (is to the left of) $R(\Lambda_2)$.

For the exponent e_j, $j \geqslant 1$, in (4) we have

$$e_j = |\{\Lambda \in D(q_j) \mid E_{q_j}^\Lambda = 1\}|. \tag{5}$$

Let R_1, R_2 be formal commutators. The predicate $R_1 \prec R_2$ is equal to 1 iff there exist commutators q_i, q_j in (4) such that $q_i = R_1$, $q_j = R_2$, $i < j$, i.e., the occurrences of R_1 were collected at an earlier stage than the occurrences of R_2 in the variant of the collection process (4).

In [1, Theorem 4.6] the following recurrence relations for the existence and precedence conditions were proved. We will use these relations further.

Suppose $\{W_j \equiv q_1^{e_1} \ldots q_j^{e_j} T_j\}_{j \geqslant 0}$ is an arbitrary variant of the collection process. Then the following recurrence relations hold (if the left-hand side of a relation is defined for $\{W_j\}_{j \geqslant 0}$):

$$E_{[Q_1, {}_u R_1]}^{\Lambda_0^1 \ldots \Lambda_u^1} = P_{Q_1, R_1}^{\Lambda_0^1 \Lambda_1^1} \bigwedge_{k=1}^{u-1} P_{R_1, R_1}^{\Lambda_k^1 \Lambda_{k+1}^1}, \quad u \geqslant 1; \tag{6}$$

$P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2}_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}$ is equal to

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}F, \qquad \text{if } u+v\geqslant 1,\ R_1=R_2,\ Q_1=Q_2; \tag{7a}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2}, \qquad \begin{array}{l}\text{if } u+v\geqslant 1,\ R_1=R_2,\ Q_1\neq Q_2,\\ u=0\Rightarrow w(Q_1)=1,\ v=0\Rightarrow w(Q_2)=1;\end{array} \tag{7b}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}, \qquad \text{if } u,v\geqslant 1,\quad R_1\prec R_2; \tag{7c}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\Lambda_0^2\ldots\Lambda_v^2}_{Q_1,[Q_2,{}_vR_2]}, \qquad \text{if } u,v\geqslant 1,\quad R_2\prec R_1; \tag{7d}$$

where $[Q_1,{}_uR_1]\neq Q_2$ for $u\geqslant 1$ and $[Q_2,{}_vR_2]\neq Q_1$ for $v\geqslant 1$,

$$\Lambda_0^1\in D(Q_1),\ \Lambda_0^2\in D(Q_2),\ \Lambda_1^1,\ldots,\Lambda_u^1\in D(R_1),\ \Lambda_1^2,\ldots,\Lambda_v^2\in D(R_2),$$

$$F=P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2}\vee(\Lambda_0^1=\Lambda_0^2)\Big((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P^{\Lambda_k^2\Lambda_k^1}_{R_1,R_2}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Big).$$

2. Universal existence condition

Let us fix a variant of the collection process $\{W_j\}_{j\geqslant 0}$. In [1, Corollary 4.8] it was proved that using relations (6)–(7) one can express the existence condition E_R by a formula containing at most the operations conjunction and disjunction, the predicates E_{a_i}, P_{a_i,a_j} and the equality relation on $\mathbb{Z}$.

Assume that we did not use relations (7c) and (7d) during the process of expressing E_R. If we change the variant of the collection process $\{W_j\}_{j\geqslant 0}$ (change the initial word or the arrangement of the collected commutators), then the process of expressing E_R will be exactly the same. Therefore, the resulting formula (as a construction of symbols $\wedge$, $\vee$, $=$, predicate symbols E_{a_i}, P_{a_i,a_j}) is an invariant with respect to a variant of the collection process. More precisely, if R arose during some collection process $\{W_j\}_{j\geqslant 0}$, then all predicate symbols E_{a_i}, P_{a_i,a_j} in the formula are defined only by the initial word W_0, and the formula in its logical values coincides with the existence condition E_R. Besides, since equality (5) holds, the exponent of R depends, perhaps, on the choice of the initial word, but not on the arrangement of the collected commutators.

Our aim is to construct such invariant formula for any commutator R. We now allow the formula to contain a symbol $\prec$. Let us replace relations (7c) and (7d) with

$$P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2}_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}=E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}\left((R_1\prec R_2)P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}\vee(R_2\prec R_1)P^{\Lambda_0^1\Lambda_0^2\ldots\Lambda_v^2}_{Q_1,[Q_2,{}_vR_2]}\right). \tag{8}$$

If we now use relations (6), (7a), (7b), (8) to express E_R, then on each step our choice of the desired relation does not depend on the arrangement of the collected commutators. However, there is a problem: the predicate symbols $P_{[Q_1,{}_uR_1],Q_2}$ and $P_{Q_1,[Q_2,{}_vR_2]}$ are not necessarily defined simultaneously. For example, R_1 was collected earlier than R_2 (i.e. $R_1\prec R_2$) during some collection process, and we have come across the predicate $P_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}$ during the process of expressing E_R. Then relation (7c) holds, but the predicate $P_{Q_1,[Q_2,{}_vR_2]}$ from (7d) is not defined if there does not exist an uncollected part containing both occurrences of Q_1 and $[Q_2,{}_vR_2]$ (see definition of the precedence condition). Thus, we can not continue the process of expressing E_R. To overcome this problem, we introduce the following definitions.

Definition 1. *For any commutators R_1, R_2, we call the interpretation of the predicate symbols*

$$E_{R_1},\ P_{R_1,R_2},\ \prec \tag{9}$$

the standard one with respect to a variant of the collection process $\{W_i\}_{i\geqslant 0}$ if they are defined according to the definitions in Section 1 formulated for $\{W_i\}_{i\geqslant 0}$.

The predicate symbol $\prec$ admits the standard interpretation with respect to any variant of the collection process $\{W_i\}_{i\geqslant 0}$. The same can not be said about the symbols E_{R_1}, P_{R_1,R_2}. In the first case, the occurrences of R_1 might not have arisen during the collection process. In the second case, the occurrences of R_1 and R_2 might not have arisen in the same uncollected part.

Definition 2. *Suppose Δ is a formula containing at most the symbols $\wedge$, $\vee$, $=$, the predicate symbols (9). We say that the standard interpretation of the formula Δ with respect to a variant of the collection process $\{W_i\}_{i\geqslant 0}$ is given if the symbol $=$ is interpreted as equality, all predicate symbols in Δ that allow standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$ are interpreted that way, the rest symbols (they can be only E_{R_1} and P_{R_1,R_2}) are interpreted as predicates defined arbitrarily on the sets $D(R_1)$ and $D(R_1)\times D(R_2)$, respectively.*

Theorem 1. *Suppose R is a formal commutator of the free group $F(a_1,\dots,a_n)$, $n\geqslant 2$. Then there exists a formula $\mathcal{E}_R$ with the following properties:*

1. *$\mathcal{E}_R$ contains at most the operations of conjunction, disjunction, and the following predicate symbols:*
$$E_{a_i},\ P_{a_i,a_j},\ \prec,\ =,\qquad i,j\in\overline{1,n}. \tag{10}$$
2. *If occurrences of R arose during some variant of the collection process, then, for the standard interpretation of $\mathcal{E}_R$ with respect to this variant of the collection process, the following equality holds:*
$$\mathcal{E}_R^{\Lambda}=E_R^{\Lambda},\qquad \Lambda\in D(R). \tag{11}$$

Proof. Consider the system of recurrence relations (6), (7a), (7b), (8) as formal relations of predicate symbols. Fix formal commutator R.

Let a formula Δ contain at most the operations of conjunction, disjunction, the symbol $=$, the predicate symbols (9). We say that Δ has property (M) if, for any variant of the collection process $\{W_i\}_{i\geqslant 0}$ during which R arose, the equality

$$\Delta_R^{\Lambda}=E_R^{\Lambda},\qquad \Lambda\in D(R), \tag{12}$$

holds for any standard interpretation of Δ_R^{Λ} with respect to $\{W_i\}_{i\geqslant 0}$.

Let us describe inductively the process of constructing the sequence of formulas $\{{}_i\Delta_R^{\Lambda}\}_{i\geqslant 0}$: 1) ${}_0\Delta_R^{\Lambda}=E_R^{\Lambda}$; 2) the formula ${}_{i+1}\Delta_R^{\Lambda}$ is obtained from ${}_i\Delta_R^{\Lambda}$ by replacing any predicate symbol of type E_{R_1} or P_{R_1,R_2}, where $w(R_1),w(R_2)\geqslant 2$, in ${}_i\Delta_R^{\Lambda}$ with the corresponding formula according to relations (6), (7a), (7b), (8). The sequence is finite and ends with the formula satisfying statement 1 of the theorem. This fact follows from the proof of Corollary 4.8 [1].

We prove that the formulas ${}_i\Delta_R^{\Lambda}$ has property (M) by induction on i. For $i=0$ the statement is true, since ${}_0\Delta_R^{\Lambda}=E_R^{\Lambda}$ and the predicate symbol E_R^{Λ} is standardly interpreted with respect to any variant of the collection process during which the commutator R arose. Assume that ${}_i\Delta_R^{\Lambda}$ has property (M) and the formula ${}_{i+1}\Delta_R^{\Lambda}$ is obtained by replacing a predicate symbol P in ${}_i\Delta_R^{\Lambda}$ with the corresponding formula.

Let $\{W_i\}_{i\geqslant 0}$ be a variant of the collection process during which the commutator R arose, and the symbol P does not allow the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$. It is known that the equality ${}_i\Delta_R^\Lambda = E_R^\Lambda$, $\Lambda \in D(R)$, is true for any standard interpretation of ${}_i\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$, in particular, the equality holds for any interpretation of the predicate symbol P. Therefore, P can be replaced with any formula at all, and we get $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$.

Now let the symbol P allow the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$. For any relation (6), (7a), (7b), if the left-hand side of the relation allows standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, then each predicate symbol in the right-hand side has the same property. Therefore, if P is replaced with the corresponding formula using one of these relations, then we have $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$. It remains to consider the case when P is replaced using relation (8).

If the left-hand side of (8) allows the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, then the same is true for the predicate symbols

$$E_{[Q_1,{}_uR_1]},\ E_{[Q_2,{}_vR_2]},\ \prec,$$

and at least for one of the symbols

$$P_{[Q_1,{}_uR_1],Q_2},\ P_{Q_1,[Q_2,{}_vR_2]}$$

in the right-hand side of (8). If $R_1 \prec R_2$, then, first, the symbol $P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}$ is standardly interpreted (according to (7c)) with respect to $\{W_i\}_{i\geqslant 0}$, second, the predicate $R_2 \prec R_1$ is false. Therefore, the equality $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ is true for any interpretation of the symbol $P_{Q_1,[Q_2,{}_vR_2]}$ at all, hence, for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$. If $R_2 \prec R_1$, the reasoning is analogous.

Thus, it has been proved that the last element of the sequence $\{{}_i\Delta_R^\Lambda\}_{i\geqslant 0}$, which we denote by $\mathcal{E}_R^\Lambda$, has property (M). Moreover, $\mathcal{E}_R^\Lambda$ allows a single standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, since it contains at most the predicate symbols (10), which are always standardly interpreted. □

Definition 3. For any formal commutator R of the free group $F(a_1,\ldots,a_n)$, $n \geqslant 2$, we call the formula $\mathcal{E}_R$ from Theorem 1 the *universal existence condition* of the commutator R.

Corollary 1. *If a commutator R was collected during some variant of the collection process $\{W_j\}_{j\geqslant 0}$, then its exponent is equal to*

$$|\{\Lambda \in D(R) \mid \mathcal{E}_R^\Lambda = 1\}|,$$

where the universal existence condition $\mathcal{E}_R^\Lambda$ has standard interpretation with respect to $\{W_j\}_{j\geqslant 0}$.

Corollary 2. *Suppose the universal existence condition $\mathcal{E}_R$ does not contain the predicate symbols $\prec$. Let $\{W_j\}_{j\geqslant 0}$, $\{V_j\}_{j\geqslant 0}$ be two variants of the collection process with the same initial word. If R was collected during both $\{W_j\}_{j\geqslant 0}$ and $\{V_j\}_{j\geqslant 0}$, then its exponent is the same in both cases.*

3. Examples

In this section we find the universal existence condition $\mathcal{E}_R$ for several series of commutators using the proof of Theorem 1. Namely, we construct a sequence of formulas that satisfy property (M). The sequence starts with E_R and ends with a formula satisfying statement 1 of Theorem 1. As a consequence, we get the exponents of these commutators in different collection formulas in an explicit form.

Lemma 1. *For $j, i_1, \dots, i_s \in \{1, \dots, n\}$ and $u_1, \dots, u_s \geqslant 1$, where $n, s \geqslant 1$, we have*

$$\mathcal{E}^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]} = \bigwedge_{k=1}^{s} P^{\Lambda_0\Lambda_1^k}_{a_j,a_{i_k}} \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_k},a_{i_k}}. \tag{13}$$

Proof. We use induction on s. For $s = 1$ we have

$$\mathcal{E}^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1}_{[a_j,\,_{u_1}a_{i_1}]} = P^{\Lambda_0\Lambda_1^1}_{a_j,a_{i_1}} \bigwedge_{h=1}^{u_1-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_1},a_{i_1}},$$

which coincides with the result of applying relation (6) to the symbol $E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1}_{[a_j,\,_{u_1}a_{i_1}]}$. Assume that equality (13) is true for some s. Let us prove (13) for $s+1$.

Using (6) replace $E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^{s+1}\dots\Lambda_{u_{s+1}}^{s+1}}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_{s+1}}a_{i_{s+1}}]}$ with the formula

$$P^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s\Lambda_1^{s+1}}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}],a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Now we use (7a) if $a_j = a_{i_{s+1}}$, otherwise we use (7b), and get the same result in both cases:

$$E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]} E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}} P^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^{s-1}\dots\Lambda_{u_{s-1}}^{s-1}\Lambda_1^{s+1}}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_{s-1}}a_{i_{s-1}}],a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Continuing this line of reasoning, after a finite number of steps we get the formula

$$\bigwedge_{k=1}^{s}\left(E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^k\dots\Lambda_{u_k}^k}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_k}a_{i_k}]} E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}}\right) P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}. \tag{14}$$

Let $\{W_j\}_{j\geqslant 0}$ be a variant of the collection process during which the commutator $[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_{s+1}}a_{i_{s+1}}]$ arose. Then all predicate symbols in (14) allow standard interpretation with respect to $\{W_j\}_{j\geqslant 0}$. For this standard interpretation, we have the following equalities of predicates for any values of variables:

$$E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} = P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}}, \qquad \bigwedge_{k=1}^{s} E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^k\dots\Lambda_{u_k}^k}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_k}a_{i_k}]} = E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]}.$$

We apply this equalities to (14) and get

$$E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Since (14) has property (M) and the reasoning above is carried out for the arbitrary variant of the collection process $\{W_j\}_{j\geqslant 0}$, then the obtained formula has property (M).

Further we should start the process of expressing the symbol $E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]}$. However, by definition of the universal existence condition, the formula $\mathcal{E}^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]}$ with standard interpretation is equal to the predicate $E^{\Lambda_0\Lambda_1^1\dots\Lambda_{u_1}^1\dots\Lambda_1^s\dots\Lambda_{u_s}^s}_{[a_j,\,_{u_1}a_{i_1},\dots,\,_{u_s}a_{i_s}]}$. Therefore, we can use the inductive assumption and get the formula

$$\bigwedge_{k=1}^{s} P^{\Lambda_0\Lambda_1^k}_{a_j,a_{i_k}} \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_k},a_{i_k}} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Collecting similar terms completes the proof. □

Lemma 2. *For $s, i, j \in \{1, \ldots, n\}$, $i \neq j$, and $u, v \geqslant 1$, where $n \geqslant 1$, we have*

$$\mathcal{E}^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[[a_s,{}_u a_i],[a_s,{}_v a_j]]} = P^{\Lambda_0^1\Lambda_1^1}_{a_s,a_i} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{a_i,a_i} P^{\Lambda_0^2\Lambda_1^2}_{a_s,a_j} \bigwedge_{k=1}^{v-1} P^{\Lambda_k^2\Lambda_{k+1}^2}_{a_i,a_i} (P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (a_j \prec a_i)(\Lambda_0^1 = \Lambda_0^2)).$$

Proof. We construct the sequence of formulas according to the proof of Theorem 1 starting with

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[[a_s,{}_u a_i],[a_s,{}_v a_j]]}.$$

Use relation (6):

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[a_s,{}_u a_i],[a_s,{}_v a_j]}.$$

Since $i \neq j$, use relation (8):

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1}_{[a_s,{}_u a_i]} E^{\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[a_s,{}_v a_j]} \left((a_i \prec a_j) P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2}_{[a_s,{}_u a_i],a_s} \vee (a_j \prec a_i) P^{\Lambda_0^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{a_s,[a_s,{}_v a_j]} \right).$$

Next we use (6) and (7a) twice:

$$P^{\Lambda_0^1\Lambda_1^1}_{a_s,a_i} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{a_i,a_i} P^{\Lambda_0^2\Lambda_1^2}_{a_s,a_j} \bigwedge_{k=1}^{v-1} P^{\Lambda_k^2\Lambda_{k+1}^2}_{a_i,a_i} \wedge \left((a_i \prec a_j) P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (a_j \prec a_i)(P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (\Lambda_0^1 = \Lambda_0^2)) \right).$$

Now we simplify the expression in brackets using logical transformations and the fact that the expression $(a_j \prec a_i) \vee (a_i \prec a_j)$ with standard interpretation is true for any variant of the collection process during which the commutator $[[a_s,{}_u a_i],[a_s,{}_v a_j]]$ arose. □

Theorem 2. *Suppose a formal commutator R was collected during some variant of the collection process $\{W_j\}_{j\geqslant 0}$ and its exponent is equal to $e(R)$. The following statements hold.*

1. *If $W_0 \equiv (a_1 \ldots a_n)^m$, $n, m \geqslant 1$, and $R = [a_j, {}_{u_1}a_{i_1}, \ldots, {}_{u_s}a_{i_s}]$, then*

$$e(R) = \sum_{\lambda_0=0}^{m-1} \prod_{\substack{k=1,\ldots,s;\\ j<i_k}} \binom{\lambda_0+1}{u_k} \prod_{\substack{k=1,\ldots,s;\\ j\geqslant i_k}} \binom{\lambda_0}{u_k}.$$

2. *If $W_0 \equiv (a_1 \ldots a_n)^m$, $n, m \geqslant 1$, and $R = [[a_s, {}_u a_i], [a_s, {}_v a_j]]$, $i \neq j$, then*

$$e(R) = \sum_{\lambda_0^1=1}^{m+\delta_{(a_j\prec a_i)}-1} \binom{\lambda_0^1 - \delta_{(a_j\prec a_i)} + \delta_{(s<i)}}{u} \binom{\lambda_0^1 + \delta_{(s<j)}}{v+1},$$

   where $\delta_A = 1$ if the proposition A is true, otherwise $\delta_A = 0$.

3. *If $W_0 \equiv a_1^{m_1} \ldots a_n^{m_n}$, $n, m_1, \ldots, m_n \geqslant 1$, and $R = [a_j, {}_{u_1}a_{i_1}, \ldots, {}_{u_s}a_{i_s}]$, then*

$$e(R) = \binom{m_j}{u+1} \prod_{k=1,\ldots,s} \binom{m_{i_k}}{u_k},$$

   where $u = u_l$ if there exists $i_l = j$, otherwise $u = 0$.

Proof. Consider a variant of the collection process with the initial word

$$W_0 \equiv a_1(1) \ldots a_n(1) \ldots a_1(m) \ldots a_n(m).$$

We have

$$P_{a_i,a_j}^{(\lambda_1,\lambda_2)} = (\lambda_1 < \lambda_2) \vee (\lambda_1 = \lambda_2)(i < j), \quad \lambda_1, \lambda_2 \in \{1,\ldots,m\},\ i,j \in \overline{1,n}.$$

Assume that the commutator $[a_j, {}_{u_1}a_{i_1}, \ldots, {}_{u_s}a_{i_s}]$ arose during the collection process. From Lemma 1 is follows that

$$E_{[a_j,{}_{u_1}a_{i_1},\ldots,{}_{u_s}a_{i_s}]}^{(\lambda_0,\lambda_1^1,\ldots,\lambda_{u_1}^1,\ldots,\lambda_1^s,\ldots,\lambda_{u_s}^s)} = \bigwedge_{k=1}^{s}((\lambda_0 < \lambda_1^k) \vee (\lambda_0 = \lambda_1^k)(j < i_k)) \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1}(\lambda_h^k < \lambda_{h+1}^k),$$

where $\lambda_0, \lambda_1^1, \ldots, \lambda_{u_1}^1, \ldots, \lambda_1^s, \ldots, \lambda_{u_s}^s \in \{1,\ldots,m\}$. Then the exponent of this commutator is equal to the number of solutions of the following system:

$$\begin{cases} 1 \leqslant \lambda_0 \leqslant m; \\ \lambda_0 \leqslant \lambda_1^k < \lambda_2^k < \ldots < \lambda_{u_k} \leqslant m, \quad k \in \overline{1,s},\ j < i_k; \\ \lambda_0 < \lambda_1^k < \lambda_2^k < \ldots < \lambda_{u_k} \leqslant m, \quad k \in \overline{1,s},\ j \geqslant i_k. \end{cases}$$

Taking into account that the number of integer sequence $(x_1,\ldots,x_m)$ that satisfy the condition $1 \leqslant x_1 \leqslant \ldots \leqslant x_m \leqslant n$ is equal to $\binom{n}{m}$, we get the number of solutions:

$$\sum_{\lambda_0=1}^{m} \prod_{\substack{k=1,\ldots,s;\\ j<i_k}} \binom{m-\lambda_0+1}{u_k} \prod_{\substack{k=1,\ldots,s;\\ j\geqslant i_k}} \binom{m-\lambda_0}{u_k} = \sum_{\lambda_0=0}^{m-1} \prod_{\substack{k=1,\ldots,s;\\ j<i_k}} \binom{\lambda_0+1}{u_k} \prod_{\substack{k=1,\ldots,s;\\ j\geqslant i_k}} \binom{\lambda_0}{u_k}.$$

Now assume that the commutator $[[a_s, {}_u a_i], [a_s, {}_v a_j]]$ for $u, v \geqslant 1$, $i \neq j$ arose during the collection process. Then by Lemma 2 we have

$$E_{[[a_s,{}_u a_i],[a_s,{}_v a_j]]}^{(\lambda_0^1,\lambda_1^1,\ldots,\lambda_u^1,\lambda_0^2,\lambda_1^2,\ldots,\lambda_v^2)} = \big((\lambda_0^1 < \lambda_1^1) \vee (\lambda_0^1 = \lambda_1^1)(s < i)\big)\big((\lambda_0^2 < \lambda_1^2) \vee (\lambda_0^2 = \lambda_1^2)(s < j)\big) \wedge$$
$$\wedge \bigwedge_{k=1}^{u-1}(\lambda_k^1 < \lambda_{k+1}^1) \bigwedge_{k=1}^{v-1}(\lambda_k^2 < \lambda_{k+1}^2)((\lambda_0^1 < \lambda_0^2) \vee (a_j \prec a_i)(\lambda_0^1 = \lambda_0^2)).$$

Therefore, the exponent of $[[a_s, {}_u a_i], [a_s, {}_v a_j]]$ is equal to the number of solutions of the following system:

$$\begin{cases} 1 \leqslant \lambda_0^1 \leqslant m; \\ 1 \leqslant \lambda_0^2 \leqslant m; \\ \lambda_0^1 - \delta_{(a_j \prec a_i)} + 1 \leqslant \lambda_0^2; \\ \lambda_0^1 - \delta_{(s<i)} + 1 \leqslant \lambda_1^1 < \lambda_2^1 < \ldots < \lambda_u^1 \leqslant m; \\ \lambda_0^2 - \delta_{(s<j)} + 1 \leqslant \lambda_1^2 < \lambda_2^2 < \ldots < \lambda_v^2 \leqslant m. \end{cases}$$

We get the following expression:

$$\sum_{\lambda_0^1=1}^{m} \sum_{\lambda_0^2=\lambda_0^1-\delta_{(a_j\prec a_i)}+1}^{m} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u}\binom{m-\lambda_0^2+\delta_{(s<j)}}{v} =$$
$$= \sum_{\lambda_0^1=1}^{m} \sum_{\lambda_0^2=\delta_{(s<j)}}^{m-\lambda_0^1+\delta_{(a_j\prec a_i)}+\delta_{(s<j)}-1} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u}\binom{\lambda_0^2}{v} =$$

since $0 \leqslant \delta_{(s<j)} \leqslant 1$ and $v \geqslant 1$, we change the lower limit of λ_0^2 to v and apply a well-known summation formula:

$$= \sum_{\lambda_0^1=1}^{m+\delta_{(a_j \prec a_i)}-1} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u}\binom{m-\lambda_0^1+\delta_{(a_j\prec a_i)}+\delta_{(s<j)}}{v+1} =$$

change the order of summation:

$$= \sum_{\lambda_0^1=1-\delta_{(a_j\prec a_i)}}^{m-1} \binom{\lambda_0^1+\delta_{(s<i)}}{u}\binom{\lambda_0^1+\delta_{(a_j\prec a_i)}+\delta_{(s<j)}}{v+1} =$$
$$= \sum_{\lambda_0^1=1}^{m+\delta_{(a_j\prec a_i)}-1} \binom{\lambda_0^1-\delta_{(a_j\prec a_i)}+\delta_{(s<i)}}{u}\binom{\lambda_0^1+\delta_{(s<j)}}{v+1}.$$

Now let us consider a variant of the collection process with the initial word

$$W_0 \equiv a_1(1)\dots a_1(m_1)\dots a_n(1)\dots a_n(m_n).$$

We have

$$P_{a_i,a_j}^{(\lambda_1,\lambda_2)} = (\lambda_1<\lambda_2)(i=j)\vee(i<j), \quad \lambda_1,\lambda_2\in\{1,\dots,m\},\ i,j\in\overline{1,n}.$$

Assume that the commutator $[a_j,\, {}_{u_1}a_{i_1},\dots,\, {}_{u_s}a_{i_s}]$ arose during the collection process. From Lemma 1 it follows that

$$E_{[a_j,\,{}_{u_1}a_{i_1},\dots,\,{}_{u_s}a_{i_s}]}^{(\lambda_0,\lambda_1^1,\dots,\lambda_{u_1}^1,\dots,\lambda_1^s,\dots,\lambda_{u_s}^s)} = \bigwedge_{k=1}^{s}((\lambda_0<\lambda_1^k)(j=i_k)\vee(j<i_k)) \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1}(\lambda_h^k<\lambda_{h+1}^k).$$

Then we get the following system:

$$\begin{cases} 1\leqslant\lambda_0\leqslant m_j; \\ \lambda_0<\lambda_1^k<\lambda_2^k<\dots<\lambda_{u_k}\leqslant m_{i_k}, \quad k\in\overline{1,s},\ j=i_k; \\ 1\leqslant\lambda_1^k<\lambda_2^k<\dots<\lambda_{u_k}\leqslant m_{i_k}, \quad k\in\overline{1,s},\ j<i_k. \end{cases}$$

The number of solutions of this system is equal to

$$\sum_{\lambda_0=1}^{m_j}\prod_{\substack{k=1,\dots,s\\ j=i_k}}\binom{m_{i_k}-\lambda_0}{u_k}\prod_{\substack{k=1,\dots,s;\\ j<i_k}}\binom{m_{i_k}}{u_k} = \sum_{\lambda_0=0}^{m_j-1}\prod_{\substack{k=1,\dots,s\\ j=i_k}}\binom{\lambda_0}{u_k}\prod_{\substack{k=1,\dots,s;\\ j<i_k}}\binom{m_{i_k}}{u_k}.$$

If none of the numbers $i_1,\dots,i_s$ is equal to j, then we get

$$m_j\prod_{k=1,\dots,s}\binom{m_{i_k}}{u_k}.$$

If some i_l is equal to j (in this case i_l is unique), then the exponent is equal to

$$\binom{m_j}{u_l+1}\prod_{k=1,\dots,s}\binom{m_{i_k}}{u_k}.$$

□

Theorem 3. *Suppose G is a group with the Abelian commutator subgroup, $a_1, \ldots, a_n \in G$, $n, m \in \mathbb{N}$. Then the following formula holds:*

$$(a_1 \ldots a_n)^m = a_1^m \ldots a_n^m \prod_{j=2}^{n} \prod_{(u_1,\ldots,u_n)\in M_{n,m}^j} [a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n]^{\sum_{k=0}^{m-1} \prod_{s=1}^{j} \binom{k}{u_s} \prod_{s=j+1}^{n} \binom{k+1}{u_s}},$$

where $M_{n,m}^j = \{(u_1, \ldots, u_n) \in \{0, \ldots, m\}^n \mid u_1 + \cdots + u_n > 0;$ the first $u_i > 0$ has $i < j\}$.

Proof. Consider the word $(a_1 \ldots a_n)^m$ of the free group $F(a_1, \ldots, a_n)$. Let us apply the collection process to this word. First, we collect letters in the following order: $a_1, \ldots, a_n$ and get the word

$$a_1^m \ldots a_n^m \prod [a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n],$$

where the product is over some non-negative integers $j, u_1, \ldots, u_n$. After that we collect the commutators $[a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n]$ in some fixed order. From Theorem 2 it follows that we get the following formula in the group G:

$$(a_1 \ldots a_n)^m = a_1^m \ldots a_n^m \prod_{j\in J} \prod_{(u_1,\ldots,u_n)\in M_{n,m}^j} [a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n]^{\sum_{k=0}^{m-1} \prod_{s=1}^{j} \binom{k}{u_s} \prod_{s=j+1}^{n} \binom{k+1}{u_s}},$$

where it remains to find the sets J and $M_{n,m}^j$. Note that the use of Theorem 2 in the case when some $u_s = 0$ is correct, since $\binom{a}{0} = 1$ for any $a \geqslant 0$.

Obviously, $J \subseteq \{2, \ldots, n\}$, since a_1 was collected first. Further, the expression in the exponent is equal to 0 when $u_s \geqslant m+1$, therefore, we have $M_{n,m}^j \subseteq \{0, \ldots, m\}^n$. At least one element of the sequence $(u_1, \ldots, u_n) \in M_{n,m}^j$ is not equal to 0, since otherwise we get the commutator a_j. Moreover, the first $u_i > 0$ has the index $i < j$, since the commutators were collected in the order $a_1, \ldots, a_n$. Thus, the following inclusions have been proved:

$$J \subseteq \{2, \ldots, n\},$$
$$M_{n,m}^j \subseteq \{(u_1, \ldots, u_n) \in \{0, \ldots, m\}^n \mid u_1 + \cdots + u_n > 0; \text{ the first } u_i > 0 \text{ has } i < j\}.$$

To prove the reverse inclusions, we assume that the expression in the exponent of $[a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n]$ is not equal to 0 for some sequence $(j, u_1, \ldots, u_n)$. From the proof of Theorem 2 it follows that there exist some values of the variables for which the formula

$$\bigwedge_{k=1}^{s} P_{a_j, a_{i_k}}^{\Lambda_0 \Lambda_1^k} \bigwedge_{k=1}^{s} \bigwedge_{h=1}^{u_{i_k}-1} P_{a_{i_k}, a_{i_k}}^{\Lambda_h^k \Lambda_{h+1}^k}$$

is equal to 1, where $1 \leqslant i_1 < \ldots < i_s \leqslant n$ and $u_{i_k} > 0$ for any k. Therefore, in the initial word $(a_1 \ldots a_n)^m$, there are u_{i_1} occurrences of a_{i_1}, u_{i_2} occurrences of a_{i_2}, etc to the right of $a_j(\Lambda_0)$. Since the letters were collected in the order $a_{i_1}, \ldots, a_{i_s}$, and $j \geqslant 2$, $i_1 < j$, the commutator $[a_j, {}_{u_{i_1}}a_{i_1}, \ldots, {}_{u_{i_s}}a_{i_s}] = [a_j, {}_{u_1}a_1, \ldots, {}_{u_n}a_n]$ arose during the collection process. □

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О собирательных формулах для положительных слов

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Аннотация. Для любого формального коммутатора R свободной группы F мы конструктивно доказываем существование логической формулы $\mathcal{E}_R$ со следующими свойствами. Во-первых, ее строение определяется структурой R, а логические значения определяются положительным словом группы F, к которому применяется собирательный процесс, и порядком сбора коммутаторов. Во-вторых, если в ходе собирательного процесса был собран коммутатор R, то его показатель степени равен количеству элементов множества $D(R)$, удовлетворяющих $\mathcal{E}_R$, где $D(R)$ определяется структурой R. В работе приведены примеры такой формулы для разных коммутаторов, как следствие, вычислены их показатели степеней для разных положительных слов F. В частности, получена в явном виде собирательная формула для слова $(a_1 \dots a_n)^m$, $n, m \geqslant 1$ в группе с абелевым коммутантом. Рассмотрен вопрос о зависимости показателя степени коммутатора от порядка сбора коммутаторов в ходе собирательного процесса.

Ключевые слова: коммутатор, собирательный процесс, свободная группа.

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On the Bipolar Classification of Endomorphisms of a Groupoid

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Abstract. In this paper, a method is obtained for calculating the bipolar type of endomorphism of an arbitrary groupoid. For groupoids with pairwise distinct left translations of elements, the described method for calculating the bipolar type of an endomorphism leads to a criterion for the fixed point of a given endomorphism. In particular, such groupoids include groupoids with a right neutral element, monoids, loops and groups. It turned out that the bipolar type of endomorphisms of a groupoid with pairwise distinct left translations ones contains all the information about the fixed points of endomorphisms of this type. A basic set of endomorphisms of a group is established, containing all regular automorphisms. A method is found for calculating the bipolar type of an inner automorphism of a monoid. We obtain upper bounds for the order of the monoid of all endomorphisms (and the group of all automorphisms) of an algebraic system with finite support that has a binary algebraic operation.

Keywords: groupoid, groupoid endomorphism, groupoid automorphism, bipolar type of endomorphism of groupoid, bipolar type of regular automorphism, bipolar type of inner automorphism, conservative estimates.

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Introduction

The work is devoted to the study of the properties of a groupoid that follow directly from the bipolar classification of endomorphisms of an arbitrary groupoid, introduced in [1] (the bipolar classification of antiendomorphisms is considered in [2]). The introduction of this classification of endomorphisms was due to the interest of various researchers in the following general problems.

Problem 1. *For a fixed groupoid G, list the elements of the monoid of all endomorphisms of this groupoid.*

Problem 2. *For a fixed groupoid G, list the elements describing the group of all automorphisms of this groupoid.*

Problem 3. *For a fixed groupoid G, give qualitative properties of all endomorphisms and all automorphisms.*

An *element-by-element description of endomorphisms* is a description of the elements of the monoid of all endomorphisms (similarly to the group of automorphisms) as transformations of the support set of the groupoid. The papers aimed at solving problems 1 and 2 include the papers [3–6], in which endomorphisms of matrix semigroups of a special form are studied; papers [7,8], which describe the group of all automorphisms of unipotent subgroups of Chevalley groups (see also works [9–13]). In [14], automorphisms of finitely presented quasigroups are studied.

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Endomorphisms of commutative (but generally not associative) finite groupoids associated with a multilayer neural network of forward signal propagation were studied in [15, 16]. Problem 1 remains open for some groupoids from [15] (see Problem 2 from [15]). Thus, Problems 1 and 2 are studied for various groupoids, semigroups, quasigroups, and groups.

By a *qualitative description of endomorphisms* we mean finding the properties that endomorphisms (or automorphisms) of various groupoids have. The results of such studies may be useful for solving Problems 1 and 2, or be of independent value. Examples of such studies can be found in [17, 18]. There are many examples of studies closely related to endomorphisms (in particular, automorphisms) of various groupoids. For example, the investigations [19, 20] in which finite groupoids are classified that have automorphism groups of a special form. Automorphism groups of finite groupoids are studied in [21]. Endomorphism semigroups for some semigroups of a special kind are studied in [22]. The concept of endomorphism (the ring of endomorphisms of an Abelian group) is intensively used in the study of Abelian groups. This extensive direction is presented, for instance, in [23].

Main results of the paper include the Theorem 2.1. According to it, for any groupoid G, any element $g \in G$ and any endomorphism ϕ of the groupoid G the following equivalences hold:

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) \in M_g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \in G \setminus M_g,$$

where $M_g := \{m \in G \mid h_m = h_g\}$. This theorem leads to a method for calculating the bipolar type of a fixed endomorphism of an arbitrary groupoid.

An important consequence of Theorem 2.1 is Theorem 2.2. This theorem for a groupoid G with different left translations gives a criterion that $g \in G$ is a fixed point of an endomorphism ϕ of the groupoid G. For any $g \in G$ and $\phi \in \mathrm{End}(G)$ the equality $\Gamma_\phi(g) = 1$ holds iff the equality $\phi(g) = g$ holds. The results of this theorem extend to all groups, monoids, loops and groupoids with a right neutral element. In particular, see Corollary 2.1 for monoids of all endomorphisms of an arbitrary groupoid. On the other hand, the Theorem 2.2 gives a practical way to calculate the bipolar type of a endomorphism of groupoid with pairwise distinct left translations. This method extends to loops, monoids, and groups. The bipolar type of endomorphism of a groupoid with different left translations is completely characterized by all fixed points of this endomorphism.

Theorem 2.3 describes the basic sets of endomorphisms of a monoid, consisting of endomorphisms of a monoid. An endomorphism of a monoid is an endomorphism of a groupoid such that the neutral element of the monoid is a fixed point of this endomorphism. The Theorem 2.4 gives a description of the basic set of endomorphisms of the group, which contains all the regular automorphisms of the group. For groups, such a base set is unique. The Theorem 2.5 gives a way to calculate the bipolar type of a inner automorphism of a monoid.

Theorem 3.1 establishes upper bounds for the order of the group of all automorphisms and the monoid of all endomorphisms of an algebraic system with finite support that has a binary algebraic operation. The resulting estimates are called *conservative estimates* of the order of the monoid of all endomorphisms and the order of the group of all automorphisms. This theorem can be used to study endomorphisms and automorphisms of such algebraic objects as rings and semifields (see [24]). The Theorem 3.1 can be used to investigate the question (**D**) from [25].

1. Basic concepts and definitions

The symmetric semigroup of all transformations of the set G will be denoted by $\mathcal{I}(G)$. The composition of transformations α_1, α_2 from $\mathcal{I}(G)$ will be denoted by $(\cdot)$ and defined by the equality

$$(\alpha_1 \cdot \alpha_2)(g) = \alpha_1(\alpha_2(g)) \quad (g \in G).$$

Endomorphisms and their compositions will be considered in the notation of a symmetric semigroup. For an arbitrary groupoid $G = (G, *)$ the *inner left translation* of an element $x \in G$ will be denoted by h_x (for any $x, y \in G$ the equality $h_x(y) = x * y$ holds).

Below is Definition 1 from [1].

Definition 1.1. By $\mathrm{Bte}(G)$ we denote the set of all possible mappings of the set G into the set $\{1, 2\}$. Mappings from this set will be called *bipolar types of endomorphism* of the groupoid G (or simply *types*). If $\gamma \in \mathrm{Bte}(G)$ and $\gamma(g) = 1$ for any $g \in G$ (similarly, $\gamma(g) = 2$), then the mapping γ will be called *first type* (similarly, *second type*). In this paper, the first type will be denoted by A, and the second type by Ω. If the mapping $\gamma \in \mathrm{Bte}(G)$ is not constant on elements of G, then γ will be called *mixed type*.

The centralizer of the transformation α in the symmetric semigroup $\mathcal{I}(G)$ will be denoted by the symbols

$$C(\alpha) := \{\beta \in I(X) \mid \alpha \cdot \beta = \beta \cdot \alpha\}.$$

Definition 2 of [1] for each $g \in G$ introduces the sets $L^{(1)}(g)$ and $L^{(2)}(g)$ (further, *type-generating sets*). Definition 3 of [1] for any type $\gamma \in \mathrm{Bte}(G)$ introduces the set $D(\gamma)$, which is called the *base set of endomorphisms of type* γ of the groupoid G. We present the sets $L^{(1)}(g)$, $L^{(2)}(g)$ and $D(\gamma)$ below:

$$L^{(1)}(g) := \{\alpha \in C(h_g) \mid h_{\alpha(g)} = h_g\}; \quad L^{(2)}(g) := \{\alpha \in \mathcal{I}(G) \mid h_{\alpha(g)} \neq h_g, \ \ \alpha \cdot h_g = h_{\alpha(g)} \cdot \alpha\};$$

$$D(\gamma) := \bigcap_{s \in G} L^{(\gamma(s))}(s); \quad D(A) := \bigcap_{s \in G} L^{(1)}(s), \quad D(\Omega) := \bigcap_{s \in G} L^{(2)}(s).$$

We present Theorem 1 from [1].

Theorem 1.1. *For any groupoid G the equality holds*

$$\mathrm{End}(G) = \bigcup_{\gamma \in \mathrm{Bte}(G)} D(\gamma). \tag{1}$$

Moreover, if τ and ω are two different types from $\mathrm{Bte}(G)$*, then the intersection of $D(\tau)$ and $D(\omega)$ is empty.*

An endomorphism ϕ has bipolar type γ from $\mathrm{Bte}(G)$ if $\phi \in D(\gamma)$ (see Definition 5 from [1]). In particular, the following terminology applies:
1) an endomorphism ϕ has the *first type* if $\phi \in D(A)$;
2) an endomorphism ϕ has *second type* if $\phi \in D(\Omega)$;
3) an endomorphism ϕ is of *mixed type* in other cases.

The above assignment of types to endomorphisms of a groupoid leads to a bipolar classification of endomorphisms.

2. Bipolar types of groupoid endomorphisms

For any element g of the groupoid G we define the set $M_g := \{m \in G \mid h_m = h_g\}$. The bipolar type of the endomorphism ϕ will be denoted by Γ_ϕ. For any endomorphism ϕ of the groupoid G the following alternative holds: either $\Gamma_\phi(g) = 1$ or $\Gamma_\phi(g) = 2$. This is deduced from the definition of bipolar type and Theorem 1.1. There is an equivalence

$$\Gamma_\phi(g) = i \Leftrightarrow \phi \in L^{(i)}(g) \quad (g \in G,\ \phi \in \mathrm{End}(G),\ i \in \{1,2\}), \tag{2}$$

which establishes a connection between $\Gamma_\phi(g)$ and type-generating sets $L^{(1)}(g)$, $L^{(2)}(g)$. This equivalence follows from Theorem 1.1, the definition of the basic set of endomorphisms, and the definition of the type of endomorphism. Note that for any $g \in G$ the intersection of the sets $L^{(1)}(g)$ and $L^{(2)}(g)$ is empty. That's why for any endomorphism ϕ of the groupoid G the following alternative holds: either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$.

Theorem 2.1. *Let G be an arbitrary groupoid. Then for an arbitrary $g \in G$ and every endomorphism ϕ of the groupoid G the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) \in M_g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \in G \setminus M_g. \tag{3}$$

Proof. Let us show that the first equivalence holds. Let the first condition of the first equivalence be satisfied for some $g \in G$. Then the endomorphism ϕ belongs to the type-generating set $L^{(1)}(g)$. Hence $h_{\phi(g)} = h_g$. Then, by the definition of the set M_g, we obtain that the element $\phi(g)$ belongs to the set M_g.

On the other hand, suppose that the condition $\phi(g) \in M_g$ is satisfied. In this case, we have the equality $h_{\phi(g)} = h_g$. Therefore, ϕ cannot belong to $L^{(2)}(g)$ (due to the definition of type-generating sets). Since ϕ is an endomorphism, then for any $g \in G$ there is an alternative: either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$. Consequently, the set $L^{(1)}(g)$ contains the endomorphism ϕ, hence we have the equality $\Gamma_\phi(g) = 1$. The first equivalence is shown.

Since for any element $g \in G$ and endomorphism ϕ the alternative hold either $\phi \in L^{(1)}(g)$ or $\phi \in L^{(2)}(g)$ and the first equivalence from (3) is proved, then we get the truth of the second equivalent from (3). The theorem is proved. □

We say that G *is a groupoid with pairwise distinct left translations* if for any $x, y \in G$ the following equivalence holds: $h_x = h_y \Leftrightarrow x = y$.

Theorem 2.2. *Let G be a groupoid with pairwise distinct left translations. Then for an arbitrary $g \in G$ and every endomorphism ϕ of the groupoid G the following equivalences hold:*

$$\Gamma_\phi(g) = 1 \Leftrightarrow \phi(g) = g, \quad \Gamma_\phi(g) = 2 \Leftrightarrow \phi(g) \neq g. \tag{4}$$

Proof. If all left translations of the elements of the groupoid G are pairwise distinct, then for any $g \in G$ the equality of the sets $M_g = \{g\}$ holds. Therefore, equivalency (4) follows from (3). The theorem is proved. □

The theorem above covers all groupoids satisfying the monoid, loop, and group axioms. In addition, groupoids with pairwise distinct left translations include groupoids with a right neutral element.

The conjunction will be denoted by the symbol ($\wedge$). Because the $\mathrm{End}(G)$ is a monoid for any groupoid G, then

Corollary 2.1. *Let G be an arbitrary groupoid and Ψ an arbitrary endomorphism from* $\mathrm{End}(\mathrm{End}(G))$ *then for any ϕ from* $\mathrm{End}(G)$ *the following equivalences hold:*

$$\Gamma_\Psi(\phi) = 1 \Leftrightarrow \Psi(\phi) = \phi, \quad \Gamma_\Psi(\phi) = 2 \Leftrightarrow \Psi(\phi) \neq \phi.$$

Moreover, if G is a groupoid with pairwise distinct left translations, then for any $g \in G$ the following implication holds:

$$(\Gamma_\Psi(\phi) = 1) \wedge (\Gamma_\phi(g) = 1) \Rightarrow [\Psi(\phi)](g) = g.$$

In the last implication, $[\Psi(\phi)](g)$ denotes the image of the element g under the endomorphism $\Psi(\phi)$, which is the image of ϕ under the action of Ψ.

Next, we formulate results on endomorphisms of a monoid, the bipolar type of a regular automorphism of a group, and the inner automorphism of a monoid with invertible elements.

Monoid endomorphisms. Let M be a monoid with a neutral element e (at the same time M is a groupoid satisfying the monoid axioms). Then, in terms of universal algebra, M is an algebra with one binary operation and one nullary operation (the distinguished element is the neutral element). Therefore, an endomorphism of the monoid M is any endomorphism ϕ of the groupoid M such that the identity $\phi(e) = e$ holds. As usual, $\mathrm{End}(M)$ is the monoid of all endomorphisms of the groupoid M and $\mathrm{End}_M(M)$ is the monoid of all endomorphisms of the monoid M.

In the set of all bipolar types $\mathrm{Bte}(M)$ of an arbitrary monoid M, select the set $\mathrm{MBte}(M)$ of bipolar types γ such that $\gamma(e) = 1$.

Theorem 2.3. *If M is a monoid, then the sets are equal:*

$$\mathrm{End}_M(M) = \bigcup_{\gamma \in \mathrm{MBte}(M)} D(\gamma).$$

Proof. Indeed, by the theorem 2.2 the endomorphism ϕ of the groupoid M satisfies the condition $\phi(e) = e$ iff $\Gamma_\phi(e) = 1$. Therefore, the endomorphism ϕ of the monoid M belongs to the base set $D(\gamma)$, where γ belongs to set $\mathrm{MBte}(M)$. The theorem is proved. □

Regular automorphisms of a group. An automorphism ϕ of a group G with a neutral element e will be called a *regular automorphism* if for any $g \in G$ different from e the condition $\phi(g) \neq g$ holds. The set of all regular automorphisms is denoted by $\mathrm{RAut}(G)$ (the identity automorphism is not contained in the constructed set). The automorphism group H of the group G will be called the *group of regular automorphisms* if H consists of regular automorphisms (that is, automorphisms occurring in $\mathrm{RAut}(G)$) and the identity automorphism. Identity automorphism we denote by ε.

In the set of all bipolar types $\mathrm{Bte}(G)$ of an arbitrary group G with neutral element e, we fix a bipolar type Λ such that

$$\Lambda(g) := \begin{cases} 1, & g = e \\ 2, & g \neq e. \end{cases}$$

Theorem 2.4. *Let G be a group. Then the set $\mathrm{RAut}(G)$ is a subset of the base set of endomorphisms $D(\Lambda)$. Moreover, if H is the group of regular automorphisms of the group G, then the inclusion $H \subseteq D(\Lambda) \cup D(A)$ holds.*

Proof. By Theorem 2.2 and the definition of the set $\mathrm{RAut}(G)$, every regular automorphism has a bipolar type γ such that $\gamma(e) = 1$ and $\gamma(g) = 2$ for any $g \in G$ different from e. Therefore $\gamma = \Lambda$, hence the inclusion $\mathrm{RAut}(G) \subseteq D(\Lambda)$ hold. The group of regular automorphisms contains the identity automorphism ε, which by virtue of the Theorem 2.2 belongs to $D(A)$, hence the inclusion $H \subseteq D(\Lambda) \cup D(A)$ holds. The theorem is proved. □

The inclusion of the element ε in the set $D(A)$ was given in Remark 1 of [1].

Inner automorphisms of a monoid. Let $G = (G, *)$ be a monoid with the set of all invertible elements G^*. For each $g \in G^*$, the permutation

$$\phi_g(m) = g^{-1} * m * g \quad (m \in G)$$

we will call the *inner automorphism of the monoid* G. For any element $g \in G^*$, the permutation ϕ_g is an automorphism of the monoid G (the proof of automorphism is similar to the proof for the case of groups). If $G^* = G$, then G is a group and the above definition coincides with the definition of an inner automorphism of a group.

Theorem 2.5. *Let G be a monoid. Then for any $g \in G^*$ and any $d \in G$ the following equivalences hold:*

$$\Gamma_{\phi_g}(d) = 1 \Leftrightarrow g * d = d * g, \quad \Gamma_{\phi_g}(d) = 2 \Leftrightarrow g * d \neq d * g.$$

Proof. Consider the first equivalence. Let $\Gamma_{\phi_g}(d) = 1$. Since G is a monoid, by virtue of the Theorem 2.2 we have the relation $\Gamma_{\phi_g}(d) = 1 \Leftrightarrow \phi_g(d) = d$. In this case, $\phi_g(d) = g^{-1} * d * g$. Since g is an invertible element of the monoid, we obtain that the equality $\Gamma_{\phi_g}(d) = 1$ is equivalent to the condition $g * d = d * g$. The first equivalence is proved. The second equivalence is derived from the truth of the first equivalence and the alternative: either $\Gamma_{\phi_g}(d) = 1$ or $\Gamma_{\phi_g}(d) = 2$. The theorem is proved. □

3. Conservative estimates for the monoid order of all endomorphisms

As usual, $|X|$ is the cardinality of the set X and $S(X)$ is the symmetric permutation group of X. An algebraic system will be denoted by $V = (V, F, P)$, where F is the set of operations of the system and P is the set of relations. The definition of a homomorphism of an algebraic system into an algebraic system of the same type can be found in [26] (see p. 49). The concept of endomorphism of an algebraic system V can be formulated as a homomorphism of an algebraic system V into itself. From the definition of an endomorphism it follows

Proposition 3.1. *Let $V = (V, F, P)$ be an algebraic system and the set F contains the binary algebraic operation $(*)$. Then the inclusions hold*

$$\mathrm{End}(V) \subseteq \mathrm{End}(V_{(*)}), \quad \mathrm{Aut}(V) \subseteq \mathrm{Aut}(V_{(*)}),$$

where $\mathrm{End}(V_{(*)})$ and $\mathrm{Aut}(V_{(*)})$ are the set of all endomorphisms and the set of all automorphisms of the groupoid $V_{(*)} := (V, *)$, respectively.

Theorem 3.1. *Let $V = (V, F, P)$ be an algebraic system with finite support V. If the system of operations F contains a binary algebraic operation $(*)$, then the inequalities hold*

$$|\mathrm{End}(V)| \leqslant \min_{g \in V}(|L^{(1)}_{(*)}(g)| + |L^{(2)}_{(*)}(g)|), \tag{5}$$

$$|\mathrm{Aut}(V)| \leqslant \min_{g \in V}(|L^{(1)}_{(*)}(g) \cap S(V)| + |L^{(2)}_{(*)}(g) \cap S(V)|), \tag{6}$$

where $L^{(1)}_{(*)}(g)$ and $L^{(2)}_{(*)}(g)$ are type-generating sets of the groupoid $V_{(*)} := (V, *)$.

Proof. For each fixed $g \in V$ we introduce the notation

$$J_1(g) := \{\gamma \in \mathrm{Bte}(V) \mid \gamma(g) = 1\}; \quad J_2(g) := \{\gamma \in \mathrm{Bte}(V) \mid \gamma(g) = 2\}.$$

For any $g \in V$, the equality $J_1(g) \cup J_2(g) = \mathrm{Bte}(V)$ and the condition $J_1(g) \cap J_2(g) = \varnothing$ holds.

By virtue of Theorem 1.1, we have the equalities

$$\mathrm{End}(V_{(*)}) = \bigcup_{\gamma \in \mathrm{Bte}(V)} D(\gamma) = \left[\bigcup_{\gamma \in J_1(g)} D(\gamma)\right] \bigcup \left[\bigcup_{\gamma \in J_2(g)} D(\gamma)\right]. \tag{7}$$

Conditions are met

$$L^{(1)}_{(*)}(g) \in \{L^{(\gamma(s))}_{(*)}(s) \mid s \in V,\ \gamma \in J_1(g)\},\quad L^{(2)}_{(*)}(g) \in \{L^{(\gamma(s))}_{(*)}(s) \mid s \in V,\ \gamma \in J_2(g)\}.$$

In this case, the relations hold

$$\bigcup_{\gamma\in J_1(g)} D(\gamma) = \bigcup_{\gamma\in J_1(g)} \bigcap_{s\in V} L^{(\gamma(s))}_{(*)}(s) \subseteq L^{(1)}_{(*)}(g),\quad \bigcup_{\gamma\in J_2(g)} D(\gamma) = \bigcup_{\gamma\in J_2(g)} \bigcap_{s\in V} L^{(\gamma(s))}_{(*)}(s) \subseteq L^{(2)}_{(*)}(g),$$

$$\mathrm{Aut}(V_{(*)}) = \mathrm{End}(V_{(*)}) \cap S(V).$$

These relations, together with the equalities (7), give inclusions

$$\mathrm{End}(V_{(*)}) \subseteq L^{(1)}_{(*)}(g) \cup L^{(2)}_{(*)}(g),\quad \mathrm{Aut}(V_{(*)}) \subseteq (L^{(1)}_{(*)}(g) \cap S(V)) \cup (L^{(2)}_{(*)}(g) \cap S(V)). \tag{8}$$

The type-generating sets $L^{(1)}_{(*)}(s)$ and $L^{(2)}_{(*)}(s)$ have an empty intersection for any $s \in V$ (follows trivially from the definition of these sets). By virtue of the Proposition 3.1, the relations (8) and the arbitrariness of g in the reasoning above, we obtain that for any $g \in G$ the inequalities are satisfied

$$|\mathrm{End}(V)| \leqslant |L^{(1)}_{(*)}(g)| + |L^{(2)}_{(*)}(g)|,\quad |\mathrm{Aut}(V)| \leqslant |L^{(1)}_{(*)}(g) \cap S(V)| + |L^{(2)}_{(*)}(g) \cap S(V)|.$$

Therefore, the inequalities (5) and (6) are satisfied. □

The estimates (5) and (6) will be called *conservative estimates* of the order of the monoid of all endomorphisms and the group of all automorphisms of the algebraic system V. Particular cases of the algebraic system V from the 3.1 theorem are such algebras as *groupoids* ($|F| = 1$, $|P| = 0$), *rings*, *quasifields*, *semifields*, etc. Note that inclusions (8) hold for non-finitary algebraic systems with a binary algebraic operation.

Proposition 3.2. *There are finite groupoids for which the estimates (5) and (6) are achievable.*

Indeed, such groupoids include finite groupoids in which all left translations are pairwise equal (as transformations). Let $G = (G, *)$ be a finite groupoid with pairwise equal left translations. Then, by definition, the type-generating set $L^{(2)}(g)$ is an empty set for any $g \in G$ and the type generating sets $L^{(1)}(g)$ pairwise coincide. Therefore, by virtue of the Theorem 1.1, we obtain the equalities of the sets

$$\mathrm{End}(G) = D(A) = \bigcap_{g\in G} L^{(1)}(g) = L^{(1)}(g^*),\quad \mathrm{Aut}(G) = D(A) \cap S(G) = L^{(1)}(g^*) \cap S(G),$$

which hold for any $g^* \in G$.

Consider examples of constructing estimates (5) for specific finite groupoids. These examples will show that there exist finite groupoids G for which the estimates (5) are better than the *natural upper bound* which is expressed the order of the symmetric semigroup $\mathcal{I}(G)$.

Example 3.1. Let $G = (G, *)$ be a groupoid with support $G = \{1, 2, 3, 4\}$ and multiplication $(*)$ defined by the Cayley table:

*	1	2	3	4
1	1	2	3	4
2	1	2	3	4
3	2	2	3	4
4	2	2	3	4

In this case $|\mathcal{I}(G)| = 4^4 = 256$. With any transformation $\alpha \in \mathcal{I}(G)$ we will associate the notation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix} = (a_1, a_2, a_3, a_4) = (\alpha(1), \alpha(2), \alpha(3), \alpha(4)).$$

Left translations have the form: $h_1 = (1, 2, 3, 4)$, $h_2 = (1, 2, 3, 4)$, $h_3 = (2, 2, 3, 4)$, $h_4 = (2, 2, 3.4)$. Let us introduce the notation $Est_g =: |L^{(1)}(g)| + |L^{(2)}(g)|$ associated with estimates (5) for the groupoid G. Computer calculations based on the enumeration principle show that the following relations hold: $|\mathrm{End}(G)| = 38$, $Est_1 = 182$, $Est_2 = 182$, $Est_3 = 56$, $Est_4 = 56$.

These relations show that for any $g \in G$ the estimates (5) are better than the natural estimate $|\mathcal{I}(G)|$.

Example 3.2. Consider the cyclic group C_5 of order 5 given by the system of left translations:

$$h_1 = (1, 2, 3, 4, 5),\ h_2 = (2, 3, 4, 5, 1),\ h_3 = (3, 4, 5, 1, 2),\ h_4 = (4, 5, 1, 2, 3),\ h_5 = (5, 1, 2, 3, 4).$$

In this case $|\mathcal{I}(G)| = 3125$. And computer calculations give the ratios:

$$|\mathrm{End}(G)| = 5,\ Est_1 = 625,\ Est_2 = Est_3 = Est_4 = Est_5 = 5.$$

In this case, the results of conservative estimates (5) are better than in the previous example, except for the estimate $Est_1 = 5^4$.

Example 3.3. Consider the Klein four-group given by the set of left translations:

$$h_1 = (1, 2, 3, 4),\ h_2 = (2, 1, 4, 3),\ h_3 = (3, 4, 1, 2),\ h_4 = (4, 3, 2, 1).$$

In this case $|\mathcal{I}(G)| = 256$, and computer calculations give the ratios:

$$|\mathrm{End}(G)| = 16,\ Est_1 = 64,\ Est_2 = Est_3 = Est_4 = Est_5 = 16.$$

The situation is similar to the previous case.

In the context of the examples above, the following is interesting question.

Question 1. When the inequality (5) turns into equality? What the conditions must a finite groupoid G satisfy for exactness of conservative estimates?

The existence of groupoids from question 1 follows from examples 3.2 and 3.3.

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О биполярной классификации эндоморфизмов группоида

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Аннотация. В работе получен способ вычисления биполярного типа эндоморфизма произвольного группоида. Для группоидов с попарно различными левыми сдвигами элементов (в частности, группоидов с правым нейтральным элементом, моноидов, луп и групп) описанный способ вычисления биполярного типа эндоморфизма приводит к критерию неподвижной точки данного эндоморфизма. Выяснилось, что биполярный тип эндоморфизмов группоида с попарно различными левыми сдвигами содержит всю информацию о неподвижных точках эндоморфизмов этого типа. Установлено базовое множество эндоморфизмов группы, содержащее все регулярные автоморфизмы. Найден способ вычисления биполярного типа внутреннего автоморфизма моноида. Получены верхние оценки порядка моноида всех эндоморфизмов (и группы всех автоморфизмов) алгебраической системы с конечным носителем, которая обладает бинарной алгебраической операцией.

Ключевые слова: группоид, эндоморфизм группоида, автоморфизм группоида, биполярный тип эндоморфизма группоида, биполярный тип регулярного автоморфизма, биполярный тип внутреннего автоморфизма, консервативные оценки.

EDN: VEETXA
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Simulation of the Process of Frost Formation on the Surface of the Heat Exchanger Fin

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Abstract. The paper considers the process of heat and mass transfer during the flow of moist air and formation of frost in the inter-fin space of an air heat exchanger at temperature below 0^oC. The mathematical model is based on a system of stationary equations of gas dynamics that describes the flow of moist air in a channel of variable cross section and on a non-stationary system of equations for determining frost deposition on a flat surface. The results of computational modelling of the formation of layer of frost on a surface of a fin with given temperature distribution are presented. It is shown that temperature distribution of the fin surface has a significant influence on frost formation.

Keywords: frost formation, air heat exchanger, heat transfer coefficient, computational modelling.

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Introduction

Frost formation is observed on low-temperature surfaces of refrigeration units in contact with moist air. The layer of frost on the working surfaces of air heat exchangers is one of the key factors affecting efficiency of their work. Frost deposition negatively affects the efficiency of refrigeration equipment due to appearance of additional thermal resistance and due to an increase in the hydraulic resistance of the heat exchanger to the air flow when the flow section of the cooled air flow is blocked by frost [1]. Accounting for the processes of frost formation is necessary to choose the design and geometric parameters of finned-plate heat exchangers. Simulation of growth process of the layer of frost allows one to set the optimal operation of the heat exchanger without preventive defrosting.

The design of a thermoelectric cooling unit for ship refrigeration units and results of the study of its characteristics were presented in [2]. The design of an air fin-plate heat exchanger of a thermoelectric cooling unit and results of the study of heat and mass transfer in the exchanger were presented in [3, 4]. Calculation of the temperature field and the heat transfer coefficient of the fin was carried out [3]. It was shown that thermal resistance of the air heat exchanger exerts primary control over cooling capacity and coefficient of performance of the thermoelectric unit. The process of frost formation on the surface of the fins was experimentally studied in [4]. It was established that frost formation has significant effect on heat transfer characteristics. The purpose of this work is to analyse and to model heat and mass transfer processes during formation of frost layer on a flat surface of an air heat exchanger fin.

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1. Computational model of the process of heat and mass transfer in the inter-fin channel

Let us consider the process of heat and mass transfer in the case of a longitudinal flow of moist air around the surface of heat exchanger fins with a temperature below 0^oC. The space region is a slit channel with constant cross section, and distance between the fin plates is $2\delta_0$ (Fig. 1). At the initial moment of time the channel is free of frost so the air flows over its entire cross section. When moist air flows on the surface of fins a layer of frost with thickness δ_f is formed. It reduces the height of the passage section of the channel $2\delta = 2\delta_0 - 2\delta_f$. The thickness of the frost layer varies both in time and along the length of the channel. The computational domain is highlighted in the figure by a dashed contour. The computational domain includes zones of the air flow and the frost layer which have movable interface. Let us define systems of equations that describe parameters of the medium in these zones.

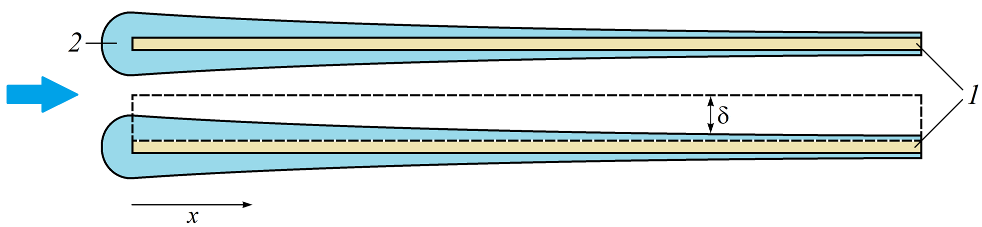

Fig. 1. General view of heat exchanger fins (*1*) with frost layer (*2*) and computational domain of the problem (highlighted by a dotted line); the arrow shows the direction of air flow

To describe the flow of moist air consider the system of one-dimensional gas dynamics equations for a channel of variable cross section [5, 6]

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial(\rho u S)}{\partial t} + \frac{\partial(\rho u^2 S)}{\partial x} + S\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial(\rho e S)}{\partial t} + \frac{\partial(\rho u e S)}{\partial x} + S\frac{\partial(p u S)}{\partial x} = q_V S, \tag{3}$$

where t — time, ρ, p, e, u — density, pressure, internal energy and air velocity, S — cross-sectional area of the channel. The volumetric capacity of the heat sink q_V is determined by the heat exchange of the air flow with the surface of the frost layer. Equations (1)–(3) are used for gas flows in flat or axisymmetric channels with a slightly changing profile.

Equations (1)–(3) are supplemented with the water vapour mass conservation equation

$$\frac{\partial(\rho_v S)}{\partial t} + \frac{\partial(\rho_v u S)}{\partial x} = -jS, \tag{4}$$

where ρ_v is the vapour density, j is the mass of vapour condensed for 1 s on the surface and recalculated per unit volume of air flow. The effect of vapour in momentum and energy conservation equations is not taken into account because of its negligible contribution ($\rho_v \ll \rho$).

The speed of sound c for the temperature conditions of the refrigerating chamber exceeds 300 m/s and the air flow velocity in the channel $u \leqslant 3$ m/s. Therefore, the air is considered in this problem as incompressible medium ($u \ll c$) [7]. In addition, the change in the thickness of the frost layer is the slowest process in the channel. Therefore, the gas-dynamic flow at each moment of time is steady (quasi-stationary) since it quickly adapts to the current profile of the frost layer. Thus, system of equations (1)–(4) is rewritten in the form of a system of ordinary differential equations

$$\frac{d(uS)}{dx} = 0, \tag{5}$$

$$\frac{d(\rho u^2/2 + p)}{dx} = 0, \tag{6}$$

$$\frac{d(c_p T + u^2/2)}{dx} = \frac{q_V}{\rho u}, \tag{7}$$

$$\frac{d\rho_v}{dx} = -\frac{j}{u}, \tag{8}$$

where c_p, T are isobaric heat capacity and air temperature. The boundary conditions for equations (5)–(8) are values of air velocity, pressure and temperature as well as the vapour density in the inlet section of the channel.

The simulation of frost deposition on a flat surface was carried out using a frost growth model based on analytical and empirical relationships [8, 9, 10]

$$\dot{m_f} = \frac{dm_f}{dt} = -\frac{\alpha(\omega - \omega_{fs})}{c_p}, \tag{9}$$

$$\delta_f = \sqrt[3]{\frac{6.34 \cdot 10^{-6} \alpha(\omega - \omega_{fs}) t^2 (T_{fs} - T_p)}{c_p \left[\alpha(\omega - \omega_{fs}) + 2c_p(T - T_{fs})\right]}}, \tag{10}$$

$$T_{fs} = T_p + A\beta \left[1 + 0.5\beta \left(0.0196A + \frac{\beta c_f \rho_f}{2c_p \lambda}\right)\right], \tag{11}$$

$$A = \frac{\alpha(T - T_{fs})\sqrt{t}}{\lambda} \left(1 + \frac{L(\omega - \omega_{fs})}{c_p(T - T_{fs})}\right), \tag{12}$$

$$\rho_f = \frac{m_f}{\delta_f}, \tag{13}$$

$$\lambda = \frac{0.0131 \left[\exp(0.0196T_{fs} + 273) - \exp(0.0196T_p + 273)\right](1 + 0.0134\rho_f)}{T_{fs} - T_p}, \tag{14}$$

where m_f is the specific mass of frost per unit surface of the fin (kg/m^2), α is the heat transfer coefficient of the frost surface (W/(m^{2}K)), ω, ω_{fs} — moisture content of air and saturated air at the surface temperature of the frost layer (kg/kg), T_p, T_{fs} — temperature of the fin plate and the surface of the frost layer (^{o}C), ρ_f — frost density (kg/m^3), $\beta = \delta_f/\sqrt{t}$ — frost layer growth factor (m/s$^{0.5}$) , L — latent heat of sublimation (J/kg). Equations (9)–(14) allow one to calculate the values of m_f, δ_f, T_{fs} for a certain point in time and the current coordinate x for given parameters α, ω, ω_{fs}, T_p and T. The values $\omega = \rho_v/\rho$ and T are calculated from the solution of equations (5)–(8). The value $q_V = \alpha(T_{fs} - T)/\delta$ in (7) is recalculated through surface heat exchange with the layer of frost. The value $j = \dot{m_f}/\delta$ in equation (8) is determined similarly. System of equations (1)–(14) is solved numerically, and distributions of gas-dynamic parameters along the channel are calculated for each time step.

2. Discussion of calculation results

The simulation of heat and mass transfer was carried out for a fin plate length of 0.15 m. Temperature, velocity, and air pressure at the inlet are $T_0 = -10^o$C, $u_0 = 3$ m/s, $p_0 = 101$ kPa. The air humidity at the inlet $\omega_0 = 1.6{\cdot}10^{-3}$ kg/kg is corresponded to the moisture content of saturated air at temperature T_0. Specified values of parameters approximately correspond to the parameters of the experimental study of frost formation [4]. The temperature distribution along the rib (along the x coordinate) can be uniform or have given profile. The temperature distribution over the fin area can be measured or obtained with the use of computational simulation [3, 11].

One of the most important parameters that affects the process of heat and mass transfer is the heat transfer coefficient of the frost surface α. This parameter is directly present in relations (9), (10), (12), and it implicitly affects the value of q_V on the right side of equation (7). Moreover, the intensity of vapour condensation $\dot{m}_f$ is directly proportional to α. In the initial state, the channel has a constant flow area along the length and the distance between fins is $2\delta_0$. With the growth of the layer of frost the flow area decreases non-uniformly along the length of the fin, and velocity distribution also becomes non-uniform. The value of the heat transfer coefficient of the frost surface α depends heavily on $D{=}2\delta$, flow velocity and thermophysical properties of the air. Calculation of the heat transfer coefficient for smooth fins of plate heat exchanger with similar process parameters was carried out in [3]. Using the same technique, the relationships between heat transfer coefficient of the surface and inter-fin spacing were obtained for various values of the flow velocity. They are shown in Fig. 2., the curves show the speeds in m/s.

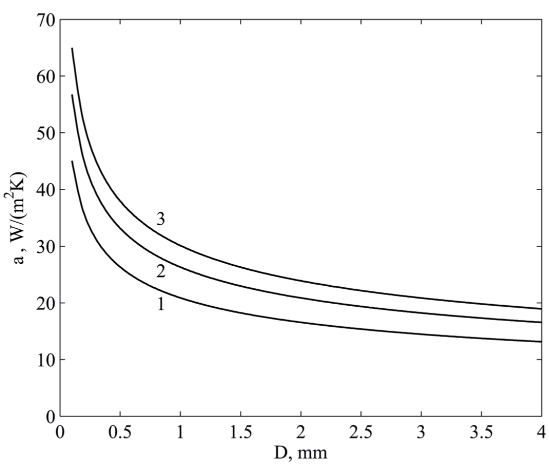

Fig. 2. Relationships between heat transfer coefficient of the surface and inter-fin distance for various values of the flow velocity (values are shown in the graph in m/s

Function $\alpha(D, u)$ is approximated using interpolation polynomials for use in the computational model. In addition, it is known that heat transfer coefficient of the frost surface differs significantly from the value that corresponds to the smooth surface of the fin. During the formation of frost, the value of heat transfer coefficient α of the surface is affected by the roughness of the frost layer. It was found that in the first hours of frost formation the heat transfer coefficient increases by 1.6–1.8 times in comparison with its value for a smooth metal plate due to surface roughness, and it is 20–25% over its value for a surface without frost in the steady state [12]. To take this into account in modelling, the value of the heat transfer coefficient of the frost surface was increased by a factor of 1.5 with respect to $\alpha(D, u)$.

The frost formation rate is characterized by the layer thickness $\delta_f(x)$ on the fin surface at different moments of times. The layer thickness as a function of x is shown in Fig. 3 for δ_0=1.5 mm and uniform temperature distribution along the fin $T_p(x) = -20^o$C. Curves correspond to the following moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min. The most rapid growth of the frost layer occurs near the inlet section where the air flow has a maximum moisture content. As the distance from the inlet section increases the thickness of the frost layer decreases monotonically because vapour content in the flow decreases due to its continuous condensation. Over time, the growth of the frost layer in the downstream part of the channel slows down significantly. This is primarily due to two factors. First, a decrease in the inlet gap $2\delta = 2\delta_0 - 2\delta_f$ due to an increase in the thickness of the frost layer δ_f leads to a proportional decrease in the total mass of steam entering the inter-fin channel. Secondly, faster growth of the frost layer at the inlet leads to a decrease in the downstream flow velocity according to (5) and a corresponding decrease in the heat transfer coefficient which also leads to slowdown in steam condensation. In the inlet part of the channel the heat transfer coefficient increases with an increase in δ_f. This contributes to intense steam condensation. As a result of the combined action of these factors the inhomogeneity of the thickness of the frost layer on the fin increases with time. At time $t = 150$ min, 93% of the cross-section of the channel at the entrance is already blocked by the layer of frost, and only 20% of the output section is blocked. In this inter-fin space an expanding channel is formed in which the air flow velocity decreases from the initial 3 m/s to 0.26 m/s at the outlet.

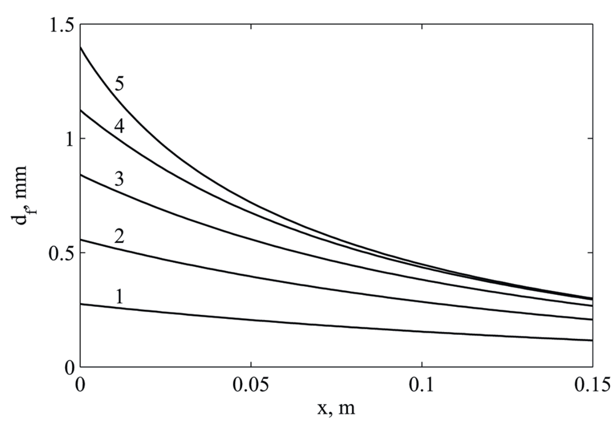

Fig. 3. Frost layer thickness versus x-coordinate at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min

Frost layer thickness δ_f monotonically decreases with x as as shown in Fig. 3. This is due to the condensation of steam which leads to steady drop in the moisture content of the air as it moves along the fin. Air humidity $\omega(x)$ as a function of x is shown in Fig. 4 at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min. A decrease in the value of ω leads to a corresponding decrease in the value of the difference $\omega - \omega_{fs}$ which, according to relations (9) and (10), determines the intensity of steam condensation and the thickness of the frost layer. The moisture content of saturated air ω_{fs} depends on the surface temperature of the frost. It is equal to $6.3 \cdot 10^{-4}$ kg/kg at $T = -20^o$C. Curve *5* asymptotically approaches this value on the chart. Such a sharp decrease in humidity as the inlet section of the inter-fin space is closed is caused by a corresponding decrease in the air flow rate and the mass of steam entering the channel. This small mass of steam is intensively condensed already at the initial section of the fin. Therefore, the air humidity drops significantly over time when approaching the outlet section, and increase in the thickness of the frost layer is insignificant.

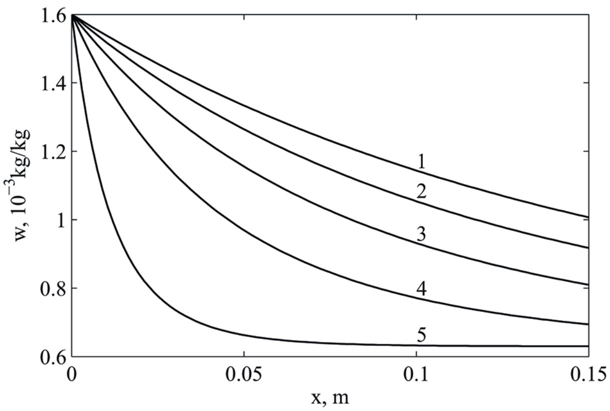

Fig. 4. Air humidity versus x-coordinate at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min

The rate of air cooling in the inter-fin channel of the heat exchanger depends on the difference $T(x) - T_{fs}(x)$. Air temperature as a function of x is shown in Fig. 5. The pattern of $T(x)$ is similar to $\omega(x)$ graphs shown in Fig. 4. This similarity is due to the similar form of relation (9) which describes the process of steam condensation, and the expression for the value of the heat flux coming from the air flow to the frost surface $q = \alpha(T_{fs} - T)$ which determines the value of the air temperature T. As the inlet section of the channel is blocked the outlet temperature approaches the value of the fin temperature.

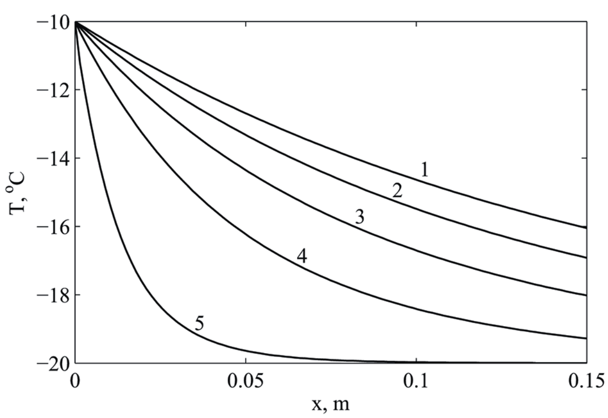

Fig. 5. Air temperature versus x-coordinate at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4*— 120 min, *5* — 150 min

Temperature $T_{fs}(x)$ is determined by the balance of convective heat exchange between the frost surface and the air flow and thermal interaction with the fin surface through the frost layer.

The frost surface temperature as a function of x is shown in Fig. 6 at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min. At the beginning of the process when the thickness of the frost layer is small, the frost surface temperature varies slowly with x, and the value of T_{fs} is close to the temperature of the fin surface. Then, the increase of T_{fs} primarily occurs at the inlet section of the channel due to improved heat exchange with the air flow and decrease in heat flow through the rapidly growing frost layer. Downstream, on the contrary, temperature T_{fs} decreases over time to values close to the temperature of the fin $T_p = -20^o$C since the air temperature here also has minimum values.

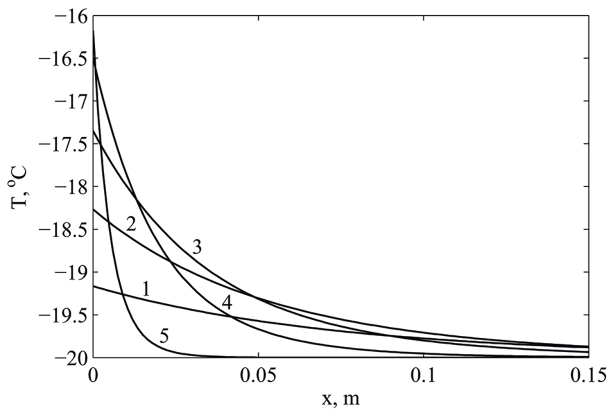

Fig. 6. Frost surface temperature versus x-coordinate at different moments of time: *1* — 30 min, *2* — 60 min, *3* — 90 min, *4* — 120 min, *5* — 150 min

The simulation results for a uniform temperature distribution along the fin showed that the overlap time of the inter-fin gap is determined by the rate of frost deposition on the inlet section. Let us consider the possibility of reducing the intensity of frost formation at the inlet section by changing the temperature profile along the fin. Influence of the temperature distribution of the fin surface on the rate of frost formation was established in experimental study [4]. Thus, in experiments with the orientation of the heat exchanger layout which provides increased temperatures at the initial section of the fin the time of overlapping the inter-fin gap with a layer of frost has significantly increased. To determine the influence of the inhomogeneity of the temperature distribution along the length of the fin $T_p(x)$ calculations were carried out with some model temperature profile. At the same time, the temperature profile in the first third of the channel changes linearly from -16^oC at the inlet to -20^oC at $x = 0.05$ m, and constant value $T_p = -20^o$C is set for the rest of the fin. The thickness of the frost layer $\delta_f(x)$ is shown in Fig. 7 at moments of time: *1* — 60 min, *2* — 120 min, *3* — 180 min, *4* — 240 min. Comparing Fig. 3 and Fig. 7, one can see significant effect of the temperature profile of the fin on the rate of the frost layer formation. The inlet section with an increased temperature makes it possible to significantly increase uniformity of the frost deposition along the length of the fin and to reduce the intensity of the frost growth at the inlet. With this temperature distribution, the time for which the channel is almost completely blocked increased from 150 min to 240 min. A similar ratio was observed in experiments [4]. The analysis of $\delta_f(x)$ in Fig. 7 shows that there is still a certain reserve for increasing this time by choosing the optimal temperature profile of the fin.

The efficiency of a heat exchanger is characterized by its ability to remove heat from the air stream. The heat power removed from the air per unit fin width for the two profiles of the fin

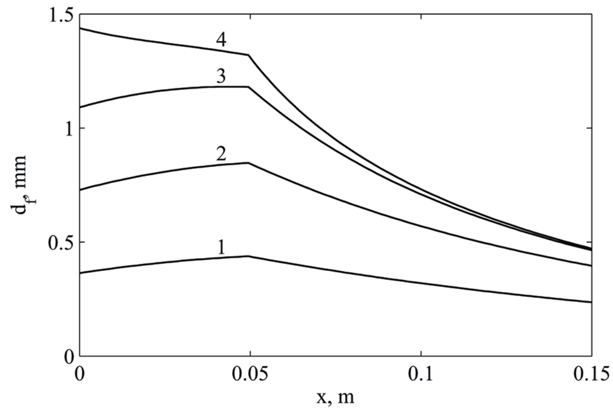

Fig. 7. Frost layer thickness versus x-coordinate at different moments of time with non-uniform temperature distribution: *1* — 60 min, *2* — 120 min, *3* — 180 min, *4* — 240 min

surface temperature considered above is shown in Fig. 8. A fin with a uniform temperature distribution has a slight advantage only for the first 30 minutes then a fin with a non-uniform temperature distribution shows a higher efficiency. It takes 81 minutes to achieve efficiency of at least 70% from the initial heat power for homogeneous profile and 136 minutes for a non-uniform distribution. Thus, setting the temperature distribution of the fin with increased values near the inlet part changes the rate of frost layer formation. It makes deposition of frost more uniform along the length of the fin. As a result, this makes it possible to increase the duration of the effective operation of a heat exchanger and to carry out preventive defrosting procedures less often to remove frost from the surface of fins.

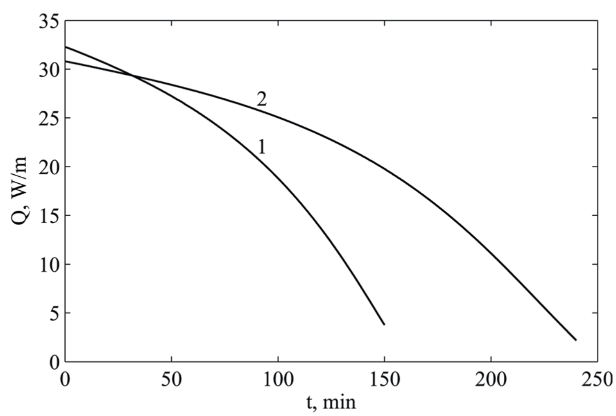

Fig. 8. Thermal power supplied from the air flow to the surface of the frost versus time for uniform (*1*) and non-uniform (*2*) temperature distributions along the fin

Conclusion

A computational model is presented to describe the process of frost formation on the surface of an air heat exchanger fin. Frost layer thickness, air temperature and humidity, frost surface temperature as functions of distance along the fin are obtained. A significant influence of the temperature profile on the rate of frost layer formation and the efficiency of heat exchange between the air flow and the fin surface was revealed. This generally corresponds to patterns identified in the experimental study of the process. Taking into account technical conditions and given operating mode, the proposed computational model can be used to optimize the design of heat exchangers in refrigeration units.

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Моделирование процесса инееобразования на поверхности ребра теплообменника

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Аннотация. В работе рассмотрен процесс тепломассообмена при течении влажного воздуха и образовании инея в межреберном промежутке воздушного теплообменника, имеющего температуру ниже 0^oC. Математическая модель основана на системе стационарных уравнений газодинамики для описания течения влажного воздуха в канале переменного сечения и нестационарной системе уравнений для определения динамики осаждения инея на плоской поверхности. Представлены результаты вычислительного моделирования процесса формирования слоя инея на поверхности ребра, имеющего заданное распределение температуры. Показано существенное влияние температурного распределения поверхности ребра на динамику инееобразования.

Ключевые слова: инееобразование, воздушный теплообменник, коэффициент теплоотдачи, вычислительное моделирование.

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On the Stability of the Solutions of Inverse Problems for Elliptic Equations

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Abstract. The inverse problems on finding the unknown lower coefficient in linear and nonlinear second-order elliptic equations with integral overdetermination conditions are considered. The conditions of overdetermination are given on the boundary of the domain. The continuous dependence of the strong solution on the input data of the inverse problem for the linear equation is proved in the case of the mixed boundary condition. As to the nonlinear equation, the continuous dependence of the strong solution on the overdetermination data is established for the inverse problem with the Dirichlet boundary condition.

Keywords: inverse problem, elliptic equation, integral overdetermination, continuous dependence on input data.

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Introduction

In this paper the stability of the solutions of two inverse problems for second order elliptic equations are considered.

Problem 1. For given functions $f(x), \beta(x), h(x), \alpha(x)$ and a constant μ find the pair of functions u and constant k, satisfying the equation

$$-\operatorname{div}(\mathcal{M}(x)\nabla u) + m(x)u + ku = f, \tag{1}$$

the boundary condition

$$\left(\frac{\partial u}{\partial \overline{N}} + \alpha(x)u\right)\Big|_{\partial\Omega} = \beta(x), \tag{2}$$

and the condition of overdetermination

$$\int_{\partial\Omega} uh(x)ds = \mu. \tag{3}$$

Problem 2. For given functions $f(x), \beta(x), h(x)$ and a constant μ find the pair of funtcions u and constant k satisfying the equation

$$-\operatorname{div}(\mathcal{M}(x)\nabla u) + m(x)u + kr(u) = f, \tag{4}$$

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the boundary condition

$$u|_{\partial\Omega} = \beta(x), \tag{5}$$

and the condition of overdetermination

$$\int_{\partial\Omega} \frac{\partial u}{\partial \overline{N}} h(x) ds = \mu. \tag{6}$$

Here $\Omega \cap \mathbf{R}^n$ is a bounded domain with a boundary $\partial\Omega \in C^2$, $\mathcal{M}(x) = m_{ij}(x)$ is a matrix of functions $m_{ij}, i, j = 1, 2, \dots, n$, $m(x)$ is a scalar funtcion, $\dfrac{\partial}{\partial \overline{N}} = (\mathcal{M}(x)\nabla, \mathbf{n})$, $\mathbf{n}$ is the unit vector of the outward normal to the boundary $\partial\Omega$.

A main goal of this paper is to establish stability (in the sense of continuous dependence on the source data) of strong generalized solutions of Problems 1 and 2. The conditions of the solvability and uniqueness of solutions to Problems 1 and 2 were established in [1,2]. The proof of the existence and uniqueness of the solutions follows the method developed by A. Sh. Lyubanova and A. Tani in [3,4] where inverse problems with integral overdetermination conditions were also considered. The method is based on the idea of reducing the inverse problem to an operator equation of the second kind for the unknown coefficient [5].

Practical interest in such inverse problems is due to many applications in the theory of diffusion and filtration [6] as well as the fact that filtration processes tend to stabilize over time [7]. The steady fluid flow in a fissured medium is described by a stationary equation in which the pressure u, coefficients and the right-hand side are independent of t. In general, the stationary equation of the compressible fluid filtration has the form

$$-\operatorname{div}(\mathbf{k}(x,u)\nabla\psi(u)) + \gamma(x,u) = f, \qquad x \in \Omega, \tag{7}$$

where $\mathbf{k}(x,u)$ is a matrix of functions, $\psi(u)$ и $\gamma(x,u)$ are scalar functions, $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega$. An example of a diffusion model is the problem of finding the concentration of a pollutant in the environment [8]

$$-\lambda\Delta u + \mathbf{v}\nabla u + ku = f, \qquad u|_{\partial\Omega} = \beta,$$

where k is a value characterizing the breakdown of a pollutant due to chemical reactions, λ is the diffusion coefficient, f is the bulk source density, $\mathbf{v}$ is the velocity vector.

The study of the inverse problems for the elliptic equations goes back to fundamental works of M.M. Lavrentiev [9–11]. Various issues related to coefficient inverse problems for the linear and nonlinear equations (7) were discussed in [11–22]. Problems of finding highest coefficients of (7) from additional boundary data on $\partial\Omega$ or on some part of $\partial\Omega$ are of particular interest. In [15, 16, 20] this problem is considered in the case of $\psi(u) = u$, $\gamma(x,u) \equiv 0$, $\mathbf{k}(x,u) = k\mathbf{E}$, $\mathbf{E}$ is the identity matrix, and function k is unknown. It is assumed that $k = k(x)$ [15, 16], or $k = k(u)$ [20]. The overdetermination condition is $k\dfrac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = \nu(x)$ for the Dirichlet boundary problem and $u|_{\partial\Omega} = \nu(x)$ for the Neumann boundary problem. The pioneering work in this line is Calderon's one [16] where the inverse problem of finding the unknown $k(x,u) = k(x)$ with such overdetermination condition was first discussed and an approximate representation was suggested for the unknown coefficient close to a constant.

Problems of recovering unknown lowest coefficients in elliptic equations have been considered by many authors. The works of [8, 12, 21–23] should be noted here. In these works, unknown coefficients are recovered from information on the values of some integral operator over the whole domain or the solution trace on some surface inside the domain in which the problem is solved. Integral conditions on the boundary were not considered in such problems.

1. The preliminaries

We use the following notations: $\|\cdot\|_R$, $(\cdot,\cdot)_R$ — the norm and the inner product in $\mathbb{R}^n$; $\|\cdot\|$, $(\cdot,\cdot)$ — the norm and the inner product in $L^2(\Omega)$; $\|\cdot\|_j$, $\langle\cdot,\cdot\rangle_1$ — the norm in $W_2^j(\Omega)$, $j=1,2$, and the duality relation between $\overset{\circ}{W}{}_2^1(\Omega)$ and $W_2^{-1}(\Omega)$, respectively, $\|\cdot\|_{j+1/2}$ — the norm in $W_2^{s+1/2}(\partial\Omega)$, $s=0,1$.

Let us introduce the linear operator $M: W_2^1(\Omega)\to(W_2^1(\Omega))^*$ of the form

$$M=-\operatorname{div}(\mathcal{M}(x)\nabla)+m(x)I,$$

where I is the identity operator. We use the notation

$$\langle Mv_1,v_2\rangle_M=\int_\Omega((\mathcal{M}(x)\nabla v_1,\nabla v_2)_R+m(x)v_1v_2)dx$$

for any $v_1,v_2\in W_2^1(\Omega)$, and reason that the following assumptions of the operator M are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i,j,l=1,2,\dots,n$, and $m(x)$ are bounded in Ω. Operator M is strongly elliptic, that is, there exist positive constants m_0 and m_1 such that for all $v\in W_2^1(\Omega)$

$$m_0\|v\|_1^2\leqslant\langle Mv,v\rangle_M\leqslant m_1\|v\|_1^2.$$

II. M is self-adjoint, that is $m_{ij}(x)=m_{ji}(x)$ for $i,j=1,\dots,n$.

We also impose restrictions on function $r(\rho)$.

III. The function $r(\rho)$ is continuous and strictly monotone $(-\infty,+\infty)$, that is for all $\rho_1,\rho_2\in(-\infty,+\infty)$, $\rho_1\neq\rho_2$,

$$(r(\rho_1)-r(\rho_2))(\rho_1-\rho_2)>0,$$

and $r(0)=0$.

IV. For all $\rho\in(-\infty,+\infty)$

$$|r(\rho)|\leqslant C_r|\rho|^{p-1}. \tag{8}$$

Here $C_r>0$, p – constants, $p>0$ when $n\leqslant 2$ and $0<p\leqslant n/(n-2)$ when $n>2$.

V. For any number $R>0$ and funtcions $v_1,v_2\in W_2^1(\Omega)$ such that $\|v_i\|_{L^{2(p-1)}(\Omega)}\leqslant R$, $i=1,2$, the inequality

$$\|r(v_1)-r(v_2)\|\leqslant c(R)\|v_1-v_2\|_1$$

is valid where constant $c(R)>0$ depends on R.

By the solution of Problem 1 is meant the pair, consisting of a function $u\in W_2^2(\Omega)$ and a constant $k>0$ which satisfies the equation (1), the boundary condition (2) and the overdetermination condition (3). By the solution of Problem 2 is meant the pair involving a function $u\in W_2^2(\Omega)$ and a constant $k>0$ which satisfies the equation (4), the boundary condition (5) and the overdetermination condition (6).

We define the auxiliary funtcions a,a^σ,b,d,d^τ and g as a solutions of the problems

$$Ma=f(x),\qquad\left(\frac{\partial a}{\partial\overline{N}}+\alpha(x)a\right)\Big|_{\partial\Omega}=\beta(x); \tag{9}$$

$$Ma^\sigma + \sigma a^\sigma = f, \qquad \left(\frac{\partial a^\sigma}{\partial \overline{N}} + \alpha(x) a^\sigma\right)\Big|_{\partial\Omega} = \beta(x); \tag{10}$$

$$Mb = 0, \qquad \left(\frac{\partial b}{\partial \overline{N}} + \alpha(x) b\right)\Big|_{\partial\Omega} = h(x); \tag{11}$$

$$Md = f(x), \qquad d|_{\partial\Omega} = \beta(x); \tag{12}$$

$$Md^\tau + \tau r(a^\tau) = f(x), \qquad d^\tau|_{\partial\Omega} = \beta(x); \tag{13}$$

$$Mg = 0, \qquad g|_{\partial\Omega} = h(x). \tag{14}$$

Here $\sigma > 0$, $\tau > 0$ — real numbers.

Existence and uniqueness theorems for strong solutions of the inverse Problems 1 and 2 were proven earlier in [1, 2]. For the sake of convenience, we give their formulations.

Theorem 1 ([1])**.** *Let $\partial\Omega \in C^2$ and assumptions I and II be fulfilled. Suppose also that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x)$, $h(x) \in W_2^{3/2}(\partial\Omega)$, $\alpha(x) \in C(\partial\Omega)$*;*

(ii) $f(x) \geqslant 0$ *almost everywhere in* Ω*;* $\beta(x) \geqslant 0$, $\alpha(x) \geqslant 0$, $h(x) \geqslant 0$ *for almost all* $x \in \partial\Omega$ *and there is a smooth piece* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geqslant \delta$ *and* $h \geqslant \delta$ *almost everywhere in* Γ.

Then Problem 1 has a solution $\{u, k\}$*, if*

$$0 \leqslant \mu - \Phi \leqslant \frac{m_0 (a, b)^2}{\|a\| \|b\|},$$

where $\Phi = \int\limits_{\partial\Omega} ah\, ds$*, and the estimates*

$$a^\sigma \leqslant u \leqslant a, \quad 0 \leqslant k \leqslant \sigma, \quad \|u\|_2 \leqslant C(\sigma + 1)\|a\| + \|a\|_2, \tag{15}$$

holds for some $\sigma > 0$*, constant* C *depends on* $mes\Omega, \sigma, m_0$ *and* m_1*. Moreover, if*

$$0 \leqslant \mu - \Phi < \frac{m_0 (a, b)^2}{\|a\| \|b\|} \tag{16}$$

then the solution of Problem 1 is unique.

Theorem 2 ([2])**.** *Let assumptions I–V be fulfilled. Suppose also that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x), h(x) \in W_2^{3/2}(\partial\Omega)$*;*

(ii) $f(x) \geqslant 0$ *almost everywhere in* $\Omega; \beta(x) \geqslant 0, h(x) \geqslant 0$ *for almost all* $x \in \partial\Omega$ *and there is a smooth piece* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geqslant \delta$ *and* $h(x) \geqslant \delta$ *almost everywhere in* Γ.

If

$$0 \leqslant Q \equiv (f, g) - \langle Md, g\rangle_1 + \mu \leqslant \frac{m_1\, (r(d), g)^2}{4c_0^p\, C_r^{p/(p-1)}\, \Psi},$$

where $\Psi = c(\|d\|_{L^{2p-2}(\Omega)})\|d\|_1\|g\|_1$, c_0 *— embedding constant* $W_2^1(\Omega)$ *in* $L^p(\Omega)$*, then the problem (4)–(6) has a solution* $\{u, k\}$*, and estimates*

$$0 \leqslant k \leqslant \tau, \qquad d^\tau \leqslant u \leqslant d, \qquad \|u\|_2 \leqslant C_M(\tau\, C_r\, \|d\|_1^{p-1} + \|d\|) + \|d\|_2. \tag{17}$$

holds for some $\tau > 0$*, with a constant* C_M*, depends on* m_0, τ *and* $mes\Omega$*. Moreover, if*

$$0 \leqslant Q \equiv (f, g) - \langle Md, g\rangle_1 + \mu < \frac{m_1\, (r(d), g)^2}{4c_0^p\, C_r^{p/(p-1)}\, \Psi},$$

then the solution is unique.

2. Stability of the solutions of inverse problems

The main results of the work are theorems on the continuous dependence of strong solutions on the input data of the above inverse problems.

Let us consider Problem 1.

Theorem 3. *Let assumptions of Theorem 1 be fulfilled and a pair $\{u_j, k_j\}$ be the unique solution of Problem 1, where $f = f_j, \beta = \beta_j, h = h_j$, and $\mu = \mu_j, j = 1, 2$. Then the estimate*

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \leqslant K(|\mu_1 - \mu_2| + \|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{3/2} + \|h_1 - h_2\|_{1/2}) \tag{18}$$

holds with a constant $K > 0$.

Proof. Let a_j, a^σ, b_j are solutions of problems (9), (10), (11), where $f = f_j, \beta = \beta_j, h = h_j$, $j = 1, 2$. It was shown in [1] that k_j is the solution of the operator equation $k_j = A_j k_j$, where $A_j k_j$ is determined as

$$A_j k_j = \frac{\mu_j - \Phi_j}{(u_j, b_j)}, \tag{19}$$

where $\Phi_j = \int\limits_{\partial\Omega} a_j h_j ds$, and σ_j is given by the relation

$$\sigma_j = \frac{\sqrt{m_0}((a_j, b_j) - \sqrt{D_j})}{2\|a_j\|\|b_j\|}, \tag{20}$$

with

$$D_j \equiv (a_j, b_j)^2 - \frac{4(\mu_j - \Phi_j)\|a_j\|\|b_j\|}{\sqrt{m_0}} \geqslant 0,$$

Estimating the right side of the difference

$$k_1 - k_2 = A_1 k_1 - A_2 k_2 = \frac{\Phi_2 - \Phi_1 + \mu_1 - \mu_2}{(u_1, b_1)} + k_2 \Big[\frac{(u_2 - u_1, b_1) - (u_1, b_1 - b_2)}{(u_1, b_1)}\Big].$$

in absolute value with (15) and the relation [1]

$$(u_1, b_1) \geqslant (a_1^\sigma, b_1) = (a_1, b_1) - (a_1 - a_1^\sigma, b_1) \geqslant (a_1, b_1) - \frac{\sigma_1}{\sqrt{m_0}}\|a_1\|\|b_1\| \geqslant 0,$$

we come to the inequality

$$|k_1 - k_2| \leqslant K_1(|\mu_1 - \mu_2| + |\Phi_1 - \Phi_2| + \|b_1 - b_2\|_1) + \frac{k_2\sqrt{m_0}\|b_1\|\|u_1 - u_2\|}{\sqrt{m_0}(a_1, b_1) - \sigma_1\|a_1\|\|b_1\|}, \tag{21}$$

where positive constant K_1 depends on m_0, mesΩ, μ_j, Φ_j, $\|a_j\|_1$, $\|b_j\|_1$, $j = 1, 2$.

On the other hand, difference $u = u_1 - u_2$ satisfies the relations (1)–(2), where $k = k_1$, $f = (k_2 - k_1)u_2 + f_1 - f_2$ and $\beta = \beta_1 - \beta_2$. Using (15) and (19), for u_j, k_j when $a = a_j$ and $\sigma = \sigma_j$, $j = 1, 2$, and also estimate [24]

$$\|v\|_2 \leqslant C_M(\|Mv\| + \|v\|), \tag{22}$$

valid for all $v \in \mathring{W}_2^1(\Omega) \bigcap W_2^2(\Omega)$, we obtain

$$\|u_1 - u_2\|_1 \leqslant \frac{1}{m_0}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\|a_1\|) + \|a_1 - a_2\|_1,$$

$$\|u_1 - u_2\|_2 \leqslant \frac{C_M(m_0+1)}{m_0}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\|a_1\|) + \|a_1 - a_2\|_2. \qquad (23)$$

Without loss of geberality, it may be suggested that $k_1 \geqslant k_2$. Then (15), (16), (20) for $j = 1$ and (21) lead to inequality

$$|k_1 - k_2| \leqslant K_2\Big[|\mu_1 - \mu_2| + |\Phi_1 - \Phi_2| + \|b_1 - b_2\|_1\Big], \qquad (24)$$

where K_2 depends on K_1, m_0, σ_1, $\|a_1\|$. For $a_1 - a_2$ and $b_1 - b_2$, we have [25, Chapter 2]

$$\|a_1 - a_2\|_j \leqslant C_2(\|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{j-1/2}), \quad j = 1, 2, \qquad (25)$$

$$\|b_1 - b_2\|_1 \leqslant C_1\|h_1 - h_2\|_{1/2}, \qquad (26)$$

where constants $C_i > 0,\ i = 1, 2$, depend on n, m_0, m_1 and $mes\Omega$. Taking into account definition of $\Phi_j,\ j = 1, 2$, and relations (23)–(26), we come to the estimate (18). Theorem is proved. □

Let us turn our attention to the theorem on stability of the strong solution of Problem 2.

Theorem 4. *Let the assumptions of Theorem 2 be fulfilled and a pair $\{u_j, k_j\}$ be a solution of Problem 2 where $\mu = \mu_j, j = 1, 2$. Then the estimate*

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \leqslant H|\mu_1 - \mu_2| \qquad (27)$$

holds with a constant $H > 0$.

Proof. Let d_j^τ be the solution of (13) with $\tau = \tau_j$, $j = 1, 2$, where

$$\tau_j = \frac{((r(d), g) - \sqrt{G_j})m_0}{2Q_j\ C_r^{p/(p-1)}\ c_0^p\ \Psi}, \qquad G_j = (r(d), g)^2 - 4Q_j\frac{C_r^{p/(p-1)}c_0^p\ \Psi}{m_0},$$

and

$$Q_j = (f, g) - \langle Md, g\rangle_1 + \mu_j.$$

As was shown in [2], k_j is a solution of the operator equation

$$k_j = B_jk_j = \frac{Q_j}{(r(u_j), g)}. \qquad (28)$$

For the sake of convenience, we denote by $\{\bar{u}, \bar{k}\}$ the difference of solutions $\{u_1, k_1\}$ and $\{u_2, k_2\}$. $\bar{k}$ is a solution of the equation

$$\bar{k} = B_1k_1 - B_2k_2 = \frac{Q_1}{(r(u_1), g)} - \frac{Q_2}{(r(u_2), g)} = \frac{(Q_1 - Q_2)(r(u_1), g) - Q_1(r(u_1) - r(u_2), g)}{(r(u_1), g)(r(u_2), g)},$$

or, by the defenition of Q_j and (28),

$$\bar{k} = \frac{Q_1 - Q_2}{(r(u_2), g)} - \frac{k_1(r(u_1) - r(u_2), g)}{(r(u_2), g)} = \frac{\mu_1 - \mu_2}{(r(u_2), g)} - \frac{k_1(r(u_1) - r(u_2), g)}{(r(u_2), g)}. \qquad (29)$$

Let us estimate the last term of the right side of the resulting relation by absolute value, taking into account (8), (17), (28), assumption V and the inequality [2]

$$(r(u_2), g) \geqslant (r(d), g) + (r(d_2^\tau) - r(d), g) \geqslant (r(d), g) - \tau_2\frac{C_r^{p/(p-1)}c_0^p}{m_0}c(\|d\|_{L^{2p-2}})\|d\|_1^{p-1}\|g\|_1.$$

Without loss of generality one may suggest that $k_1 \leqslant k_2$. We have

$$\Big|\frac{k_1(r(u_1)-r(u_2),g)}{(r(u_2),g)}\Big| = Q_2 \cdot \frac{|(r(u_1)-r(u_2),g)|}{(r(u_2),g)^2} \leqslant$$

$$\leqslant \frac{m_0(r(d),g)^2}{4C_r^{p/(p-1)}\, c_0^p\Psi} \cdot \frac{c(\|d\|_{L^{2p-2}})\|u_1-u_2\|_1\|g\|}{((r(d),g)-\tau_2 C_r^{p/(p-1)} c_0^p m_1{}^{-1} c(\|d\|_{L^{2p-2}}(\Omega))\|d\|_1^{p-1}\|g\|_1)^2} =$$

$$= \frac{m_0(r(d),g)^2}{C_r^{p/(p-1)} c_0^p \|d\|_1^{p-1}((r(d),g)+\sqrt{G_2})^2} \cdot \|u_1-u_2\|_1. \tag{30}$$

On the other hand, difference $\{\bar u, \bar k\}$ satisfies the equation

$$M\bar u + k_1(r(u_1)-r(u_2)) = (k_2-k_1)r(u_2) \tag{31}$$

and the boundary condition

$$\bar u|_{\partial\Omega} = 0. \tag{32}$$

Then multiplying (31) by $\bar u$ in terms of the inner product in $L^2(\Omega)$ and integrating by parts in the first term with regard to (32) give

$$\langle M\bar u, \bar u\rangle_1 + k_1(r(u_1)-r(u_2),\bar u) = (k_2-k_1)(r(u_2),\bar u). \tag{33}$$

We estimate the right-hand side of (33) by the absolute value using the embedding theorem $W_2^1(\Omega)$ in $L^p(\Omega)$ and (8).

$$|(k_2-k_1)(r(u_2),\bar u)| \leqslant C_r^{p/(p-1)}|k_2-k_1|\|u_2\|^{p-1}_{L^p(\Omega)}\|\bar u\|_{L^p(\Omega)} \leqslant$$

$$\leqslant C_r^{p/(p-1)} c_0^p\, |k_2-k_1|\|d\|_1^{p-1}\|\bar u\|_1 \leqslant \frac{C_r^{2p/(p-1)} c_0^{2p}}{2m_0}|k_2-k_1|^2\|d\|_1^{2p-2} + \frac{m_1}{2}\|\bar u\|_1^2.$$

By the assumptions I – V, the equalty (33) and the last relation lead us to the inequality

$$\|\bar u\|_1 \leqslant \frac{C_r^{p/(p-1)} c_0^p}{m_1}\|d\|_1^{p-1}|k_2-k_1|. \tag{34}$$

Combining (29), (30) and (33) we obtain the estimate

$$|\bar k| \leqslant \frac{|\mu_1-\mu_2|}{(r(d),g)+\sqrt{G_2}} + \frac{(r(d),g)^2}{((r(d),g)+\sqrt{G_2})^2}|\bar k|,$$

which implies that

$$|\bar k| \leqslant \frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2|. \tag{35}$$

Inequalities (34) and (35) give us an estimates

$$\|\bar u\|_1 \leqslant \frac{C_r^{p/(p-1)} c_0^p}{m_1}\|d\|_1^{p-1}\frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2|$$

and in view of IV

$$\|r(u_1)-r(u_2)\| \leqslant c(\|d\|_{L^{2p-2}(\Omega)})\|u_1-u_2\| \leqslant$$

$$\leqslant \frac{C_r^{p/(p-1)} c_0^p}{m_0}\|d\|_1^{p-1} c(\|d\|_{L^{2p-2}(\Omega)})\frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2| \equiv C_3|\mu_1-\mu_2|.$$

We now multiply (31) by $M\bar{u}$ in terms of inner product in $L_2\Omega$.

$$\|M\bar{u}\|^2 = (k_2 - k_1)(r(u_2), M\bar{u}) - k_1(r(u_1) - r(u_2), M\bar{u}).$$

We estimate the right hand side of the last relation taking into account (35). This gives

$$|(k_2 - k_1)(r(u_2), M\bar{u}) - k_1(r(u_1) - r(u_2), M\bar{u})| \leqslant \frac{1}{2}(\tau_1 C_3|\mu_1 - \mu_2| + |\bar{k}|\|r(u_2)\|)^2 + \frac{1}{2}\|M\bar{u}\|^2 \leqslant$$

$$\leqslant \frac{1}{2}\left(\tau_1 C_3 + \frac{r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})}\right)^2 |\mu_1 - \mu_2|,$$

whence, due to the inequality (22), we obtain the estimate

$$\|\bar{u}\|_2 \leqslant C_M\left(\tau_1 C_3 + \frac{(r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})} + \frac{C_r^{p/(p-1)} c_0^p}{m_0}\|d\|_1^{p-1}\frac{(r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})}\right)|\mu_1 - \mu_2|.$$

Theorem is proved. □

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Об устойчивости решений некоторых обратных задач для эллиптических уравнений

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Аннотация. В работе рассматриваются обратные задачи отыскания неизвестного младшего коэффициента в линейном и нелинейном эллиптических уравнениях второго порядка с интегральными условиями переопределения на границе исследуемой области. Для линейного уравнения доказана непрерывная зависимость сильного решения обратной задачи от ее исходных данных в случае смешанного граничного условия. Для нелинейного уравнения установлена непрерывная зависимость сильного решения обратной задачи с граничным условием первого рода от данных переопределения.

Ключевые слова: обратная задача, эллиптическое уравнение, интегральное переопределение, непрерывная зависимость от входных данных.

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Collatz Hypothesis and Planck's Black Body Radiation

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Abstract. The Collatz conjecture is considered and the density of values is compared to Planck's black body radiation, showing a remarkable agreement with each other. We also briefly discuss a generalisation of Collatz conjecture.

Keywords: Collatz conjecture, Black body radiation, Collatz conjecture generalisation.

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1. The Collatz conjecture

The Collatz conjecture starts from a very simple function. For $N \in \mathbb{N}$ define $C(N)$ as

$$C(N) = \begin{cases} \frac{N}{2} & \text{if } N \text{ even,} \\ 3N+1 & \text{if } N \text{ odd.} \end{cases} \tag{1}$$

This function applied recursively creates a sequence. Translating $C(N)$ to a sequence $\{a_i\}_{i\in\mathbb{N}}$, applying recursively the operation starting from a positive integer N one could write a_i as follows:

$$a_i = \begin{cases} N & \text{for } i = 0 \\ C(a_{i-1}) & \text{for } i > 0 \,, \end{cases} \tag{2}$$

so that $a_i = [C(N)]^i$.

For instance, starting from $N = 7$, the obtained sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 \,,$$

while starting from $N = 1236$ the sequence is

$$1236, 618, 309, 928, 464, 232, 116, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40,$$
$$20, 10, 5, 16, 8, 4, 2, 1 \,.$$

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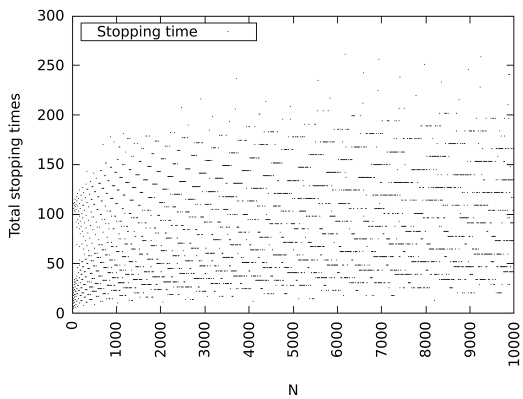
The sequence is concluded when it reaches the number 1. For the examples above, for 7 the sequence is concluded after 17 steps, while 27 steps are needed for 1236. Clearly, for a number of the type $N = 2^k$, $k \in \mathbb{N}$, 1 is reached after k steps. The number of steps necessary for reaching 1 is called total stopping time.

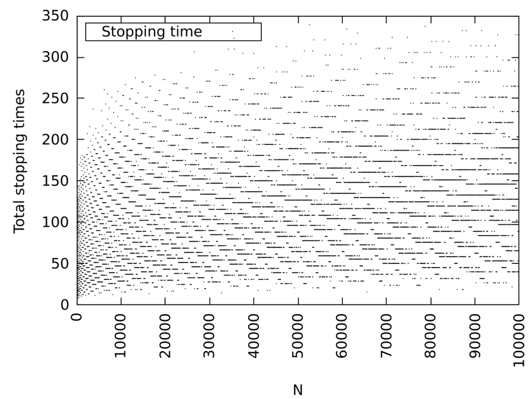
The conjecture of Collatz (1937) [1] states that function (1) starting from any natural number has a finite stopping time. Until today the conjecture has not been neither proved nor disproved.

There is plenty of literature on the subject, see for instance [2–6].

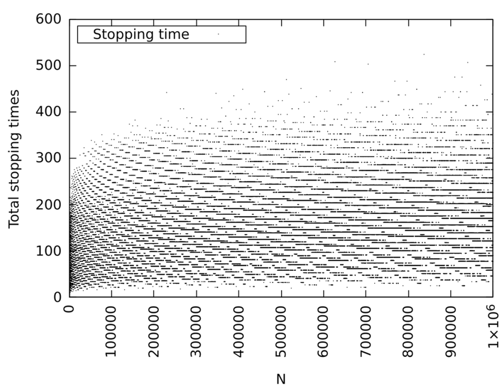
We are here concerned with the problem of total stopping times and its distribution. In Fig. 1 we have shown total stopping times for different starting values of N.



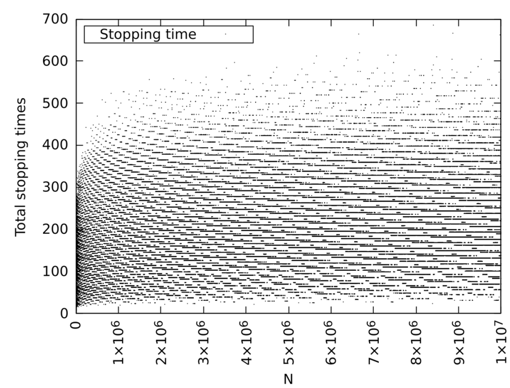
(a) $N = 10^4$



(b) $N = 10^5$



(c) $N = 10^6$



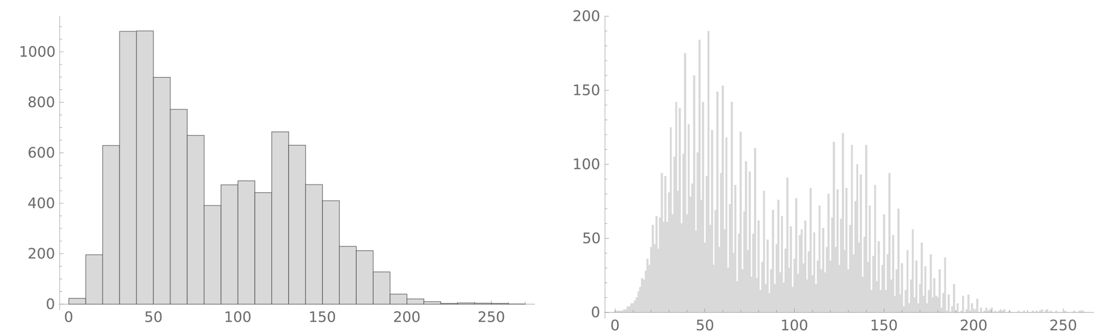
(d) $N = 10^7$

Fig. 1. Total stopping times for different starting N

A different kind of plot for total stopping times is shown in Fig. 2. For fixed N, total stopping times are shown with respect to frequencies. The latter results suggest a parallel to a well-known problem in physics.

2. Planck's radiation

Suppose to have electromagnetic radiation (that is, photons) at equilibrium inside a cavity of volume V at a temperature T. This system is known as "blackbody cavity". As is well-known, the free electromagnetic field can be written as a sum of harmonic oscillators at a fixed frequency ν. From the quantum theory each oscillator, that is each photon of frequency ν can only have



(a) Bin width=10

(b) Bin width=1

Fig. 2. Histograms for $N = 10^4$ of total stopping times with respect to frequencies for different bin widths

energies $\left(n' + \frac{1}{2}\right)h\nu$, for $n' \in \mathbb{N}$. The partition function of this system at the temperature T, $\beta = 1/k_B T$ can be written as

$$Z = \sum_{n'=0}^{+\infty} \exp(-\beta h\nu n') = \frac{1}{1 - \exp(-\beta h\nu)} \ . \tag{3}$$

The average energy for a photon of frequency ν is given by

$$\langle E\rangle = -\frac{d}{d\beta} \ln Z = \frac{h\nu \exp(-\beta h\nu)}{1 - \exp(-\beta h\nu)} = \frac{h\nu}{\exp(\beta h\nu) - 1} = \frac{h\nu}{\exp[h\nu/(k_B T)] - 1} \ . \tag{4}$$

The photon of frequency ν has a momentum $\vec{p} = (h/2\pi)\vec{k}$, and $k = |\vec{k}| = (2\pi/c)\nu$. For a volume V the number of momenta between k and $k + dk$ is given by

$$\frac{V}{(2\pi)^3} 8\pi k^2 dk = 8\pi \frac{V}{c^3} \nu^2 d\nu \ .$$

The internal energy U of the system is given by

$$U = 8\pi \frac{V}{c^3} \int_0^{+\infty} \langle E\rangle \nu^2 d\nu = 8\pi \frac{V}{c^3} \int_0^{+\infty} \frac{h\nu^3}{\exp[h\nu/(k_B T)] - 1} d\nu \ ,$$

so that the internal energy per unit volume is

$$\frac{U}{V} = \int_0^{+\infty} u(\nu, T) d\nu \ , \tag{5}$$

with

$$u(\nu, T) = 8\pi \left(\frac{h}{c^3}\right) \frac{\nu^3}{\exp[h\nu/(k_B T)] - 1} \ , \tag{6}$$

this is famous Planck's radiation law of photon energy density at frequency ν [7]. Performing integral (5) one observes that U/V increases with temperature as T^4, obtaining the Stefan–Boltzmann law for the power radiated from a black body.

Planck's function (6) has, remarkably, the same behaviour as the total stopping times with respect to frequencies shown in Fig. 2.

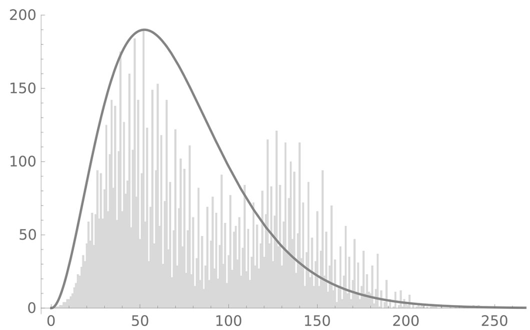
In Tab. 1 we have shown the equivalence between Collatz distribution of total stopping times and Planck's radiation density.

Table 1. Translation table Collatz–Planck

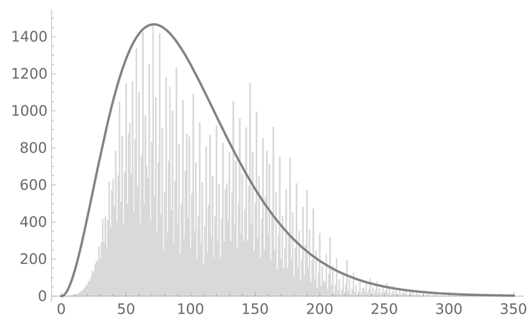
Collatz	$\longleftrightarrow$	Planck
Frequency	$\longleftrightarrow$	Photon frequency
Total stopping time	$\longleftrightarrow$	Black body radiation density

The results of the comparison of total stopping times and Planck's radiation density are shown in Fig. 3 for different values of starting N.

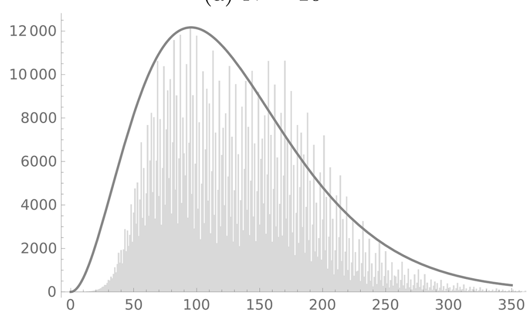
One could observe that the agreement between the two functions increases with increasing N, and also that Planck's radiation overestimates a little the decrease of stopping times with respect to N.



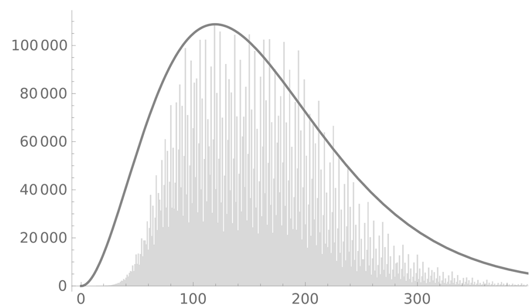
(a) $N = 10^4$



(b) $N = 10^5$



(c) $N = 10^6$



(d) $N = 10^7$

Fig. 3. Histograms: Total stopping times with respect to frequencies; Functions: Planck's black body radiation density with respect to photon frequency

From those results, formula (6), and with the aid of Tab. 1 we could infer that total stopping time goes to zero with increasing photon frequency ν at least as

$$\text{Total stopping time} \sim \nu^3 \exp(-\nu) \ . \tag{7}$$

Fig. 4 show the scaling of total stopping times with respect to the logarithm of starting N, $n = \log(N)(\approx 2.3 \ln(N))$, and the scaling of temperature T of Planck's radiation density with respect to n, respectively. The temperature is related to the longest stopping time, that is the peak of graphs in Fig. 3. Fig. 4a shows a fairly linear scaling of total stopping times with the logarithm of starting N, that is

$$\text{Total stopping times} \sim \log(N) \ . \tag{8}$$

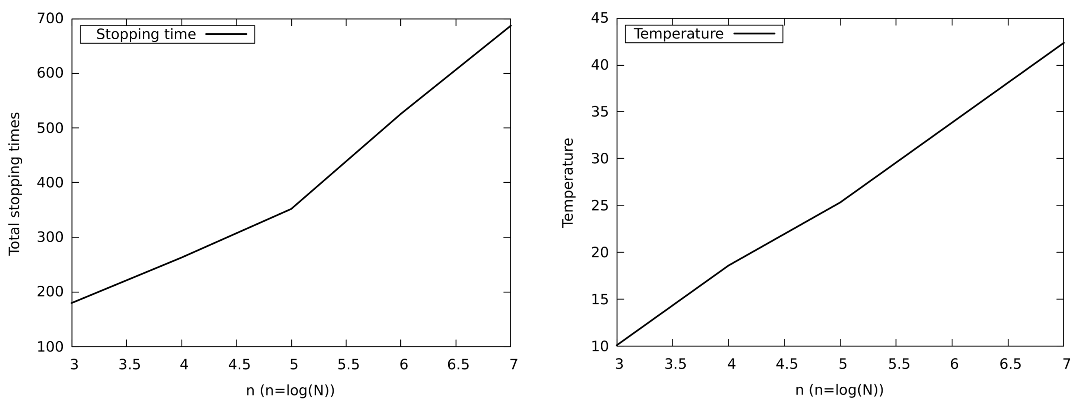
Fig. 4b shows an almost perfect linear scaling for temperatures, or maximum values of total stopping times with the logarithm of starting N:

$$\text{Temperatures} \sim \log(N) \; . \tag{9}$$

From Stefan–Boltzmann's law we could also deduce that the sum of total stopping times scales with the fourth power of $\log(N)$:

$$\sum (\text{Total stopping times}) \sim (\log(N))^4 \; . \tag{10}$$

All these results, although they are not proofs, show that for a finite N total stoppings times are finite, and their sum is finite as well, thus supporting Collatz's conjecture.



(a) Total stopping times

(b) Temperatures

Fig. 4. (a): Scaling of total stopping times with respect to starting value of n, $n = \log(N)$; (b): Scaling of temperatures with respect to starting value of n, $n = \log(N)$

3. Collatz generalisation

Define the generalisation of Collatz $C(q, N)$ as

$$C(q,N) = \begin{cases} \dfrac{N}{2} & \text{if } N \text{ even,} \\ qN+1 & \text{if } N \text{ odd.} \end{cases} \tag{11}$$

wich has fixed points $x = 0$ and $x = -1/(q-1)$ respectively. For even N behaves as usual, so we have to focus only on odd N. Let q be even, $q = 2j$. For odd $N = 2m + 1$ one has

$$2j(2m+1) + 1 = 2[j(2m+1)] + 1$$

which is again odd and grows without bounds. Therefore, q has to be odd, $q = 2j + 1$.

For $N = 2m + 1$ we obtain

$$(2j+1)(2m+1) + 1 = 2[(2j+1)m + (j+1)] \tag{12}$$

which is even. Using the notation $\rightarrow$ as "transforms to" we obtain Tab. 2. Actually the original Collatz function loops on the sequence $4, 2, 1$ for any N, so one could say that the conjecture is verified when any $N \in \mathbb{N}$ enters this loop.

For $q > 3$ the situation is different, probably due to the fact that observing Tab. (2), and (12), the ratio

$$\frac{2j+1}{j+1}$$

is larger than 3/2 so the sequence reaches fewer odd numbers with respect to even numbers.

Table 2. Transformation table for Collatz generalisation

j	q	$(2m+1) \rightarrow$
1	3	$2(3m+2)$
2	5	$2(5m+3)$
3	7	$2(7m+4)$
4	9	$2(9m+5)$

In the case $q = 5$ we have various different loops than the usual $4, 2, 1$.

For $N = 5$:

$$5, 26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, \ldots$$

For $N = 13$:

$$13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, 66, \ldots$$

For $N = 15$:

$$15, 76, 38, 19, 96, 48, 24, 12, 6, 3, 16, 8, 4, 2, 1, 6, 3, 16, 8, 4, \ldots$$

For $N = 17$:

$$17, 86, 43, 216, 108, 54, 27, 136, 68, 34, 17, 86, \ldots$$

While for other values $C(q, N)$ diverges, like $N = 7, 9, 11, 14, 18, \ldots$.

For $q > 5$ the situation worsens drastically, and $C(q, N)$ diverges already for $N = 5$, $C(q, 5)$, for values of $q = 9, 11, 13, 15, 17, 19, 21, \ldots, 211, \ldots$.

One could conjecture that only the Collatz case $q = 3$ has no divergencies and an unique loop, $4, 2, 1$, while for $q > 3$ there exist more loops separated by some divergent points.

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Гипотеза Коллатца и излучение черного тела Планка

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Аннотация. Рассматривается гипотеза Коллатца, и плотность значений сравнивается с излучением черного тела Планка, демонстрируя замечательное согласие друг с другом. Мы также кратко обсудим обобщение гипотезы Коллатца.

Ключевые слова: гипотеза Коллатца, излучение черного тела, обобщение гипотезы Коллатца.

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The Impact of Inclination Angle and Thermal Load on Flow Patterns in a Bilayer System Taking into Account the Mass Transfer

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Abstract. Bilayer convective flows of liquid and gas-vapour mixture in an inclined channel are modelled when heat and mass transfer at the thermocapillary interface is taken into account. Mathematical modelling is based on the exact solutions of special type of the Navier-Stokes equations in the Oberbeck-Boussinesq approximation with the Soret and Dufour effects in the gas-vapour layer. Inclined or horizontal position of a channel and direction of the boundary thermal load determine a form of exact solution and algorithm of its construction. Examples of velocity profiles, temperature and vapour concentration distributions in the «ethanol — nitrogen» system are presented. Results of comparative analysis of the two-layer flow in the system positioned horizontally and by small inclination from the horizontal level are also presented. The influence of the thermal load intensity on the flow and mass transfer characteristics is studied.

Keywords: exact solution, bilayer flow, thermocapillary interface, convection, mass transfer, inclined channel.

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Introduction

Convective flows of liquids are often observed in the natural and technological processes. One of the main mechanisms that determines the character of such flows is the heat and mass transfer. In addition, the reciprocal effects of the diffusive thermal conductivity and influence of thermodiffusion on fluid flows induced both by temperature and concentration inhomogeneities [1].

The interest in construction of exact solutions describing convective flows with interfaces is motivated by possibility to analyse the influence of various parameters of the problem on the processe. This allows one to identify the dominant mechanisms that influence the flow topology, to improve experimental techniques, and to predict the results of experimental study of dynamics of liquids and co-current gases [2]. Because problems of convective heat and mass transfer are

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generally non-linear construction of their exact solutions can be rather interesting problem from the mathematical point of view.

Nowadays, there are quite many works devoted to the construction of exact solutions that describe flows of thermally conductive fluids. Special solutions for one-directional flows were proposed [3, 4]. The group nature of solutions obtained in [4] was revealed in [5]. Generalization of the proposed solutions for the non-stationary case in a plane layer and in rotating tube was proposed [6]. The example of exact solution describing flow of one-directional binary fluid was proposed in [7].

Due to non-linearity of the system of the Oberbeck-Bussinesq equations, the introduction of additional effects or parameters complicates the process of problem solving. Construction of exact solutions in such case is connected with generalization of known solutions. Construction of exact solutions describing the two-layer flows in a «liquid-liquid» system subject to mass transfer in a horizontal channel was presented [8]. The process of liquid vaporization into the gas-vapour layer under closed flow conditions was considered in [9]. Modelling of the bilayer flows in relation to given gas flow rate in the upper layer was carried out in [10]. The diffusive thermal conductivity effect in the gas-vapour layer and thermodiffusion process were taken into account in [11]. The stability issues of the presented exact solutions were studied [12, 13]. Comparison of analytical results concerning liquid evaporation into gas-vapour layer with experimental data was presented in [11].

In addition, geometry of a flow domain complicates construction of exact solutions of the Navier-Stokes equations in the Oberbeck–Boussinesq approximation. Mathematical modelling in horizontal, vertical and inclined layers was carried out [14]. A variant of such solution was proposed for the problem of fluid flow in an inclined channel with moving solid boundaries on which longitudinal temperature gradient is given [15]. Bilayer «liquid-gas» systems with evaporation at the thermocapillary interface and given inclination angle of the channel were considered [16]. Here, it was assumed that condition of total vapour absorption is satisfied on the upper channel wall.

This paper presents exact solutions of special type of the Navier–Stokes equations in the Boussinesq approximation. They describe flows in an inclined channel filled by liquid and gas-vapour mixture. It is assumed that thermocapillary interface is non-deformable (see [12, 17]). The Soret and Dufour effects are taken into account in the upper layer of the system, and the gas flow rate is given. The condition of zero vapour flux is chosen on the upper channel wall. The obtained solution is compared with the solution for the horizontal layer [11, 18]. The impact of physical and geometrical parameters of the problem on the flow patterns is studied.

1. Problem statement of convection in an inclined channel in the case of non-deformable interface

1.1. Governing equations

The stationary flow in «liquid–gas» system in an inclined layer with solid impermeable walls is studied. A viscous incompressible liquid and a bi-component mixture of gas and vapour occupy an infinite channel. The thicknesses of the layers are fixed and equal to l and h, respectively (see Fig. 1). The Cartesian coordinate system is positioned in such a way that interface which remains non-deformable is determined by the equation $y = 0$. The mass force vector $\mathbf{g}$ is directed at the angle φ with respect to the substrate ($\mathbf{g} = (g\cos\varphi, -g\sin\varphi)$). Note that inclination angle from the horizontal plane is defined as $(\pi/2 - \varphi)$. Vapour is a passive impurity in the gas phase, i.e., it does not affect the properties of the gas.

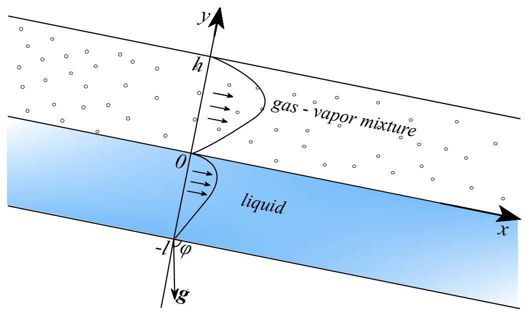

Fig. 1. Flow area geometry

Mathematical modelling of the liquid and gas-vapour mixture flows is carried out by using the Navier–Stokes equations in the Oberbeck–Boussinesq approximation. The vapour concentration function Φ satisfies the diffusion equation. The Soret and Dufour effects are taken into account in the upper layer.

The exact solutions of the governing equations are constructed in the special Ostroumov-Birikh form [3, 4] (see [8]):

$$u_i = u_i(y), \quad v_i = 0, \quad T_i = Ax + \vartheta_i(y), \quad \Phi = -Bx + \psi(y), \quad p_i' = p_i'(x, y), \tag{1}$$

where u_i and v_i are projections of the velocity vector on the Cartesian coordinate system axes, p_i' is modified pressure or deviation from the hydrostatic pressure ($p_i' = p_i - \rho_i\, \mathbf{g} \cdot \mathbf{x}$, $\mathbf{x} = (x, y)$, p_i — pressure, ρ_i — density), T_i — temperature, Φ — vapour concentration in the gas, A, B — parameters defining longitudinal gradients of temperature and vapour concentration, ϑ_i, ψ — terms included in the definition of functions T_i and Φ, and they depend only on the longitudinal coordinate. Hereinafter, i defines the number of the system layer: functions and parameters describing the fluid flow are marked by $i = 1$, gas-vapour mixture by $i = 2$. It should be noted that Birikh solution which is a special case of solution (1) has experimental and numerical confirmation (see, for example, [19]). In addition, the results obtained using solution (1) are confirmed by experimental data in [2, 11].

The system of equations describing flows in the lower layer filled with one-component fluid is written as follows (1):

$$\frac{1}{\rho_1} p'_{1\,x} = \nu_1 u_{1\,yy} - g \cos\varphi \beta_1 T_1, \qquad \frac{1}{\rho_1} p'_{1\,y} = g \sin\varphi \beta_1 T_1, \qquad u_1 T_{1x} = \chi_1 T_{1\,yy}. \tag{2}$$

In the upper layer containing gas and vapour mixture the system of governing equations should be supplemented by the diffusion equation for the vapour concentration function:

$$\begin{gathered} \frac{1}{\rho_2} p'_{2\,x} = \nu_2 u_{2\,yy} - g \cos\varphi(\beta_2 T_2 + \gamma\Phi), \qquad \frac{1}{\rho_2} p'_{2\,y} = g \sin\varphi(\beta_2 T_2 + \gamma\Phi), \\ u_2 T_{2x} = \chi_2 T_{2\,yy} + \chi_2 \delta \Phi_{yy}, \qquad u_2 \Phi_x = D\Phi_{yy} + \alpha D T_{2\,yy}. \end{gathered} \tag{3}$$

In equations (2), (3) the following notations are used: ν_i — kinematic viscosity coefficients, β_i — thermal expansion coefficients, χ_i — heat diffusivity coefficients, γ — concentration density coefficient, D — vapour diffusion coefficient in the gas, coefficients α and δ characterize the Soret and Dufour effects, respectively.

1.2. Boundary conditions

Let us formulate the conditions on the boundaries of the system. The no-slip condition for velocity should be fulfilled on solid walls of the channel:

$$u_1|_{y=-l} = 0, \quad u_2|_{y=h} = 0. \tag{4}$$

The linear thermal regime is set on channel walls:

$$T_1|_{y=-l} = Ax + \vartheta^-, \quad T_2|_{y=h} = Ax + \vartheta^+. \tag{5}$$

Here ϑ^- and ϑ^+ are some fixed constants.

Vapour concentration function Φ satisfies the condition of zero vapour flux at the boundary $y = h$:

$$(\Phi_y + \alpha T_{2\,y})|_{y=h} = 0. \tag{6}$$

Conditions of continuity of the velocity and temperature functions are satisfied at the thermocapillary interface $y = 0$:

$$u_1|_{y=0} = u_2|_{y=0}, \quad T_1|_{y=0} = T_2|_{y=0}. \tag{7}$$

The kinematic condition ($v_1 = 0$ and $v_2 = 0$) is satisfied automatically by virtue of exact solution (1). The projection of the dynamic condition onto the tangent vector to the interface is written as follows

$$\rho_1\nu_1 u_{1\,y} = \rho_2\nu_2 u_{2\,y} + \sigma_T T_{1\,x}|_{y=0}, \tag{8}$$

where σ_T is the temperature coefficient of surface tension σ. The linear relationship between surface tension and temperature is assumed: $\sigma = \sigma_0 + \sigma_T(T - T_0)$, σ_0 — surface tension at some reference temperature T_0, $\sigma_T = const$, $\sigma_T < 0$. The dynamic condition expresses the tangential stress balance at the interface.

Taking into account the diffusion mass flux of vaporizing liquid at the interface M and the diffusive thermal conductivity effect, the heat flux balance takes the form [10, 11, 20]

$$\kappa_1 T_{1\,y} - \kappa_2 T_{2\,y} - \delta\kappa_2\Phi_y|_{y=0} = -LM, \quad M = -D\rho_2(\Phi_y|_{y=0} + \alpha T_{2\,y}|_{y=0}). \tag{9}$$

Here the following notations are accepted: L is the latent heat of evaporation, M is the mass velocity of liquid evaporating from a unit surface area per unit time ($M = const$), κ_1 and κ_2 are thermal conductivity coefficients of liquid and gas-vapour mixture, respectively.

The saturated vapour concentration at the interface is determined according to the following relation (see [11])

$$\Phi|_{y=0} = \Phi_*(1 + \varepsilon(T_2|_{y=0} - T_0)), \tag{10}$$

where ε is a parameter depending on the characteristic temperature and physical and chemical properties of the medium, Φ_* is the concentration of saturated vapour at $T_2 = T_0$.

The problem is solved under the condition of given liquid flow rate Q_1 and gas flow rate Q_2.

$$Q_1 = \int_{-l}^{0} \rho_1 u_1(y)dy, \quad Q_2 = \int_0^h \rho_2 u_2(y)dy. \tag{11}$$

2. Constructing exact solution of a special type

The exact solution of the problem is constructed by substituting (1) into differential equations (2), (3). Functions p_i' are eliminated by cross differentiation of the first two relations in (2), (3). Performing further subsequent differentiation of the obtained expressions with respect to variable y, one can obtain $u_{1\,y}^{(4)} + \lambda_1 u_1 = 0$ and $u_{2\,y}^{(4)} + \lambda_2 u_2 = 0$. The solutions of obtained equations are functions u_1 and u_2 which determine the longitudinal velocities in each layer of the system. Coefficients λ_1 and λ_2 depend on geometric, physical and chemical parameters of the media. They have the following form: $\lambda_1 = -Ag\cos\varphi\beta_1/(\chi_1\nu_1)$, $\lambda_2 = -Ag\cos\varphi E$, where $E = [D(\beta_2 - \alpha\gamma) - \chi_2 C_*\varepsilon(\delta\beta_2 - \gamma)]/[\chi_2\nu_2 D(1-\alpha\delta)]$. Function ϑ_1 is found by integration from the heat transfer equation (see the third expression in (2)). Similarly, functions ϑ_2 and ψ are recovered from the heat transfer and diffusion equations (see the third and fourth expressions in (3)). The inequalities $\lambda_1 < 0$ and $\lambda_2 > 0$ when $A > 0$, and $\lambda_1 > 0$, $\lambda_2 < 0$ when $A < 0$ are true for systems of «ethanol — nitrogen» type. Parameter A determines the longitudinal temperature gradient. In the first case, the desired functions are represented as the following analytic expressions [22] (see also [21]):

$$\begin{aligned}
&u_1 = C_1\sin(k_1y) + C_2\cos(k_1y) + C_3\,\mathrm{sh}(k_1y) + C_4\,\mathrm{ch}(k_1y),\\
&u_2 = \overline{C}_1\sin(m_1y)\,\mathrm{sh}(m_1y) + \overline{C}_2\cos(m_1y)\,\mathrm{sh}(m_1y) + \overline{C}_3\sin(m_1y)\,\mathrm{ch}(m_1y) +\\
&\qquad + \overline{C}_4\cos(m_1y)\,\mathrm{ch}(m_1y),\\
&T_1(x,y) = Ax + \frac{F_1}{k_1^2}\Big(-C_1\sin(k_1y) - C_2\cos(k_1y) + C_3\,\mathrm{sh}(k_1y) + C_4\,\mathrm{ch}(k_1y)\Big) + C_5y + C_6,\\
&T_2(x,y) = Ax + \frac{F_2}{2m_1^2}\Big(-\overline{C}_1\cos(m_1y)\,\mathrm{ch}(m_1y) + \overline{C}_2\sin(m_1y)\,\mathrm{ch}(m_1y) -\\
&\qquad - \overline{C}_3\cos(m_1y)\,\mathrm{sh}(m_1y) + \overline{C}_4\sin(m_1y)\,\mathrm{sh}(m_1y)\Big) + \overline{C}_5y + \overline{C}_6,\\
&\Phi(x,y) = -Bx + \frac{G}{2m_1^2}\Big(-\overline{C}_1\cos(m_1y)\,\mathrm{ch}(m_1y) + \overline{C}_2\sin(m_1y)\,\mathrm{ch}(m_1y) -\\
&\qquad - \overline{C}_3\cos(m_1y)\,\mathrm{sh}(m_1y) + \overline{C}_4\sin(m_1y)\,\mathrm{sh}(m_1y)\Big) + \overline{C}_7y + \overline{C}_8.
\end{aligned} \tag{12}$$

When $A < 0$, functions describing the flow patterns take the form [22]

$$\begin{aligned}
&u_1(y) = C_1\sin(k_2y)\,\mathrm{sh}(k_2y) + C_2\cos(k_2y)\,\mathrm{sh}(k_2y) +\\
&\qquad + C_3\sin(k_2y)\,\mathrm{ch}(k_2y) + C_4\cos(k_2y)\,\mathrm{ch}(k_2y),\\
&u_2(y) = \overline{C}_1\sin(m_2y) + \overline{C}_2\cos(m_2y) + \overline{C}_3\,\mathrm{sh}(m_2y) + \overline{C}_4\,\mathrm{ch}(m_2y),\\
&T_1(x,y) = Ax + \frac{F_1}{2k_2^2}\Big(-C_1\cos(k_2y)\,\mathrm{ch}(k_2y) + C_2\sin(k_2y)\,\mathrm{ch}(k_2y) -\\
&\qquad - C_3\cos(k_2y)\,\mathrm{sh}(k_2y) + C_4\sin(k_2y)\,\mathrm{sh}(k_2y)\Big) + C_5y + C_6,\\
&T_2(x,y) = Ax + \frac{F_2}{m_2^2}\Big(-\overline{C}_1\sin(m_2y) - \overline{C}_2\cos(m_2y) + \overline{C}_3\,\mathrm{sh}(m_2y) + \overline{C}_4\,\mathrm{ch}(m_2y)\Big) +\\
&\qquad + \overline{C}_5y + \overline{C}_6,\\
&\Phi(x,y) = -Bx + \frac{G}{m_2^2}\Big(-\overline{C}_1\sin(m_2y) - \overline{C}_2\cos(m_2y) + \overline{C}_3\,\mathrm{sh}(m_2y) + \overline{C}_4\,\mathrm{ch}(m_2y)\Big) +\\
&\qquad + \overline{C}_7y + \overline{C}_8.
\end{aligned} \tag{13}$$

Coefficients k_s, m_s, F_1, F_2, G are calculated in terms of problem parameters: $k_1 = \sqrt[4]{Ag\cos\varphi\beta_1/(\chi_1\nu_1)}$, $k_2 = \sqrt[4]{-Ag\cos\varphi\beta_1/(4\chi_1\nu_1)}$, $m_1 = \sqrt[4]{Ag\cos\varphi E/4}$, $m_2 = \sqrt[4]{Ag\cos\varphi E}$, $F_1 = A/\chi_1$, $F_2 = A(D-\delta\chi_2 C_*\varepsilon)/[\chi_2 D(1-\alpha\delta)]$, $G = A(\alpha D - \chi_2 C_*\varepsilon)/[\chi_2 D(\alpha\delta - 1)]$. The index s defines solutions for $A > 0$ ($s = 1$) and $A < 0$ ($s = 2$). Coefficients C_i and $\overline{C}_i$ ($i = 1,\dots,8$) are different integration constants for each system (12), (13). These constants satisfy the system of algebraic equations resulting from conditions on the solid walls and interface (4)–(10) and expressions (11) that determine the gas and liquid flow rates (see [22]). The resulting system is non-closed. For closure, constants C_6 and $\overline{C}_6$ which are included as free terms in the temperature function can be assumed, for instance, to be zero. The pressure functions p_i' are recovered by their partial derivatives from the first two equations of systems (2), (3). Note that expression defining the saturated vapour concentration at the interface (10) defines the condition of compatibility of the problem parameters defining the longitudinal gradients of temperature and vapour concentration: $B = -\Phi_*\varepsilon A$.

Let us consider a special case when $\varphi = 90°$, i.e., the channel position becomes horizontal (see Fig. 1). Then components of the mass force vector are (0, -g). Exact solutions of special type were proposed [10, 11, 18], where temperature and vapour concentration functions are written as $T_i = (A+\bar{A}y)x + \vartheta_i(y)$, $\Phi = -(B+\bar{B}y)x + \psi(y)$. Taking into account (1) with $\bar{A} = \bar{B} = 0$, solutions given in [10, 11, 18] are written in the following simplified form

$$u_1 = \frac{g\beta_1}{\nu_1}\frac{y^3}{6}A + \frac{y^2}{2}c_1 + yc_2 + c_3, \quad u_2 = \frac{g}{\nu_2}\frac{y^3}{6}(\beta_2 A + \gamma B) + \bar{c}_1\frac{y^2}{2} + \bar{c}_2 y + \bar{c}_3,$$

$$T_1 = Ax + \frac{y^5}{120}\frac{g\beta_1(A)^2}{\nu_1\chi_1} + \frac{y^4}{24}\frac{c_1 A}{\chi_1} + \frac{y^3}{6}\frac{c_2 A}{\chi_1} + \frac{y^2}{2}\frac{c_3 A}{\chi_1} + yc_4 + c_5,$$

$$T_2 = Ax + + \frac{y^5}{120}\frac{g}{\nu_2}\Big(\frac{A}{\chi_2} - \delta\frac{B}{D}\Big)(\beta_2 A + \gamma B) + \frac{y^4}{24}\Big(\frac{A}{\chi_2} - \delta\frac{B}{D}\Big)\bar{c}_1 + $$
$$+ \frac{y^3}{6}\Big(\frac{A}{\chi_2} - \delta\frac{B}{D}\Big)\bar{c}_2 + \frac{y^2}{2}\Big(\frac{A}{\chi_2} - \delta\frac{B}{D}\Big)\bar{c}_3 + y\bar{c}_4 + \bar{c}_5, \tag{14}$$

$$\Phi = Bx + \frac{y^5}{120}\frac{g}{\nu_2}\frac{B}{D}(\beta_2 A + \gamma B) + \frac{y^4}{24}\frac{B}{D}\bar{c}_1 + \frac{y^3}{6}\frac{B}{D}\bar{c}_2 + \frac{y^2}{2}\frac{B}{D}\bar{c}_3 + y\bar{c}_6 + \bar{c}_7,$$

where c_i, $\bar{c}_i$ are integration constants determined by conditions at the boundaries of the system. The first relation (7) entails the equality $c_3 = \bar{c}_3$. The relationship between constants c_2 and $\bar{c}_2$ is determined from dynamic condition (8): $c_2 = (\rho_2\nu_2)/(\rho_1\nu_1) + (\sigma_T A)/(\rho_1\nu_1)$. Unknowns c_1, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ are found using the system of linear algebraic equations that follows from (4), (11). The equality of constants c_5, $\bar{c}_5$ follows from the second relation in (7). The expression $\bar{c}_7 = \Phi_* + \Phi_*\varepsilon(\bar{c}_5 - T_0)$ is obtained using condition (10). The mass of liquid evaporating from the interface is calculated using the relation $M = -D\rho_2(\bar{c}_6 + \alpha\bar{c}_4)$ (see mass balance equation (9)). Constant c_4 is determined from condition (9) as follows $c_4 = (LD\rho_2/\kappa_1 + \delta\kappa_2)\bar{c}_6 + (LD\rho_2\alpha/\kappa_1 + \kappa_2/\kappa_1)\bar{c}_4$. Unknowns $\bar{c}_4$, $\bar{c}_5$, $\bar{c}_6$ are found using the system of linear algebraic equations that follows from conditions (5) and (6).

3. Examples of flows

Let us consider ethanol as the liquid filling the lower layer and nitrogen as the gas. Chemical parameters of the medium are given in the order {ethanol, nitrogen} (or only ethanol) according to [23]: $\rho = \{7.89\cdot10^2, 1.2\}$ kg/m^3; $\nu = \{2\cdot10^{-6}, 0.15\cdot10^{-4}\}$ m^2/s; $\beta = \{1.079\cdot10^{-3}, 3.67\cdot10^{-3}\}$

K^{-1}; $\chi = \{8.9 \cdot 10^{-8},\ 0.3 \cdot 10^{-4}\}$ m^2/s; $\kappa = \{0.1705,\ 0.02717\}$ W/(m·K); $\sigma_T = -0.8 \cdot 10^{-2}$ N/(m·K); $D = 1.02 \cdot 10^{-4}$ m^2/s; $L = 217$ W·s/kg; $\Phi_* = 0.1$ (corresponding to the equilibrium temperature $T_0 = 293.15$ K); $\gamma = -0.62$; $\varepsilon_* = 0.06$ K^{-1}. The thicknesses of liquid and gas-vapour layers are assumed to be 5 mm. The value of gas flow rate in the upper layer of the system is $3.6 \cdot 10^{-5}$ kg/(m·s). The Soret and Dufour coefficients are assumed to be 10^{-4} K^{-1} and 10^{-4} K, respectively. The value Q_1 that determines the liquid flow rate in the lower layer is assumed to be equal to zero. This corresponds to the condition of the closed flow in the lower layer which is physically correct for small values of the inclination angle $(\pi/2 - \varphi)$ relative to the horizontal position of the channel.

3.1. The impact of channel inclination angle on the flow patterns

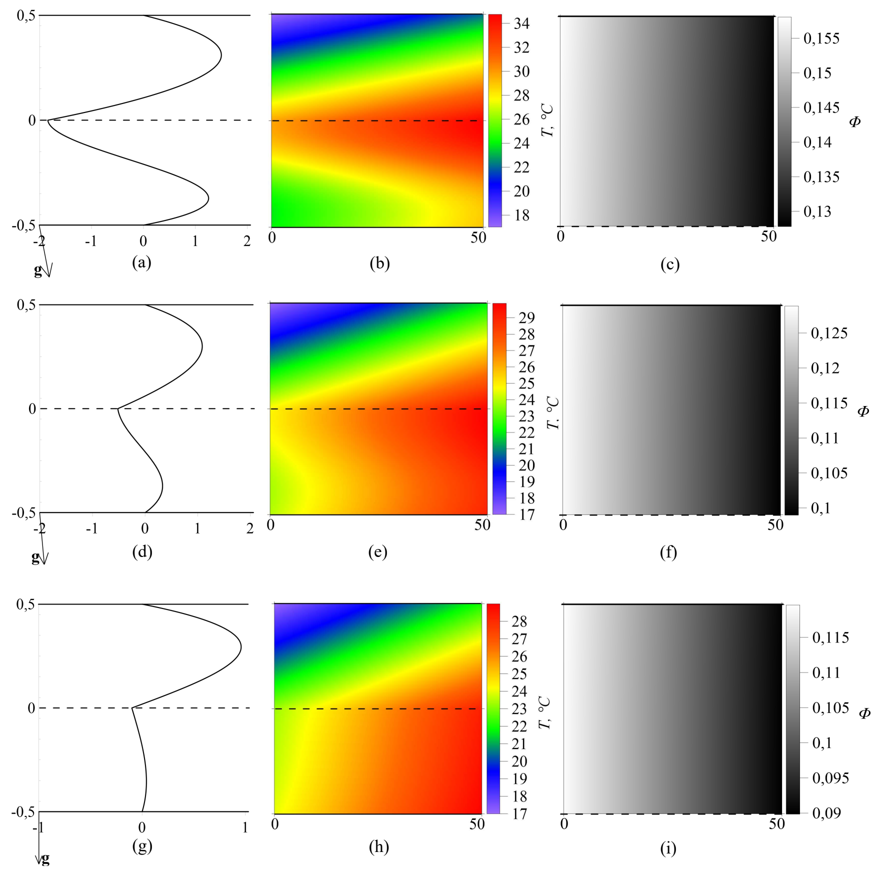

Fig. 2. Velocity profiles (a, d, g), temperature (b, e, h) and vapour concentration (c) distributions in the system: $A = 10$ K/m, $\vartheta^- = 24\ ^\circ C$, $\vartheta^+ = 17\ ^\circ C$, $\alpha = 10^{-4}$ K^{-1}, $\delta = 10^{-4}$ K, $g = 9.81$ m/s^2: (a,b,c) — $\varphi = 80^\circ$ $M = 2.124 \cdot 10^{-6}$ kg/(m^2·c), (d,e,f) — $\varphi = 85^\circ$ $M = 2.124 \cdot 10^{-6}$ kg/(m^2·c), (g,h) — $\varphi = 90^\circ$ $M = 2.124 \cdot 10^{-6}$ kg/(m^2·c)

Let us consider the impact of angle φ on the character of flow in the system as well as on the intensity of the liquid evaporation into the gas-vapour layer under conditions of normal gravity. Profiles of longitudinal velocity (a, d), temperature distribution (b, e), and vapour concentration (c, f) are presented in Fig. 2. Here, solutions (12) are used as a calculation model. For the case of horizontal layer the flow characteristics are based on formulas (14) (g, h, k). As the value of angle φ increases from 80° to 90° which corresponds to a decrease in the channel inclination angle with respect to the horizontal plane the intensity of the return flow near the interface decreases. Temperature distributions change both qualitatively and quantitatively. In the case when φ is 80° the formation of a thermocline near the interface is observed. When angle $(\pi/2-\varphi)$ decreases zone of the highest temperature moves to the lower wall of the channel. This is caused by its additional heating ($\vartheta^- = 24\ °C$). The vapour concentration distribution for various values of parameter φ qualitatively remains constant but there are some quantitative changes. As φ increases function Φ decreases. Note that liquid evaporation intensity does not depend on the channel inclination angle.

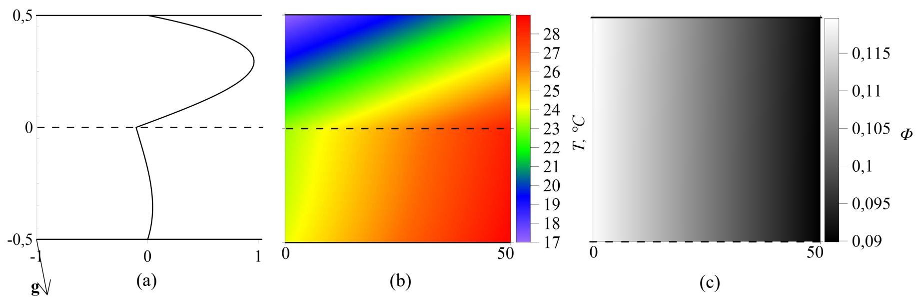

Fig. 3. Velocity profile (a), temperature (b) and vapour concentration (c) distributions in the system: $A = 10$ K/m, $\vartheta^- = 24\ °C$, $\vartheta^+ = 17\ °C$, $g = 9.81 \cdot 10^{-2}$ m/s^2: (a,b,c) – $\varphi = 80°$ $M = 2.124 \cdot 10^{-7}$ kg/(m^2·c)

Functions characterizing liquid and gas flows are weakly dependent on the channel inclination angle under microgravity conditions. The flow patterns both qualitatively and quantitatively remain close to those obtained using formulas (14) (see Fig. 3 and Fig. 2 (g, h, k)) when velocity profiles, temperature and vapour concentration distributions are obtained using exact solutions (12) that describe flows in the inclined layer. However, the mass flow rate of evaporation also decreases in the case of decreasing gravity.

3.2. The impact of thermal load on flow patterns

Figs. 4 and 5 show the results illustrating the impact of the longitudinal temperature gradient on velocity profiles, temperature and vapour concentration distributions and on the processes of liquid evaporation and condensation in an inclined channel. The angle φ here is 70°. Let us consider the case when $A > 0$ (Fig. 4). The growth of parameter A has only quantitative effect on the longitudinal velocity in the system while the qualitative character of the flow remains unchanged (Fig. 4 (a, d, g)). The temperature distribution changes significantly both qualitatively and quantitatively. In the case of small values of the longitudinal gradient A the highest values of temperature are observed at the lower wall of the channel. This is is a consequence of its heating. As parameter A increases the value $\vartheta^- = 27\ °C$ has less influence on the character of temperature distributions. The highest values of temperature are observed near the interface. This effect is accompanied by more intensive evaporation of the lower liquid and by increasing of vapour concentration. The character of distribution of Φ function remains unchanged.

In the case when longitudinal temperature gradient have negative value (the heater is located on the right side, see Fig. 5) some qualitative changes in the velocity profile appear. The intensity

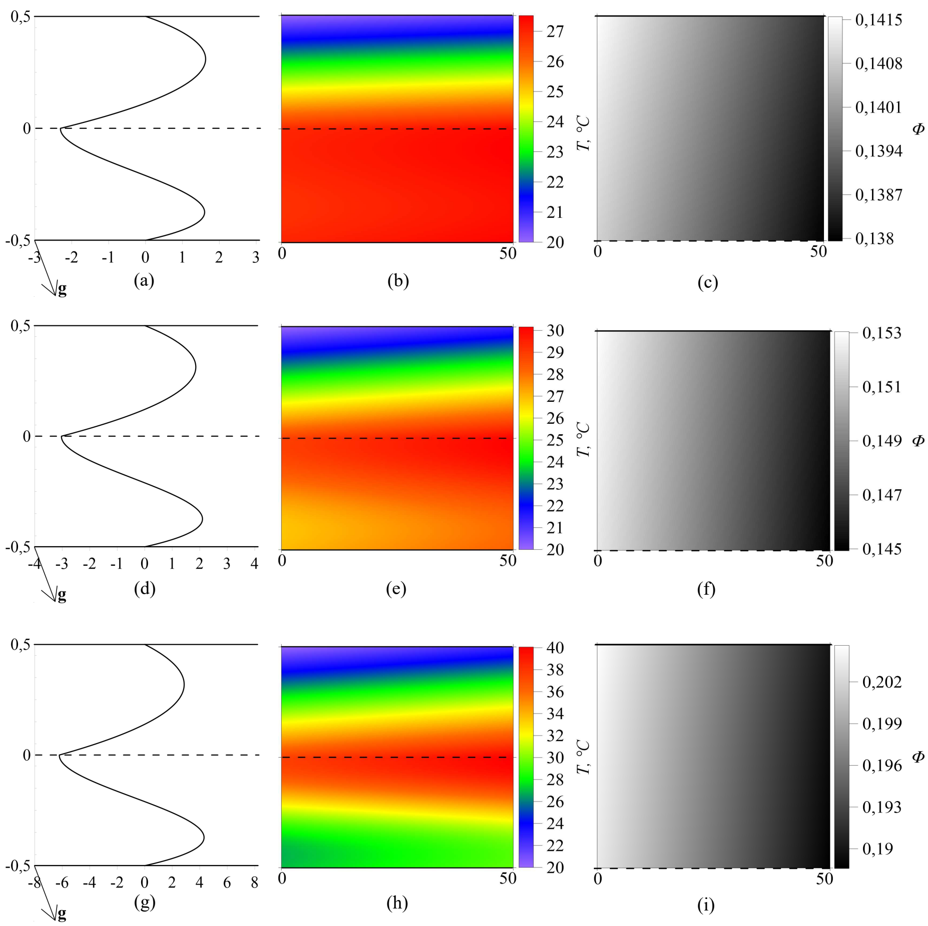

Fig. 4. Velocity profiles (a,d,g), temperature (b,e,h) and vapour concentration (c) distributions in the system: $\vartheta^- = 27\ °C$, $\vartheta^+ = 20\ °C$, $g = 9.81$ m/s^2, (a,b,c) — $A = 1$ K/m, $M = 0.212 \cdot 10^{-6}$ kg/(m^2·c), (d,e,f) — $A = 2.5$ K/m, $M = 0.531 \cdot 10^{-6}$ kg/(m^2·s), (g,h) — $A = 5$ K/m, $M = 1.062 \cdot 10^{-6}$ kg/(m^2·s))

of return current near the interface decreases with increasing modulus of A. The quantitative characteristics of the temperature distribution is changed insignificantly but the character of the function itself is changed. The highest values of temperature are observed near the lower wall of the channel for all A but the influence of heater increases. The distribution of vapour concentration is also changed qualitatively with respect to the case when $A > 0$. As the value of $|A|$ increases the drop in the value of function Φ also increases. Note that there is a process of condensation of liquid ($M < 0$) in this case.

Conclusions

The work presents exact solutions of a special type of Navier–Stokes equations in the Boussinesq approximation. Mathematical model of stationary bilayer convective flows subject to heat and mass transfer at a non-deformable thermocapillary interface is presented. The effects of thermodiffusion and diffusive thermal conductivity are considered in the gas-vapour layer. The

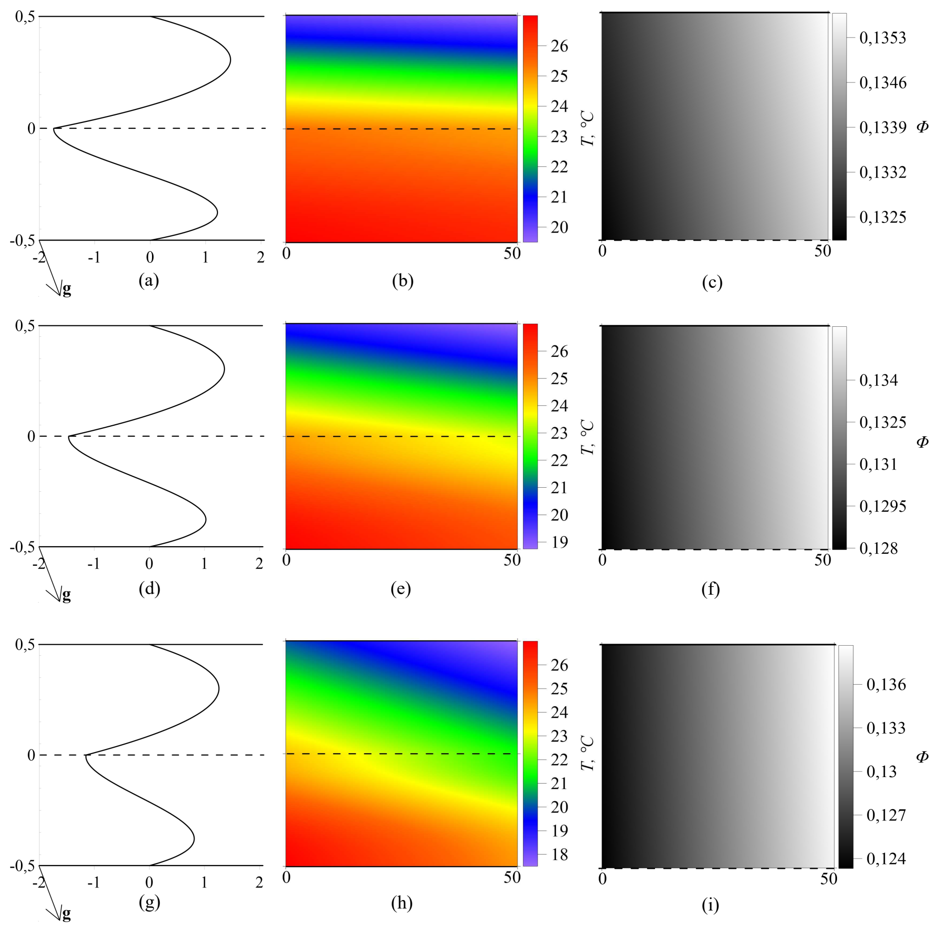

Fig. 5. Velocity profiles (a,d,g), temperature (b,e,h) and vapour concentration (c) distributions in the system: $\vartheta^- = 27\,°C$, $\vartheta^+ = 20\,°C$, $g = 9.81$ m/s^2, (a,b,c) — $A = -1$ K/m, $M = -0.212\cdot 10^{-6}$ kg/(m^2·c), (d,e,f) — $A = -2.5$ K/m, $M = -0.531 \cdot 10^{-6}$ kg/(m^2·c), (g,h) — $A = -5$ K/m, $M = -1.062 \cdot 10^{-6}$ kg/(m^2·c)

condition of zero vapour flux is set on the upper wall of the channel. Flows are considered in both inclined and horizontal channels. It is shown that in the case of an inclined layer the location of the heater on the system boundaries (the sign of the longitudinal temperature gradient) affects the type of the exact solution.

Examples of longitudinal velocity profiles, distributions of temperature and vapour concentration for the «ethanol — nitrogen» system are given. Comparison of results obtained by means of exact solutions describing the flow in inclined and horizontal layers is presented. The impact of the channel inclination angle, gravitation level and thermal load on the character of flows has been studied. It is shown that in the case of normal gravity an increase in the inclination angle of the system significantly changes the character of the flow while in conditions of microgravity such effect is not observed.

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Влияние угла наклона и тепловой нагрузки на характер течения в двухслойной системе с учетом массопереноса

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Аннотация. Изучаются двухслойные конвективные течения жидкости и парогазовой смеси в наклонном канале с учетом тепло- и массопереноса на термокапиллярной границе раздела. Математическое моделирование проводится на основе точных решений специального вида уравнений Навье–Стокса в приближении Обербека-Буссинеска с учетом эффектов Соре и Дюфура в газопаровом слое. Наклонное или горизонтальное положение канала, а также направление граничной тепловой нагрузки определяют вид точного решения и алгоритм его построения. Приведены примеры профилей скорости, распределения температуры и концентрации пара в системе «этанол — азот». Представлены результаты сравнительного анализа двухслойных течений в системе, расположенной горизонтально и под небольшим наклоном относительно горизонтального положения. Изучено влияние интенсивности тепловой нагрузки на характер течения и массоперенос на границе раздела.

Ключевые слова: точное решение, двухслойное течение, термокапиллярная граница раздела, конвекция, массоперенос, наклонный канал.

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Finding Power Sums of Zeros of an Entire Function of Finite Order of Growth

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Abstract. Formulas are given for finding power sums of zeros to a negative power for entire functions of finite order of growth.

Keywords: power sum of zeros, entire function of finite order of growth.

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Let the function $f(z)$

$$f(z) = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad b_0 = 1,$$

be an entire function of finite growth order and having zeros at $\alpha_1,\ \alpha_2,\ \ldots, \alpha_n, \ldots$ (each root is counted as many times as its multiplicity). There can be a finite or infinite number of zeros. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha|_n \leqslant \ldots$.

Let us recall the Hadamard expansion for such functions (see, for example, [1, Chapter 8, Theorem 8.2.4], [2, Chapter 7]).

Theorem 1. *If $f(z)$ is an entire function of finite order ρ, then*

$$f(z) = z^s e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \ldots + \frac{z^p}{p\alpha_n^p}}, \tag{1}$$

where $Q(z)$ is a polynomial whose degree q is not higher than ρ, s is the multiplicity of zero of f at the point 0, and $p \leqslant \rho$.

The infinite product in (1) converges absolutely and uniformly in $\mathbb{C}$. (Recall that a sequence of holomorphic functions converges uniformly in the open set U, if it converges uniformly on every compact set in U.)

In what follows we assume that $f(0) = 1$. We will write the polynomial $Q(z)$ in the form

$$Q(z) = \sum_{j=1}^{q} d_j z^j.$$

Here $d_0 = 0$, since $f(0) = 1$.

The expression

$$\Phi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \ldots + \frac{z^p}{p\alpha_n^p}} \tag{2}$$

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is called *the canonical product*, and the number p is *the genus of the canonical product. The genus of the entire function* $f(z)$ is the number $\max\{q, p\}$. If we denote by ρ' the order of the canonical product (2), then $\rho = \max\{q, \rho'\}$.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^{\gamma}}. \tag{3}$$

The infimum of positive numbers γ for which the series (3) converges is called the *index of convergence* of zeros of the canonical product $\Phi(z)$.

It is well known (see, for example, [1, Chapter 8, paragraph 8.2.5], [2, Chapter 7]) that the index of convergence of zeros of a canonical product is equal to its order ρ' .

Therefore *power sums of zeros to a negative power*

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

are absolutely convergent series for $k > \rho'$, i.e. and for $k > \rho$. It is also known that $\rho' - 1 \leqslant p \leqslant \rho'$ (see, for example, [1]).

In what follows, we will consider power sums with positive integer exponents k.

For polynomials, the recurrent formulas of Newton and Waring are well known, connecting the usual power sums of the roots of a polynomial and its coefficients (see, for example, [3–5]).

Now we will connect the integrals in the formula

$$\int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} \tag{4}$$

and power sums of zeros σ_k. Here

$$\gamma_r = \{z : \ |z| = r\}, r > 0\},$$

Let us express this integral in terms of power sums of zeros using Hadamard's formula. Let us restrict ourselves to the case when $s = 0$.

In a sufficiently small neighborhood of the origin we have (by Hadamard's formula (1))

$$\varphi(z) = \ln f(z) = Q(z) + \sum_{n=1}^{\infty} \ln\left[\left(1 - \frac{z}{\alpha_n}\right) e^{P_n(z)}\right],$$

where $P_n(z) = \dfrac{z}{\alpha_n} + \dfrac{z^2}{2\alpha_n^2} + \ldots + \dfrac{z^p}{p\alpha_n^p}$.

The series for $\varphi(z)$ converges uniformly and absolutely in a sufficiently small neighborhood of the origin, since the zeros of α_j are separated from the origin.

Let us find the integrals in (4) for each term. Obviously,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{z^k} \cdot dQ(z) = \begin{cases} kd_k & \text{при} \quad 1 \leqslant k \leqslant q, \\ 0 & \text{при} \quad k > q. \end{cases}$$

Let us transform the expression

$$d \ln\left[\left(1 - \frac{z}{\alpha_n}\right) e^{P_n(z)}\right] = \frac{d\left[\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}\right]}{\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}} =$$

$$= \frac{d\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}} + \left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}} d\left(\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}\right)}{\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}} =$$

$$= \frac{d\left(1-\frac{z}{\alpha_n}\right)}{\left(1-\frac{z}{\alpha_n}\right)} + d\left(\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}\right) =$$

$$= \frac{dz}{z-\alpha_n} + \left(\frac{1}{\alpha_n}+\frac{z}{\alpha_n^2}+\cdots+\frac{z^{p-1}}{\alpha_n^p}\right)dz =$$

$$= \frac{dz}{z-\alpha_n} + \frac{1}{\alpha_n}\left[\frac{\left(\frac{z^p}{\alpha_n^p}-1\right)}{\left(\frac{z}{\alpha_n}-1\right)}\right]dz = \frac{dz}{z-\alpha_n} + \frac{(z^p-\alpha_n^p)dz}{\alpha_n^{p-1}(z-\alpha_n)} = \frac{z^p dz}{\alpha_n^p(z-\alpha_n)}.$$

Then

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r}\frac{d\left[\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}\right]}{z^k\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r}\frac{z^{p-k}dz}{\alpha_n^p(z-\alpha_n)} =$$

$$= \begin{cases} 0, \quad \text{если} \quad k \leqslant p, \\ -\sum\limits_{n=1}^{\infty}\frac{1}{\alpha_n^k} = -\sigma_k, \quad \text{если} \quad k > p. \end{cases}$$

Thus, the equality is true if $q \leqslant p$, then

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \text{for} \quad k \leqslant q, \\ 0 \quad \text{for} \quad q < k \leqslant p, \\ -\sigma_k \quad \text{for} \quad k > p. \end{cases}$$

Let $q > p$, then

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \text{for} \quad k \leqslant p, \\ kd_k - \sigma_k \quad \text{for} \quad p < k \leqslant q, \\ -\sigma_k \quad \text{for} \quad k > q. \end{cases}$$

Thu,s we obtain the statement

Theorem 2. *For $q \leqslant p$ the following equalities are true:*

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \textit{for} \quad k \leqslant q, \\ 0 \quad \textit{for} \quad q < k \leqslant p, \\ -\sigma_k \quad \textit{for} \quad k > p. \end{cases}$$

For $q > p$ the following equalities are true:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \textit{for} \quad k \leqslant p, \\ kd_k - \sigma_k \quad \textit{for} \quad p < k \leqslant q, \\ -\sigma_k \quad \textit{for} \quad k > q. \end{cases}$$

This theorem generalizes Proposition 1.4.1 from [6].

Corollary 1. *The equality is true*

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = -\sigma_k, \quad \textit{if} \quad k > \rho.$$

Corollary 2. *The formulas are valid*

$$\sigma_k = -\frac{(-1)^{k-1}}{b_0^k}\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} \quad \textit{for} \quad k > \rho. \tag{5}$$

To prove it, it is enough to multiply the second column in the formula (5) by b_1, the third by b_2, etc., then add them to the first column.

These formulas relate the power sums σ_k and the Taylor coefficients of the function f.

Example 1. Consider the function

$$f(z) = \cos z \cdot e^z.$$

This is a function of the first order of growth ($\rho = 1$). Then

$$f(z) = \cos z \cdot e^z = 1 + z + \frac{z^3}{3!} + \dots .$$

Then from (5) we obtain $\sigma_2 = 1$, which corresponds to the known equalities.

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Нахождение степенных сумм нулей целых функций конечного порядка роста

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Аннотация. Получены формулы для нахождения степенных сумм нулей в отрицательной степени целых функций конечного порядка роста.

Ключевые слова: степенные суммы нулей, целая функция конечного порядка роста.